Somewhat quasireflexive Banach spaces

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The question of "What kind of subspaces must a nonreflexive Banach space X have?" has received a lot of attention. Pelczynski [23] (in 1962) has given the most general answer to date: X contains a basic sequence which is not shrinking (and hence spanning a nonreflexive space). For special cases more is known. Johnson and Rosenthal [8] have shown that X and X^* contain reflexive subspaces if X^{**} is separable. (This was extended to the case when X^{**}/X is separable by Clark [2].) In another direction, Davis and Johnson [5] have shown that if X^{**}/X is infinite dimensional then X contains a basic sequence that spans a nonquasireflexive subspace. Perhaps the main reason for this interest are the following two long open questions:

(1) Does each Banach space contain an unconditional basic sequence?
(2) Does each Banach space contain a subspace isomorphic to c_0, l_1 or to a reflexive space?

Indeed, James [6] has shown that a positive answer to (1) implies a positive answer to (2). And clearly these results are partial answers to (2).

On the other hand, consider the collection of spaces to which the special cases apply. James [7], Lindenstrauss [9], Davis, Figiel, Johnson and Pelczynski [4] and the author [1] show how to construct an X so that X^{**}/X is a pregiven Z (with restrictions on Z). All these constructions depend on reflexivity or quasireflexivity in a strong way and the constructed X has lots of quasireflexive subspaces.

This paper attempts to unite these results. It is shown that if X^{**}/X is separable then each element of X^{**}/X is "reachable" by an order one quasireflexive subspace $Z\subset X$, so that Z has a shrinking basis (Theorem 8). If X^{**} is separable, both X and X^* have subspaces and quotients which are order one quasireflexive with bases (Theorem 9). And if X^* is separable then X has a nonreflexive quotient

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with a basis (Corollary 10). Thus most of the known results about reflexive subspaces in the first paragraph now follow from the 1957 paper of Civin and Yood [3] (the result is contained in Corollary 11). However, as Newton, we stand on giants, we use these earlier results to obtain our results.

Define a Banach space X to be somewhat quasireflexive (resp. quowhat quasireflexive, somewhat reflexive, quowhat reflexive) if each nonreflexive subspace Y (resp. nonreflexive quotient, any subspace, any quotient) has a further subspace (resp. quotient, subspace, quotient) which is order one quasireflexive (resp. ditto, reflexive, reflexive) with a basis. It is convenient to use these notions in stating theorems. Note that question (2) above can be equivalently stated as

(2′) Does each nonreflexive Banach space contain an isomorph of c_0 or l_1 or must it be somewhat reflexive?

This question is implied (for positive answers) by

(3) Does each nonreflexive Banach space contain an isomorph of c_0 or l_1 or must it be somewhat quasireflexive?

Furthermore, with the exceptional case of X^*, when X^{**}/X is separable, but X is not separable, each known positive answer to (2) or (2′) has a positive answer to (3) as well.

There seem to be two ways to look at these results. Either "Quasireflexive spaces abounded in nature!" or "These spaces are all rather special". We leave to the reader this decision.

§ 0. Preliminaries

Our notation is fairly standard and generally follows that of [10] where undefined terms and unproved statements may be found. In particular, we write (x_n) or $(x_n)_n$ for $(x_n)_{n=1}^{\infty}$, $\sum x_n$ or $\sum_n x_n$ for $\sum_{n=1}^{\infty} x_n$ and $[x_n]$ for the closed linear span of (x_n). Continuous duals, biduals and triduals of X are written X^*, X^{**}, and X^{***} respectively. Our main departure for convention is that we do not require that a basis (x_n) satisfy $\|x_n\|=1$, but require the weaker condition that $0<\inf\|x_n\|\leqq \sup\|x_n\|<\infty$.

A sequence $(x_n)\in X$ is *a basis* if for each $x\in X$, there is a unique scalar sequence (α_n) so that $x=\lim_N \sum_1^N \alpha_n x_n$. Equivalently, there is a constant K so that

$$\left\|\sum_p^{p+q} \alpha_n x_n\right\| \leqq K\left\|\sum_1^{p+q+r} \alpha_n x_n\right\|$$

for all (α_n), p, q and r. If $K=1$, then (x_n) is said to be *bimonotone*. (*Monotone* is $K=1$ when p is restricted to be one.) A space X with a basis can always be renormed so that the basis is bimonotone. A sequence $(x_n)\subset X$ is a *basis sequence* if it is a basis for $[x_n]$.

If (x_n) is a basic sequence, then define $(x_n)^{\mathrm{LIM}}$ to be the space of all scalar sequences (α_n) so that

$$(*)\qquad \sup_N \left\|\sum_1^N \alpha_n x_n\right\| < \infty.$$

We will often write the sequence (α_n) as the (formal) sum $\sum \alpha_n x_n$. If (x_n) is monotone the left side of $(*)$ will give the norm on $(x_n)^{\mathrm{LIM}}$. (Otherwise it is an equivalent norm on $[x_n]$ which is monotone.)

The basic sequence (x_n) is *boundedly complete* if $[x_n]=(x_n)^{\mathrm{LIM}}$. The basic sequence (x_n) is *shrinking* if and only if the sequence of coefficient functionals (x_n^*) is boundedly complete. Note that $(x_n^*)^{\mathrm{LIM}}$ is isomorphic to $[x_n]^*$ for any basic sequence (x_n). If (x_n) is shrinking then $[x_n]^{**}$ is isomorphic to $(x_n)^{\mathrm{LIM}}$. Note $[x_n]$ is reflexive if and only if (x_n) is both shrinking and boundedly complete.

A space X is (order k) quasireflexive if X^{**}/X is finite dimensional (k dimensional). A basis (x_n) is *k-shrinking* (*k-boundedly complete*) if $(x_n^*)^{\mathrm{LIM}}/[x_n^*]$ ($(x_n)^{\mathrm{LIM}}/[x_n]$) is k-dimensional. For notational reasons we will assume that *quasireflexive implies nonreflexive.*

Finally, we include a result of Pelczynski [13] (or see Marti [11, pp 75, 58]). For readability, we include an outline of the proof.

Lemma 1. *If $(x_n)\subset X$, $x^{**}\in X^{**}\setminus X$ and $x_n\to x^{**}$ in the $\sigma(X^{**},X^*)$ topology, then (x_n) has a subsequence which is basic.*

Proof. Since $(x^{**}-x_n)$ is a weak-star null sequence and X^* norms X^{**}, a minor modification in the usual Mazur product construction yields a basic subsequence. Thus we may assume the whole sequence is basic and $x_1=0$. It P is the projection of $[x^{**}-x_n]$ onto the span of $x^{**}-x_1=x^{**}$, then $T=I-P\colon [x_n]_2^\infty\to [x^{**}-x_n]_2^\infty$ is one-one and onto. Hence $(x_n)_2^\infty$ is basic, since $Tx_n=-(x^{**}-x_n)$.

§ 1. Bimonotone basic sequences

This section is mostly preliminary in nature. However it is the largest part of the paper and many of the results may be of interest by themselves.

Lemma 2. *If (e_n) is a bimonotone basis, then the quotient space $(e_n)^{\mathrm{LIM}}/[e_n]$ is $(e_n)^{\mathrm{LIM}}$ modulo the equivalence relation*

$$\sum \alpha_n e_n \sim \sum \beta_n e_n \Leftrightarrow \lim_N \lim_M \left\|\sum_N^M (\beta_n-\alpha_n)e_n\right\| = 0$$

with norm given by

$$\|\sum \alpha_n e_n\| = \lim_N \lim_M \|\sum_N^M \alpha_n e_n\|.$$

Proof. Straightforward.

Remark. Note that bimonotoneness is required to obtain the quotient norm isometrically.

Lemma 3. *Let (e_n) be a basis with coefficient functionals (f_n). Suppose further that $\sum e_n \in (e_n)^{\mathrm{LIM}}$ and $(f_n)^{\mathrm{LIM}}$ is separable. Then there is an increasing sequence of integers $(N(i))$ with $N(0)=0$ so that if*

$$b_i = \sum_{N(i-1)+1}^{N(i)} e_n,$$

then (b_i) is shrinking.

Proof. Assume $(f_n)^{\mathrm{LIM}} \setminus [f_n]$ is non-empty else there is nothing to prove. Let $(\theta_m) \in (f_n)^{\mathrm{LIM}}$ be a countable dense subset. The method of proof is to diagonalize subsequences of subsequences of ...etc. to obtain the $(N(i))$.

Indeed, let (c_i) be the coefficient functionals to the (b_i) and $J: [b_i] \to [e_n]$ be the injection. Then $J^*: (f_n)^{\mathrm{LIM}} \to (c_i)^{\mathrm{LIM}}$ is the quotient map. So we want J^* to map each θ_m into $[c_i]$. For then $(c_i)^{\mathrm{LIM}} \in J^*((f_n)^{\mathrm{LIM}}) \in [c_i]$ and thus (b_i) will be shrinking.

Thus it suffices to show that for each θ_m and any subsequence $(M(i))$, there is a subsequence $(P(i))$ so that for any further subsequence $(N(i))$ then $J^*(\theta_m) \in [c_i]$. To see that this can be done, observe $(\|\sum_1^{M(i)} e_n\|)_i$ is bounded so that $(\theta_m(\sum_1^{M(i)} e_n))_i$ has a convergent subsequence. Thus pick $P(i+1)$ (for large i) so that

$$|\theta_m(\sum_{P(i)+1}^{P(i+1)} e_n)| < 2^{-i}.$$

It is straightforward to complete the proof.

Proposition 4. *If (e_n) is a bimonotone basis with $(e_n)^{\mathrm{LIM}}$ separable and $\sum e_n \in (e_n)^{\mathrm{LIM}}$, then there is an increasing sequence of integers $(N(i))$ with $N(0)=0$ so that if*

$$b_i = \sum_{N(i-1)+1}^{N(i)} e_n$$

then $(b_n)^{\mathrm{LIM}} = [\{\sum b_i\} \cup (b_i)]$.

Proof. Identify $(b_i)^{\mathrm{LIM}}$ with the obvious subspace of $(e_n)^{\mathrm{LIM}}$. Suppose $\theta \in (e_n)^{\mathrm{LIM}}$ which is not in the span of $\sum e_n + [e_n]$. We will find an increasing function $f: \mathbf{N} \to \mathbf{N}$ so that if $(N(i))$ is an increasing sequence of integers with $N(0)=0$ and so that $(N(i+1) \geqq f(N(i))$ is true for large i, then θ will not be in $(b_i)^{\mathrm{LIM}}$. So that a diagonalization will complete the proof, f must be chosen so that dist $(\theta, (b_i)^{\mathrm{LIM}})$ is "large".

Let $Z = (e_n)^{\mathrm{LIM}}/[e_n]$ and let $\varphi: (e_n)^{\mathrm{LIM}} \to Z$ be the quotient map. Let $W = [\varphi(\sum e_n)]$ in Z and let $K = \mathrm{dist}(\varphi(\theta), W)$ in Z. Since $\theta \notin \sum e_n + [e_n]$, K is positive.

We show how f can be chosen so that

$$(\#) \qquad \operatorname{dist}(\theta, (b_i)^{\mathrm{LIM}}) > K/2.$$

Write $\theta=\sum \theta_n e_n$ and let $\sigma=\sum e_n$. Let $S=\|\sigma\| \geqq \sup \|e_n\|$ and $s=\inf \|e_n\|>0$. Let $\varepsilon>0$ to be determined later. Choose $\{\delta_t\}_{t=1}^m$ to be (ε/S)-net in the scalar set $\{\alpha: |\alpha| \leqq (K+\|\theta\|)/S\}$.

For each t, $\|\varphi(\theta)=\varphi(\delta_t \sigma)\|_Z \geqq K$. Lemma 2 implies

$$\lim_N \lim_M \left\|\sum_N^M (\theta_n-\delta_t) e_n\right\| \geqq K.$$

Thus for each positive integer N, we can find $M'=f(N)$ so that for all $1 \leqq t \leqq m$ and $M \geqq M'$

$$(*) \qquad \left\|\sum_{N+1}^M (\theta_n-\delta_t) e_n\right\| > K-\varepsilon.$$

This completes the construction of the function f. Let $(N(i))$ and (b_i) be the resulting objects.

Consider
$$E_i = \left\|\sum_{N(i-1)+1}^{N(i)} (\theta_n-\beta_i) e_n\right\|.$$

If $|\beta_i|>(K+\|\theta\|)/S$, then since

$$\left\|\sum_{N(i-1)+1}^{N(i)} \theta_n e_n\right\| \leqq \|\theta\| \quad \text{and} \quad \left\|\beta_i \sum_{N(i-1)}^{N(i)} e_n\right\| \geqq |\beta_i| s$$

we have $E_i>K$. On the other hand if $\|\beta_i| \leqq (K+\|\theta\|)/S$, there is a t with $|\beta_i-\delta_t|<\varepsilon/S$ and hence $\left\|(\beta_i-\delta_t) \sum_{N(i-1)+1}^{N(i)} e_n\right\|<\varepsilon$. Combining with $(*)$ yields $E_i>K-2\varepsilon$ for large i. Thus if $\sum \beta_i b_i \in (b_i)^{\mathrm{LIM}}$, then

$$\|\theta-\sum \beta_i b_i\| \geqq \left\|\varphi(\theta)-\varphi\left(\sum \beta_i b_i\right)\right\|_Z \geqq \limsup E_i \geqq K-3\varepsilon.$$

This completes $(\#)$.

The diagonalization goes as follows. Let f_m be the function for $(\#)$ for $\theta_m \in (e_n)^{\mathrm{LIM}} \setminus [\{\sum e_n\} \cup (e_n)]$, where (θ_m) is dense in this subset. Let $f: \mathbf{N} \to \mathbf{N}$ be defined via $f(n)=\max \{f_m(n): 1 \leqq m \leqq n\}$. This function f will exclude all unwanted elements. For if θ and K are as above, choosing m so that $\|\theta-\theta_m\|<K/4$, yields

$$\operatorname{dist}(\theta, (b_n)^{\mathrm{LIM}}) > 3K/8-K/4 > 0.$$

Obviously $\sum b_i=\sum e_n \in (b_n)^{\mathrm{LIM}}$.

Corollary 5. *If (e_n) is basis which is not boundedly complete but $(e_n)^{\mathrm{LIM}}$ is separable, then there is a block basic sequence (b_i) of (e_n) with*

$$(b_i)^{\mathrm{LIM}} = [\{\sum b_i\} \cup (b_i)].$$

Proof. By renorming, (e_n) can assumed to be bimonotone. If $\sum \beta_n e_n \in (e_n)^{\mathrm{LIM}}/[e_n]$, then for some $(N(i))$ with $N(0)=0$

$$b_i = \sum_{N(i-1)+1}^{N(i)} \beta_n e_n$$

will have $\sum b_i \in (b_i)^{\mathrm{LIM}}$. The proposition can now be applied.

The dual result to Proposition 4 is much easier to prove.

Proposition 6. *If (e_n) is a basis with coefficient functionals (f_n) so that $(f_n)^{\mathrm{LIM}}$ is separable and $\sum f_n \in (f_n)^{\mathrm{LIM}}$, then for some subsequence (b_i) of (e_n) with coefficient functionals (c_i), $(c_i)^{\mathrm{LIM}} = [\{\sum c_i\} \cup (c_i)]$.*

Proof. If $\theta = \sum \theta_n f_n \in (f_n)^{\mathrm{LIM}}$, then for any subsequence of the integers $(M(i))$ we can find a further subsequence $(N(i))$ so that

(1) $\mathrm{Lim}_i\, \theta_{N(i)} = L$ and
(2) $|\theta_{N(i)} - L| < 2^{-i}$, for large i.

Doing this for a countable dense set in $(f_n)^{\mathrm{LIM}}$ and diagonalizing to obtain $(N(i))$ that satisfy (1) and (2) for each element of the dense set.

Let $b_i = e_{N(i)}$ and $\varphi\colon [b_i] \to [e_n]$ be the injection and hence $\varphi^*\colon (f_n)^{\mathrm{LIM}} \to (c_i)^{\mathrm{LIM}}$ is the quotient. By the above a dense set of $(f_n)^{\mathrm{LIM}}$ gets mapped into $[\{\sum c_i\} \cup (c_i)]$, hence everything does. Clearly $\sum c_i = \varphi^*(\sum f_n) \in (c_i)^{\mathrm{LIM}}$.

Corollary 7. *If (e_n) is a non-shrinking basis and $[e_n]^*$ is separable, then there is a block basis sequence (b_i) of (e_n) with coefficient functionals (c_i) so that, $(c_i)^{\mathrm{LIM}} = [\{\sum c_i\} \cup (c_i)]$.*

Proof. Assume (e_n) is bimonotone by renorming and let (f_n) be the coefficient functionals. If $\sum \beta_n f_n \in (f_n)^{\mathrm{LIM}} \setminus [f_n]$. Choose $(N(i))$ and (α_n) so that if $d_i = \sum_{N(i-1)+1}^{N(i)} \beta_n f_n$ and $b_i = \sum_{N(i-1)+1}^{N(i)} \alpha_n e_n$ then $\|b_i\| = 1$, the $(\|d_i\|)$ converges to a nonzero and (b_i, d_i) is biorthogonal. Thus if (c_i) is the sequence of coefficient functions to (b_i) we have $\sum c_i \in (c_i)^{\mathrm{LIM}}$ and $(c_i)^{\mathrm{LIM}}$ as a quotient of $(f_n)^{\mathrm{LIM}}$ is separable. The proposition completes the proof.

Remark. There is a nice duality here. Let us say X has a *minimal shrinking basis* if it has a shrinking basis (e_n) with $(e_n)^{\mathrm{LIM}} = [\{\sum e_n\} \cup (e_n)]$. If X has a minimal shrinking basis, then X^* has a minimal shrinking basis and there is a space Y with a minimal shrinking basis with X isomorphic to Y^*.

Indeed, if (f_n) are the coefficient functionals to the minimal shrinking basis (e_n), then (g_n) is a minimal shrinking basis for X^* where $g_1 = f_1$ and $g_n = f_n - f_{n-1}$ for $n \geqq 2$. (That (g_n) is a basis is proved similar to the construction in Theorem 8 below. The other conditions are straightforward.)

Now $(\sum_n^\infty e_i)_n$ are the coefficient functionals to (g_n). Furthermore, $([g_2]_n^\infty) = [\sum_2^\infty e_i]_{n=1}^\infty$ is isomorphic to $X = [e_n]$, since their intersection is codimension one in both spaces.

§ 2. Results and applications

Theorem 8. *If X^{**}/X is separable and $x^{**}\in X^{**}\setminus X$, then there is a shrinking basic sequence $(e_n)\subset X$ so that*

(1) $[e_n]^{**}=[\{\sum e_n\}\cup(e_n)]$, *in particular* $[e_n]$ *is order one quasireflexive, and*
(2) $S^{**}(\sum e_n)=x^{**}$, *where* S *is the injection*: $[e_n]\to X$.

Proof. Let X and x^{**} be given as above. By McWilliams [12, p. 122], there is a bounded sequence $(x_n)\subset X$ so that $x_n\to x^{**}$ in the $\sigma(X^{**}, X^*)$-topology. Let $y^*\in X^*$ so that $x^{**}(y^*)=1$. Using a small perturbation we may assume $y^*(x_n)=1$ for each n. By Lemma 1, we may also assume (x_n) is basic.

Let $e_1=x_1$ and $e_n=x_n-x_{n-1}$ for $n>1$. Clearly $\sum_1^n e_i\to x^{**}$ in $\sigma(X^{**}, X^*)$. We claim (e_n) is also basic. Note $[e_n]=[x_n]$. If $x\in[x_n]$ is written $\sum\alpha_n x_n$ then $(\|\sum_N^\infty \alpha_n x_n\|)\to 0$, hence $f_n(x)=y^*(\sum_{i=n}^\infty \alpha_i x_i)\to 0$. Let $\beta_n=f_n(x)$. Now

$$x = \lim \textstyle\sum_1^N \alpha_i x_i = \lim \sum_1^N (f_i(x)-f_{i+1}(x))x_i$$
$$= \lim\left(\textstyle\sum_1^N \beta_i e_i-\beta_{N+1}x_N\right).$$

Thus $x=\lim\sum_1^N\beta_i e_i$. Since (e_i, f_i) is biorthogonal the sequence (β_i) is the unique one so that $x=\sum\beta_i e_i$. Therefore (e_i) is basic.

Since $(e_n)\subset X$, $(e_n)^{\text{LIM}}/[e_n]$ is a subspace of X^{**}/X and hence $[e_n]^{**}$ and $[e_n]^*$ are separable. Thus by Lemma 3 we may assume (e_n) is shrinking and still have $\sum_1^N e_i\to x^{**}$ in $\sigma(X^{**}, X^*)$. Thus Proposition 4 implies that in addition we may assume $(e_i)^{\text{LIM}}=[\{\sum e_i\}\cup(e_i)]$. This completes (1), (2) follows since $\sum_1^N e_i\to\sum e_i$ in $\sigma([e_n]^{**}, [e_n]^*)$ and S^{**} is weak-star continuous.

Theorem 9. *If X^{**} is separable, then both X and X^* are both somewhat quasireflexive and quowhat quasireflexive.*

Proof. We made assume the spaces are nonreflexive or there is nothing to prove. Now X has such subspaces and X^* such quotients by Theorem 8. So we turn to subspaces of X^*. Let $X^\perp$ be the subspace of X^{***} which annihilates each element of $X\subset X^{**}$. Let $x^{***}\in X^\perp\setminus\{0\}$, and hence by McWilliams [12, p. 123], there is a bounded $(x_n^*)\subset X^*$ so that $x_n^*\to x^{***}$ in the $\sigma(X^{***}, X^{**})$-topology. Since $x^{***}\in X^\perp$, $x_n^*\to 0$ in the $\sigma(X^*, X)$-topology. Let $y^{**}\in X^{**}$ so that $x^{***}(y^{**})=1$. Again we may assume $y^{**}(x_n^*)=1$ with a small perturbation.

Now [10, pp. 11, 13] implies that by passing to a subsequence we may assume (x_n^*) is both weak-star basic and boundedly complete. Let (f_n) be the coefficient functionals to (x_n^*) in $[x_n^*]^*$. For $x^*\in[x_n^*]$ and integer N, let $x^*=\sum\alpha_n x_n^*$ and note

$$\left|\left(\sum_1^N f_n\right)(x^*)\right| = \left|\left(\sum_1^N f_n\right)\left(\sum_1^N \alpha_n x_n^*\right)\right| = \left|y^{**}\left(\sum_1^N \alpha_n x_n^*\right)\right|$$
$$\leqq \|y^{**}\| \left\|\sum_1^N \alpha_n x_n^*\right\| \leqq K\|y^{**}\| \|x^*\|,$$

where K is the basis constant of (x_n^*). Thus $\sum f_n \in (f_n)^{\mathrm{LIM}}$ and hence Proposition 6 implies that we may assume $(f_n)^{\mathrm{LIM}} = [\{\sum f_n\} \cup (f_n)]$ by passing to a subsequence.

We have (x_n^*) is weak-star basic, boundedly complete and 1-shrinking. Hence X has a quotient with a shrinking basis (e_n) whose coefficient functionals are (x_n^*). Clearly $[e_n]$ and $[e_n]^*$ are order one quasireflexive.

Remark. In some sense we can still reach x^{***} as it remains the $\sigma(X^{***}, X^{**})$-limit of (x_n^*).

Corollary 10. *If X is nonreflexive and X^* is separable, then X has a nonreflexive quotient with a shrinking basis.*

Proof. If l_i is not a subspace of X^*, then each element of X^{***} is the weak-star limit of a sequence in X^* [10, p. 101]. Thus we follow the proof of Theorem 9 to obtain (x_n^*) boundedly complete and weak-star basic, which gives us our quotient [10, p. 11]. The quotient is nonreflexive since the above proof of the theorem shows (x_n^*) isn't shrinking.

If l_1 is a subspace of X^*, then c_0 is a quotient of X [10, p. 104]. Clearly c_0 isn't reflexive and has a shrinking basis.

Remark. This seems to be the most general result about the existence of nonreflexive quotients with basis in the literature.

Corollary 11 (Civin and Yood [3]). *If X is quasireflexive, then X is both somewhat reflexive and quowhat reflexive.*

Proof. If (e_n) is shrinking and $(e_n)^{\mathrm{LIM}} = [\{\sum e_n\} \cup (e_n)]$, then (e_{2n}) spans a reflexive subspace. An appeal to Theorem 8 shows somewhat reflexivity. The quowhat reflexive property follows by duality. (See the remarks at the end of Section One.)

Corollary 12 (Johnson and Rosenthal [8]). *If X^{**} is separable, then X and X^* are both somewhat reflexive and quowhat reflexive.*

Proof. Combine Theorem 9 with Corollary 11.

Corollary 13 *If X^{**}/X is separable, then there is a finite or infinite sequence (X_k) of separable subspaces of X so that*

(1) *X_{k+1}/X_k is order one quasireflexive with a shrinking basis, and*
(2) *$X/[X_k]$ is reflexive.*

Proof. Let (z_n) be dense in X^{**}/X and pick X_{k+1} inductively Theorem 8 on X/X_k so that $(z_n)_1^{k+1} \in X_{k+1}^{**}$. And continue this until $X/[X_k]$ is reflexive.

Remark. This includes both results of Sternbach [14] and a result of McWilliams [12].

Corollary 14 (Clark [2]). *If X^{**}/X is separable, then X and X^* are somewhat reflexive.*

Proof. For X use Theorem 9 and Corollary 11. For X^*, if it's separable use Corollary 12, else use the dual of the subspace in Corollary 13 (2).

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