On the snow flake domain

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Introduction

We show that *on the boundary of the snow flake domain, harmonic measure lies completely on a set of Hausdorff dimension less than that of the entire boundary.* It is surprising because the snow flake domain is highly symmetric.

Øksendal has proved that in the plane, harmonic measure is always singular with respect to area measure, and conjectured that harmonic measure is singular with respect to α-dimensional Hausdorff measure for any $\alpha > 1$, see [4]. Our example hints that his conjecture may be true.

According to Lehto and Virtanen [3], the construction of the snow flake domain is due to G. Piranian.

Let T_0 be an equilateral triangle with side length 1 and center P. We subdivide each side of T_0 into three equal subintervals of length 1/3 each; and build an equilateral triangle over each middle subinterval, exterior to T_0 and with one side coinciding with that interval; call these triangles $S_{1,i}$, $i=1, 2, 3$ and let $T_1 = T \cup \bigcup_{i=1}^{3} S_{1,i}$.

Suppose T_j has been constructed, and is a polygon with 3×4^j sides each, of side length 3^{-j}. We subdivide each side of T_j into three equal subintervals of length 3^{-j-1} each; and build an equilateral triangle over each middle subinterval, exterior to T_j and with one side coinciding with that subinterval; call these triangles $S_{j+1,i}$, $1 \leqq i \leqq 3\times 4^{j+1}$ and let $T_{j+1} = T_j \cup \bigcup_{i=1}^{3\times 4^{j+1}} S_{j+1,i}$. Let

$$T = \bigcup_{j=1}^{\infty} T_j, \quad \bar{\Omega} = \bar{T} \quad \text{and} \quad \Omega = \bar{\Omega}^0.$$

Let f be a homeomorphism from $\partial\Omega$ onto $[0, 3]$ (mod 3), such that the three vertices of T_0 are mapped to $1, 2, 3 \equiv 0$ (mod 3), and any two endpoints of a side of T_j are mapped to points on $[0, 3]$ of distance 4^{-j} to each other. Since vertices of T_j are in $\partial\Omega$, we identify vertices of T_j with points in $[0, 3]$ (mod 3) by their quaternary expansion whenever it is convenient to do so. The vertices of T_j are exactly those points in $\partial\Omega$ whose quaternary expansion terminates.

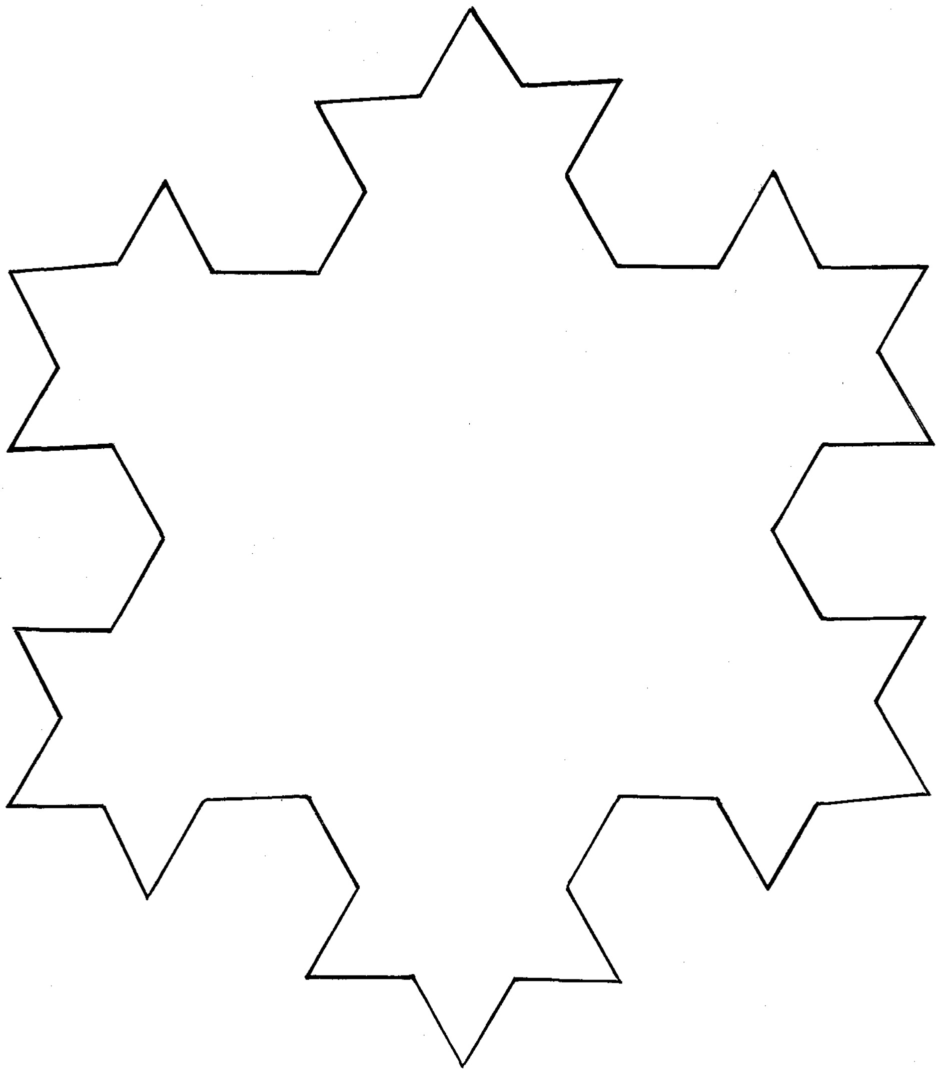

Figure: T_2

On $\partial\Omega$, f satisfies the Hölder condition:

$$|f(x)-f(y)| \leqq C|x-y|^{\frac{\log 4}{\log 3}}.$$

Therefore the $\frac{\log 4}{\log 3}$-dimensional measure of $\partial\Omega$ is positive, and it is easily seen that the $\frac{\log 4}{\log 3}$-dimensional measure of $\partial\Omega$ is finite. Hence $\dim(\partial\Omega)=\frac{\log 4}{\log 3}$.

Proposition. *Let I and J be any two neighboring vertices of T_j with*

$$f(I) = a+\sum_{i=1}^{j} a_i 4^{-i}, \quad \text{where} \quad a, a_i = 0, 1, 2 \quad \text{or} \quad 3,$$

$$f(J) = f(I)+4^{-j}.$$

Let α_1 and α_2 be arcs joining neighboring vertices of T_{j+2}, defined by

$$\alpha_1 = \left\{x\in\partial\Omega\colon f(I)+4^{-j-1} \leqq f(x) \leqq f(I)+4^{-j}\times\frac{5}{16}\right\}, \quad \text{and}$$

$$\alpha_2 = \left\{x\in\partial\Omega\colon f(I)+4^{-j}\times\frac{7}{16} \leqq f(x) \leqq f(I)+4^{-j}2^{-1}\right\}.$$

Then there exists an absolute constant $C>1$ so that

$$\frac{\omega(\alpha_1)}{\omega(\alpha_2)} > C, \tag{1}$$

where ω is the harmonic measure with respect to Ω at P.

A better result can be proved with more effort, but this suffices for the example.

To *prove the Proposition,* we let A be the endpoint of α_1 with $f(A)=f(I)+4^{-j-1}$, B be the endpoint of α_2 with $f(B)=f(I)+4^{-j}2^{-1}$ and C be the vertex of T_{j+1} adjacent to B with $f(C)=f(I)+4^{-j-1}\times 3$. From the construction, B is not on T_j, hence A, B and C form an equilateral triangle. Let γ be the subarc of $\partial\Omega$ that contains B and has endpoints A and C, Γ be the arc in Ω joining A to C, such that the domain G bounded by $\gamma\cup\Gamma$ is similar to Ω. Let Q be the center of G, and Ω_0 denote the component of $\Omega\setminus(\overline{AQ}\cup\overline{QC})$ that contains P.

We claim that

$$\omega^Y(\alpha_1, \Omega) \geqq \omega^Y(\alpha_2, \Omega) \quad \text{for} \quad Y\in\overline{AQ}\cup\overline{QC}. \tag{2}$$

By symmetry,

$$\omega^Y(\Gamma, G) < 1/2 \quad \text{for every} \quad Y\in\overline{AQ}\cup\overline{QC}.$$

Also for $Y\in\overline{AQ}\cup\overline{QC}$,

$$\omega^Y(\alpha_i, \Omega) = \omega^Y(\alpha_i, G)+\int_\Gamma \omega^Z(\alpha_i, \Omega)\, d\omega^Y(Z, G); \tag{3}$$

and for $Z\in\Gamma$,

$$\omega^Z(\alpha_i, \Omega) = \int_{\overline{AQ}\cup\overline{QC}} \omega^{Y'}(\alpha_i, \Omega)\, d\omega^Z(Y', \Omega_0). \tag{4}$$

Let R be the point on $\overline{AQ}$ with $|\overline{AR}|=\frac{1}{100}|\overline{AQ}|$, then there exists $C_0>1$ so that

$$\omega^Y(\alpha_1, G) \geqq C_0\omega^Y(\alpha_2, G) \quad \text{for} \quad Y\in\overline{AR}, \tag{5}$$

and

$$\omega^Y(\alpha_1, G) \geqq \omega^Y(\alpha_2, G) \quad \text{for} \quad Y \in \overline{AQ} \cup \overline{QC}. \tag{6}$$

It follows from (3), (4) and (6) that if $Y \in \overline{AQ} \cup \overline{QC}$, then

$$\omega^Y(\alpha_2, \Omega) - \omega^Y(\alpha_1, \Omega)$$
$$\leqq \int_\Gamma \Big[\int_{\overline{AQ}\cup\overline{QC}} (\omega^{Y'}(\alpha_2, \Omega) - \omega^{Y'}(\alpha_1, \Omega))\, d\omega^Z(Y', \Omega_0)\Big]\, d\omega^Y(Z, G)$$
$$\leqq 1/2 \max_{Y' \in \overline{AQ}\cup\overline{QC}} (\omega^{Y'}(\alpha_2, \Omega) - \omega^{Y'}(\alpha_1(\Omega)).$$

This gives (2).

In fact, from (2) and the maximum principle, we see that

$$\omega^Z(\alpha_1, \Omega) \geqq \omega^Z(\alpha_2, \Omega) \quad \text{for every} \quad Z \in \Omega_0. \tag{7}$$

Next we observe that

$$\omega^P(\alpha_i, \Omega) = \int_{\overline{AQ}\cup\overline{QC}} \omega^Y(\alpha_i, \Omega)\, d\omega^P(Y, \Omega_0) \tag{8}$$
$$= \int_{\overline{RQ}\cup\overline{QC}} \omega^Y(\alpha_i, \Omega)\, d\omega^P(Y, \Omega_0)$$
$$+ \int_{\overline{AR}} \Big[\omega^Y(\alpha_i, G) + \int_\Gamma \omega^Z(\alpha_i, \Omega)\, d\omega^Y(Z, G)\Big]\, d\omega^P(Y, \Omega_0).$$

From (5), (6), (7) and (8), we see that

$$\omega^P(\alpha_1, \Omega) \geqq \omega^P(\alpha_2, \Omega) + (C_0 - 1) \int_{\overline{AR}} \omega^Y(\alpha_2, G)\, d\omega^P(Y, \Omega_0).$$

To complete the proof, we need only show that for some $c > 0$,

$$\int_{\overline{AR}} \omega^Y(\alpha_2, G)\, d\omega^P(Y, \Omega_0) \geqq c\omega^P(\alpha_2, \Omega). \tag{9}$$

Because $\partial\Omega_0$ is a K-quasiconformal circle for some positive K independent of j, the harmonic measure on $\partial\Omega_0$ satisfies the doubling property [2]. Therefore, there exists $c > 0$ so that

$$\omega^P(\overline{A'R}, \Omega_0) > c\omega^P(\overline{AQ} \cup \overline{QC}, \Omega_0), \tag{10}$$

where A' is the midpoint of $\overline{AR}$. By symmetry

$$\omega^Q(\alpha_2, G) = \frac{1}{12}. \tag{11}$$

Hence by (10), (11), the Harnack principle and the maximum principle,

$$\int_{\overline{AR}} \omega^Y(\alpha_2, G)\, d\omega^P(Y, \Omega_0) \geqq \int_{\overline{A'R}} \omega^Y(\alpha_2, G)\, d\omega^P(Y, \Omega_0)$$

$$\geqq c\int_{\overline{A'R}} \omega^Q(\alpha_2, G)\, d\omega^P(Y, \Omega_0) \geqq c\omega^P(\overline{A'R}, \Omega_0)$$

$$\geqq c\omega^P(\overline{AQ}\cup\overline{QC}, \Omega_0) \geqq c\omega^P(\alpha_2, \Omega).$$

This proves (9) and hence the Proposition.

To prove the main assertion, let γ_0 denote any of the three subarcs of $\partial\Omega$ joining the vertices of the basic triangle T_0, γ_n any of the 3×16^n subarcs into which $\partial\Omega$ is divided by the vertices of T_{2n}, so that the diameter of an arc γ_n is $c9^{-n}$. Let E be a subset of the boundary, and suppose that for large n, E is contained in the union of cr^n arcs γ_n, where $0<r<16$. Then plainly E has Hausdorff dimension at most $\log r/\log 9$.

We find a set F, with $\omega(F)=1$, but $F=\bigcup_1^\infty E_k$, and each E_k has the property named above, with a common value $r<16$; so that $\dim F\leqq\log r/\log 9$.

The curve $\partial\Omega$ is endowed with a natural sequence $F_0\subseteqq F_1\subseteqq F_2\subseteqq F_3\subseteqq\dots$ $\dots\subseteqq F_n\subseteqq\dots$ of σ-algebras: F_n is the discrete algebra determined by the arcs γ_n, so that F_0 has 8 elements; F_n has $2^{3\times 16^n}$ elements for $n\geqq 1$. Suppose now that in each arc γ_n, we choose a distinguished subarc γ_{n+1}^0 in a certain way, and let $\chi_{n+1}(X)=1$ if $X\in\gamma_{n+1}^0$, $\chi_{n+1}(X)=0$ elsewhere on γ_n. The conditional expectation with respect to ω, $E(\chi_{n+1}|F_n)$ can be described as follows

$$E(\chi_{n+1}|F_n) = \omega(\gamma_{n+1}^0)/\omega(\gamma_n) \quad \text{on} \quad \gamma_n. \tag{12}$$

Then Lemma 1 below shows that

$$\textstyle\sum_1^N \chi_{n+1} = \sum_1^N E(\chi_{n+1}|F_n)+o(N) \quad \omega\text{-almost everywhere.} \tag{13}$$

We consider two extreme ways to choose the distinguished arcs γ_{n+1}^0: when γ_{n+1}^0 is the subarc of γ_n of greatest ω-measure, we write χ_{n+1}^+; when $\omega(\gamma_{n+1}^0)$ is least, we write χ_{n+1}^-. Using the Proposition and (12), (13) we get

$$\textstyle\sum_1^N \chi_{n+1}^+ \geqq \sigma\sum_1^N \chi_{n+1}^- - o(N) \ \ \omega\text{-almost everywhere,} \quad \text{with} \quad \sigma>1, \tag{14}$$

and since $E(\chi_{n+1}^+|F_n)\geqq 1/16$ we have further

$$\textstyle\sum_1^N \chi_{n+1}^+ \geqq N/16-o(N) \quad \omega\text{-almost everywhere.} \tag{15}$$

Let E_k be the set on $\partial\Omega$ defined by

$$\textstyle\sum_1^N \chi_{n+1}^+ \geqq \sigma^{1/2}\sum_1^N \chi_{n+1}^- \quad \text{for all} \quad N\geqq k. \tag{16}$$

From (14), (15) and (16) it follows that $\omega(E_k)$ increases to 1. From (16) and Lemma 2, we see that for large N, E_k is contained in a union of cr^N arcs γ_N, with a number $r<16$ dependent only on α. Hence $\dim E_k \leqq \log r/\log 9$; and with $F=\bigcup_1^\infty E_k$, we have $\dim F < \log 4/\log 3$ but $\omega(F)=1$. This proves our assertion.

A limit theorem. Let (X, P, F) be a probability space and $F_0 \subseteqq F_1 \subseteqq F_2 \subseteqq \ldots \subseteqq F_n \ldots$ an increasing sequence of σ-algebras contained in F; let f_n be F_n-measurable for $n \geqq 1$ and $0 \leqq f_n \leqq 1$, and let $g_n = E(f_n|F_{n-1})$, $n \geqq 1$.

Lemma 1. $f_1+\ldots+f_n-\sum_1^n g_j = o(n)$, a.e.

This result can be improved with a quantitative bound for $o(n)$, but this would have no effect on the main result. To prove it we write $h_n = f_n - g_n$, $n \geqq 1$, so that $-1 \leqq h_n \leqq 1$, $\int h_n^2 \leqq 1/4$, and $h_1+h_2+\ldots+$ is an orthogonal series. Hence by Chebyshev's inequality

$$P(|h_1+\ldots+h_{N^4}| > N^3) < N^{-2} \quad (N = 1, 2, 3, \ldots)$$

hence $h_1+\ldots+h_{N^4} = 0(N^3)$ a.e. Since $-1 \leqq h_n \leqq 1$, we also get $h_1+\ldots+h_N = 0(N^{3/4})$, a.e.

A combinatorial result. Let $\delta>0$, let N be a large integer, and let $A(N, \delta)$ be a collection of sequences $n_1, \ldots, n_N$ defined as follows: each $n_i = 0, 1, 2, \ldots, 15$, and the number of occurrences of 0 is at least $1+\delta$ times the number of occurrences of 1. Let $C(N, \delta)$ be the cardinal number of $A(N, \delta)$.

Lemma 2. *For large N,*

$$\log C(N, \delta) \leqq N \log 16 - \eta N,$$

where $\eta>0$ depends only on δ.

Proof. $C(N, \delta)$ is a sum $N!\Sigma'(r_0!\, r_1! \ldots r_{15}!)^{-1}$, where Σ' means that the sum is extended over those integers $r_0, r_1, r_2, \ldots, r_{15}$ such that $\sum_0^{15} r_j = N$ and $r_0 \geqq (1+\delta) r_1$. Since the number of ways to choose $r_0, r_1, \ldots, r_{15}$ is certainly $<(N+1)^{15}$, it will be sufficient to obtain the bound claimed for each individual term $N!(r_0!, r_1!, \ldots, r_{15}!)^{-1}$. If r_j is not too small, a rough application of Stirling's formula yields

$$\log r_j! = r_j \log r_j - r_j + 0(\log N),$$

and this is true at $r_j = 0, 1$ if $r_j \log r_j$ is defined to be 0. Writing $r_j = t_j N$ we have

$$\log (r_0!\, r_1! \ldots r_{15}!) = N \log N - N + N \sum_0^{15} t_j \log t_j + 0(\log N),$$

and therefore our claim is reduced to the estimate $\sum_0^{15} t_j \log t_j \geqq -\log 16 + \eta$; in verifying this claim we allow $t_0, \ldots, t_{15}$ to assume real values, subject to the obvious restraints. Now $t \log t$ is strictly convex on [0, 1], so its minimum for all possible real values, namely $-\log 16$, is attained only when $\frac{1}{16}=t_0=t_1=\ldots=t_{15}$. On any set of values, whose closure misses this set, the minimum must be strictly larger, as claimed, that is $\Sigma t_j \log t_j \geqq -\log 16+\eta$ if $t_0 \geqq (1+\delta) t_1$.

This Lemma is suggested by a calculation in [1].

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Added in proof: Atter submitting this paper, we received a letter from L. Carleson, containing substantial improvements of our result. More recently N. G. Makarov proved Øksendal's conjecture and solved Carleson's problem on harmonic measure [Duke Math. J., 40 (1973), 547—559].