A variant of Hall's lemma and maximal functions on the unit n-sphere

Peter M. Knopf

0. Introduction

The use of potential theory to solve the Carleman—Milloux problem and certain extremal problems was first developed by A. Beurling [1] and R. Nevanlinna [6] in 1933. They obtained lower bounds on the harmonic measure, $\omega_E(z)$, of a sufficiently nice set E in the unit disc in $\mathbf{R}^2$ evaluated at a point z in the disc.

More precisely, take the unit disc to be centered at the origin. The set E is projected on the line segment which is drawn from the origin to the unit circle such that the segment does not contain the point z but its linear extension passes through the point z (see figure 1). The projection of $\zeta\in E$ is accomplished by rotating the point ζ about the origin at a fixed distance $|\zeta|$ until it intersects ζ^* on the line segment (see

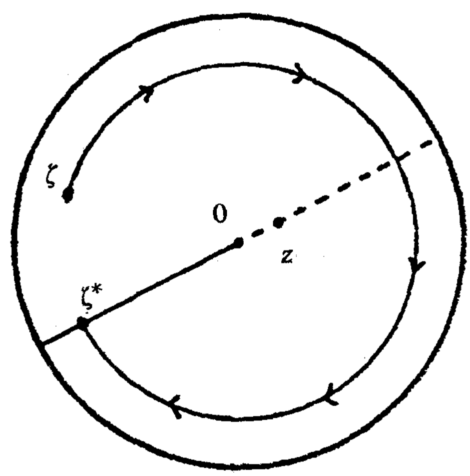

Figure 1

figure 1). Let E^* denote the set of points ζ^*. Beurling and Nevanlinna showed that $\omega_E(z) \geqq \omega_{E^*}(z)$.

A few years later, in 1937, T. Hall [2] showed an analogous result in the upper half-plane of the complex plane, $\mathbf{C}$, which will be stated after introducing some notation.

Let $\omega_E(x+iy)$ be the harmonic measure of a sufficiently nice set E evaluated at a point $x+iy$ in the upper half-plane of $\mathbf{C}$. Let E^* be the set of points obtained by rotating the points of E about the origin onto the positive x-axis. Hall showed that $\omega_E(x+iy) \geqq k\omega_{E^*}(-|x|+iy)$, where k is a constant such that $2/3 \leqq k \leqq 1$.

This paper obtains a variant of these results in the unit ball in $\mathbf{R}^n$ for $n \geqq 2$. Let E be a closed set in the interior of the unit ball in $\mathbf{R}^n$ such that the points of E are regular and let $\omega_E(z)$ be the harmonic measure of E at z. It is shown that $\omega_E(0) \geqq (c/\sqrt{n})\omega_{E^*}(0)$, where c is a positive constant independent of n, by the use of a simple formula involving the Green's function. The author conjectures that $\omega_E(0) \geqq c\omega_{E^*}(0)$ with c independent of n. The methods in this paper are not sharp enough to obtain this result.

E. Stein and J. O. Strömberg [7] have recently shown that the Hardy—Littlewood maximal function on $\mathbf{R}^n$ is weak-type (1,1) with a constant cn with c independent of n using the idea of subordination and applying the Hopf maximal ergodic theorem. The same result is obtained here by means of a new proof via the previously stated theorem on harmonic measure.

Furthermore, it is shown that the radial maximal function of the unit sphere in $\mathbf{R}^n$ is weak-type (1,1) with constant $c\sqrt{n}$. From this it easily follows that the Hardy—Littlewood maximal function on the unit sphere in $\mathbf{R}^n$ is weak-type (1,1) with constant $cn\sqrt{n}$.

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1. Notation and definitions

Let B_n stand for the closed unit ball in $\mathbf{R}^n$ centered at the origin, $\mathring{B}_n$ for the open unit ball, and ∂B_n for its boundary (the unit sphere). If $E \subseteqq B_n$, define E^* to be the radial projection from the origin of E onto ∂B_n; that is, $\zeta \in E$ implies that $\zeta^* = \frac{\zeta}{|\zeta|} \in E^*$. Let $|E^*|$ signify the Lebesgue measure of E^* on ∂B_n. The surface area of the unit sphere in $\mathbf{R}^n$ will be denoted by ω_{n-1}.

A point $z \in E$ will be said to be regular if there exists a barrier function at z. The definitions of a regular point and a barrier function are given in Hayman and Kennedy's book entitled *Subharmonic functions* [3, p. 58] in the beginning of Section 2.6.2. Suppose E is a closed set contained in $\mathring{B}_n$ such that each point of $E \cup \partial B_n$ is regular and such that $B_n \setminus (E \cup \partial B_n)$ is connected. The harmonic measure of E, $\omega_E(z)$, is defined to be the solution of the Dirichlet problem on $B_n \setminus (E \cup \partial B_n)$ with boundary values 1 on E and 0 on ∂B_n in the sense stated in [3, p. 58] in Theorem 2.10. The above conditions guarantee that $\omega_E(z)$ is well-defined.

The Poisson kernel of B_n is given by

$$P(z, \zeta^*) = \frac{1-|z|^2}{\omega_{n-1}|z-\zeta^*|^n}.$$

If $f \in L^1(\partial B_n)$, then

$$u(z) = \int_{\partial B_n} P(z, \zeta^*) f(\zeta^*)\, d\Sigma(\zeta^*) \tag{1}$$

defines a harmonic function in B_n, where $z \in B_n$ and $d\Sigma$ is the Lebesgue measure on the unit sphere. If E^* is a closed set in ∂B_n, the harmonic measure of E^*, $\omega_{E^*}(z)$ is defined to be

$$\omega_{E^*}(z) = \int_{\partial B_n} P(z, \zeta^*)\, X_{E^*}(\zeta^*)\, d\Sigma(\zeta^*) \tag{2}$$

where X_{E^*} is the characteristic function of the set E^*.

The radial maximal function of f is defined to be

$$u^*(z^*) = \sup_{0 \leq r \leq 1} |u(rz^*)| \tag{3}$$

with $z^* \in \partial B_n$. The Hardy—Littlewood maximal function of f on the unit sphere is

$$Mf(z^*) = \sup_{0 \leq t \leq 2} \int_{\partial B_n} \frac{1}{|S(z^*, t)|} X_{S(z^*, t)}(\zeta^*) |f(\zeta^*)|\, d\Sigma(\zeta^*), \tag{4}$$

where $S(z^*, t) = \{\zeta^* \in \partial B_n \colon |z^* - \zeta^*| \leq t\}$ and $z^* \in \partial B_n$.

The symbol c will stand for a positive constant that may be different at different appearances but will always lie between 10^{-6} and 10^6.

2. The results

We begin by stating the main theorem on harmonic measure.

Theorem. *If E is a closed subset of $\mathring{B}_n$ such that $B_n \setminus (E \cup \partial B_n)$ is connected and every point of $E \cup \partial B_n$ is regular, then $\omega_E(0) \geq (c/\sqrt{n})\omega_{E^*}(0)$ for $n \geq 2$.*

The conclusion of the theorem can be restated to be

$$\omega_E(0) \geqq \frac{c}{\omega_{n-1}\sqrt{n}}|E^*| \tag{5}$$

for $n\geqq 2$. This we now verify. Letting $z=0$ in (2), we obtain $\omega_{E^*}(0)=\frac{1}{\omega_{n-1}}|E^*|$, which establishes inequality (5).

There is a striking connection between the theorem above and weak-type (1,1) inequalities for the Hardy—Littlewood maximal function and the radial maximal function on ∂B_n as we shall see in the following two applications.

Corollary 1. *If* $f\in L^1(\partial B_n)$ *and* $n\geqq 2$, *then*

$$\left|\{z^*\in\partial B_n:\ u^*(z^*)\geqq\lambda\}\right| \leqq \frac{c\sqrt{n}}{\lambda}\|f\|_1$$

for all $\lambda>0$.

Proof. The technical part of this proof consists in constructing for almost all $\lambda>0$ certain corresponding sets $S=S(\lambda)$; two of their properties are that $B_n\setminus(S\cup\partial B_n)$ are connected open subsets in B_n and every point of S is a regular point for every S. These two properties will insure that $\omega_S(z)$ is well-defined and has well-behaved boundary values for every S. The rest of the proof of this corollary is an elegant argument due to L. Carleson.

Without loss of generality we can assume that $f\geqq 0$ and $f\in C^\infty(\partial B_n)$. Fix $\lambda>0$. Consider the closed sets $E=\{z\in B_n:\ u(z)\geqq\lambda\}$ and $E^*=\{z^*\in B_n:\ u^*(z^*)\geqq\lambda\}$. Suppose $|E\cap E^*|\geqq\frac{1}{2}|E^*|$. On the set $E\cap E^*$ we have $u(z^*)=f(z^*)\geqq\lambda$. This implies that

$$|E^*| \leqq 2|E\cap E^*| \leqq 2\int_{E\cap E^*}\frac{f(z^*)}{\lambda}\,d\Sigma(z^*) \leqq \frac{2\|f\|_1}{\lambda}$$

as desired. We are left to consider the case when $|(E\cap\mathring{B}_n)^*|\geqq\frac{1}{2}|E^*|$. Define the set F to be $F=\{z\in\mathring{B}_n:\ u(z)=\lambda\}$. Suppose $|F^*|<|(E\cap\mathring{B}_n)^*|$. In this case there exists a $z^*\in\partial B_n$ such that $u(rz^*)>\lambda$ for all $0\leqq r<1$. In particular, $u(0)>\lambda$ and thus $\|f\|_1/\omega_{n-1}=u(0)>\lambda$. Clearly we would then have $|E^*|\leqq\omega_{n-1}\leqq\|f\|_1/\lambda$. So we are left with the case $|F^*|=|(E\cap B_n)^*|$ and $|F^*|\geqq\frac{1}{2}|E^*|$.

Define the set D_1 to be $D_1=\{z\in B_n:\ u(z)>\lambda\}$. Fix a point $z_0\in\mathring{B}_n$ in D_1. Consider the component of D_1 which contains the point z_0. We claim that the closure of this component must intersect ∂B_n. If not, then the boundary of the component is contained in $\mathring{B}_n$. By the maximum principle $u(z_0)=\lambda$, which contradicts the assumption. The same argument implies that the closure of every component of the set $D_2=\{z\in\mathring{B}_n:\ u(z)>\lambda\}$ intersects ∂B_n.

Pick $\varepsilon>0$ so that the set $S=F\cap\{z\in B_n: 0\leqq|z|\leqq 1-\varepsilon\}$ has the property that $|S^*|\geqq\frac{1}{4}|E^*|$ which clearly can be done. Since the closure of every component of $D_1\cup D_2$ intersects ∂B_n, then it is obvious that $B_n\setminus S$ is a connected open set in B_n.

Since $f\in C^\infty(\partial B_n)$, then $u: \mathring{B}_n\to\mathbf{R}$ is a C^∞-function. By Sard's theorem [5, p. 16], we have for almost all λ that every point of S has a non-zero gradient of u. Clearly it is enough to prove the corollary for these λ. Since the gradient of u is non-zero at every point of S, then it is obvious that every point of S satisfies the "cone" condition (a) or (d) in Theorem 2.11 in [3, p. 61]. This guarantees that each point of S is a regular point. Since dist $(S, \partial B_n)\geqq\varepsilon$, every point of ∂B_n is also a regular point.

We have now shown that $B_n\setminus(S\cup\partial B_n)$ is an open connected set and every point of $S\cup\partial B_n$ is a regular point. By Theorem 2.10 in [3, p. 58] we know that $\omega_S(z)$ is well-defined and $\lambda\omega_S(z)\leqq u(z)$. By the maximum principle we have

$$\lambda\omega_S(0)\leqq u(0)=\frac{1}{\omega_{n-1}}\|f\|_1.$$

By the theorem we have

$$\omega_S(0)\geqq\frac{c}{\omega_{n-1}\sqrt{n}}|S^*|\geqq\frac{c}{\omega_{n-1}\sqrt{n}}|E^*|,$$

which completes the proof of the corollary.

Corollary 2. *If $f\in L^1(\partial B_n)$ and $n\geqq 2$, then*

$$\left|\{z^*\in\partial B_n: M_f(z^*)\geqq\lambda\}\right|\leqq\frac{cn\sqrt{n}}{\lambda}\|f\|_1 \quad \textit{for all} \quad \lambda>0.$$

Proof. For any $z^*\in\partial B_n$ and $0\leqq r\leqq 1$, define $z=rz^*$. Fix z^*. By (1), (3), (4), and Corollary 1, it is enough to show that for every t, $0\leqq t\leqq 2$, there exists an r depending only on n and t such that

$$Mf(z^*)\leqq cn\,u(rz^*)\equiv cn\,u(z).$$

This implies that it is enough to show that

$$\frac{1}{|S(z^*,t)|}X_{S(z^*,t)}(\zeta^*)\leqq cn\,P(z,\zeta^*)$$

for every $\zeta^*\in\partial B_n$. Since $X_{S(z^*,t)}$ is supported on $S(z^*,t)$ and equals 1 there and $P(z,\zeta^*)$ decreases as $|z^*-\zeta^*|$ increases, it is enough to show

$$I\equiv nP(z,\zeta^*)|S(z^*,t)|\geqq c$$

for every ζ^* such that $|\zeta^*-z^*|=t$ with $0\leqq t\leqq 2$. Using spherical coordinates it is easy to see that

$$|S(z^*, t)| = \omega_{n-2}\int_0^{2\arcsin(t/2)} \sin^{n-2} u\, du \geqq \omega_{n-2}\int_0^{2\arcsin(t/2)} \sin^{n-2} u \cos u\, du$$

$$\geqq \frac{c\omega_{n-2}}{n}[t^2(1-t^2/4)]^{(n-1)/2},$$

as long as $0 \leqq t \leqq \sqrt{2}$. By the law of cosines it is a straightforward calculation to verify that

$$|z-\zeta^*|^2 = (1-|z|)^2+|z|t^2.$$

From this we obtain

$$I \geqq c\frac{\omega_{n-2}(1-|z|)}{\omega_{n-1}t\sqrt{1-t^2/4}}\left[\frac{t^2(1-t^2/4)}{(1-|z|)^2+|z|t^2}\right]^{n/2}$$

when $0 \leqq t \leqq \sqrt{2}$. If $1/\sqrt{n} \leqq t \leqq \sqrt{2}$, choose $1-|z|=t^2/2$. In this case $I \geqq c\omega_{n-2}t/\omega_{n-1} \geqq c$ since $\omega_{n-2}/\omega_{n-1} \geqq c\sqrt{n}$ and $t \geqq 1/\sqrt{n}$. If $0<t<1/\sqrt{n}$, choose $1-|z|=t/\sqrt{n}$. One then has

$$I \geqq c\frac{\omega_{n-2}}{\omega_{n-1}\sqrt{n}}\left[\left(1-\frac{1}{4n}\right)\Big/\left(1+\frac{1}{n}\right)\right]^{n/2} \geqq c.$$

The case $\sqrt{2} \leqq t \leqq 2$ is trivial to handle by picking $1-|z|=1$ and observing that

$$\tfrac{1}{2}\omega_{n-1} \leqq |S(z^*, t)| \leqq \omega_{n-1},$$

whenever $\sqrt{2} \leqq t \leqq 2$.

Proof of the theorem. Without loss of generality we can assume that $E \cap \{0\} = \emptyset$ since otherwise there is nothing to prove. Furthermore we can assume that E has the property that every ray from the origin intersects E at most once by the following argument. Let $F=\{z \in B_n\colon z=Rz^*,\ z^* \in E^*,$ and $R=\sup_{0 \leqq r \leqq 1} r$ such that $rz^* \in E\}$. Since E is a closed set in B_n, then F^* is a closed set in ∂B_n and $|E^*|=|F^*|$. Since $F \subseteqq E$, we have $\omega_E(0) \geqq \omega_F(0)$ and thus it suffices to show

$$\omega_F(0) \geqq \frac{c}{\omega_{n-1}\sqrt{n}}|F^*|.$$

The first part of the proof is not cumbersome and consists of reducing the problem to a one variable maximization problem. We wish to show $\omega_E(0) \geqq (c/\omega_{n-1}\sqrt{n})|E^*|$. The idea of the proof is to construct a harmonic function, say V, explicitly, from which it will immediately be seen that $V(0)=\frac{1}{\omega_{n-1}}|E^*|$ and which, but this takes some work, satisfies $\omega_E(0) \geqq \frac{c}{\sqrt{n}}V(0)$.

First consider the case when $n \geqq 3$. We define

(6)
$$V(z) = \frac{1}{\omega_{n-1}} \int_E G(z, \zeta) \frac{|\zeta|^{n-2}}{1-|\zeta|^{n-2}} d\bar{\Sigma}(\zeta),$$

where $G(z, \zeta)$ is the Green's function for B_n and $d\bar{\Sigma}$ is the Borel measure on E such that $\bar{\Sigma}(F) = |F^*|$ for any Borel set $F \subseteqq E$. Since $E \subseteqq \mathring{B}_n$ and every ray from the origin intersects E at most once, (6) is certainly well-defined. It is well-known that

(7)
$$G(z, \zeta) = \begin{cases} \dfrac{1}{|z-\zeta|^{n-2}} - \dfrac{1}{\left| z|\zeta| - \dfrac{\zeta}{|\zeta|} \right|^{n-2}}, & \zeta \neq 0 \\ \dfrac{1}{|z|^{n-2}} - 1, & \zeta = 0 \end{cases}$$

is harmonic in $B_n \setminus \{z\}$ and equals zero on ∂B_n. It is easy to see that $V(z)$ is harmonic in $B_n \setminus E$ and equals zero on ∂B_n.

Suppose we succeed in showing $V(z) \leqq c\sqrt{n}$ for every $z \in E$. Since $V(z) = 0$ on ∂B, we would have $V(z) \leqq c\sqrt{n}\,\omega_E(z)$ for $z \in E \cup \partial B_n$. By the maximum principle we could conclude $V(z) \leqq c\sqrt{n}\,\omega_E(z)$ everywhere in B_n. Picking $z = 0$, we would have $V(0) \leqq c\sqrt{n}\,\omega_E(0)$. Since

(8)
$$V(0) = \frac{1}{\omega_{n-1}} \int_E G(0, \zeta) \frac{|\zeta|^{n-2}}{1-|\zeta|^{n-2}} d\bar{\Sigma}(\zeta) = \frac{1}{\omega_{n-1}} \int_E d\bar{\Sigma}(\zeta) = \frac{|E^*|}{\omega_{n-1}},$$

the theorem would then follow. So we are left to show $V(z) \leqq c\sqrt{n}$ for $z \in E$.

Independently, T. J. Lyons, K. B. Mac Gibbon, and J. C. Taylor [4] constructed and studied the same function $V(z)$ and have shown that the function is bounded independently of z and E. Their bound on the function grows exponentially with n.

We can decompose the set E into two parts, $E = E_1 \cup E_2$, where $E_1 = \{\zeta \in E: 0 < |\zeta| \leqq 1 - 1/n\}$ and $E_2 = \{\zeta \in E: 1 - 1/n < |\zeta| < 1\}$. Clearly we have

(9)
$$V(z) = V_1(z) + V_2(z),$$

where

(10)
$$V_1(z) = \frac{1}{\omega_{n-1}} \int_{E_1} G(z, \zeta) \frac{|\zeta|^{n-2}}{1-|\zeta|^{n-2}} d\bar{\Sigma}(\zeta)$$

and

(11)
$$V_2(z) = \frac{1}{\omega_{n-1}} \int_{E_2} G(z, \zeta) \frac{|\zeta|^{n-2}}{1-|\zeta|^{n-2}} d\bar{\Sigma}(\zeta).$$

Let t be the distance from the point $\dfrac{z}{|z|}$ to the point $\dfrac{\zeta}{|\zeta|}$. By the law of cosines, we have

(12) $$|z-\zeta|^2 = (|z|-|\zeta|)^2+|z||\zeta|t^2$$

and

(13) $$\left|z|\zeta|-\frac{\zeta}{|\zeta|}\right|^2 = (1-|z|\,|\zeta|)^2+|z|\,|\zeta|t^2.$$

Thus we can view $V_i(z)$ for $i=1$ or 2 in the following way:

(14) $$V_i(z) = \int_{E_i} H(|z|, |\zeta|, t)\, d\Sigma(\zeta),$$

where H is a function of $|z|$, $|\zeta|$, and t. Clearly it is enough to find for each fixed z the set F_z^i which maximizes

(15) $$\int_{F_z^i} H(|z|, |\zeta|, t)\, d\Sigma(\zeta)$$

and then to show that expression (15) is less than $c\sqrt{n}$ where F_z^i has the property that any ray from the origin intersects F_z^i at most once and furthermore

$$F_z^1 \subseteqq \{\zeta\in B_n\colon 0 < |\zeta| \leqq 1-1/n\}$$

and

$$F_z^2 \subseteqq \{\zeta\in B_n\colon 1-1/n < |\zeta| < 1\}.$$

Certainly we can assume $(F_z^i)^*=\partial B_n$. Since $d\Sigma$ is Lebesgue surface measure on ∂B, it is a matter of choosing F_z^i so that $H(|z|, |\zeta|, t)$ is maximized for each $\zeta^*\in\partial B_n$ with $\zeta=|\zeta|\zeta^*$; that is, we wish to find $|\zeta|$ so that $H(|z|, |\zeta|, t)$ is maximized for each $\zeta^*\in\partial B_n$ with $\zeta=|\zeta|\zeta^*$. We may regard $|\zeta|$ as a function of ζ^* in (15). Since $H(|z|, |\zeta|, t)$ depends on $|\zeta|$ and t for fixed z, we can regard $|\zeta|$ as a function of t alone. By what we have said $F_z^i=\{\zeta\in B_n\colon |\zeta|=g_z^i(t)\}$ where g_z^i is the appropriate function of t to maximize (15) subject to the constraint $0<g_z^1(t)\leqq 1-1/n$ if $i=1$ or $1-1/n< g_z^2(t)<1$ if $i=2$. Thus (15) can be rewritten as an integral in t. The surface area on the unit sphere of the spherical cap, $S\left(\frac{z}{|z|}, t\right)$, of radius t about the point $\frac{z}{|z|}$ is

$$\left|S\left(\frac{z}{|z|}, t\right)\right| = \int_0^{2\arcsin(t/2)} \omega_{n-2}\sin^{n-2}u\, du$$

with $0\leqq t\leqq 2$. Consequently the surface area element is

$$d\left|S\left(\frac{z}{|z|}, t\right)\right| = \omega_{n-2}t^{n-2}(1-t^2/4)^{(n-3)/2}\, dt.$$

Hence (15) can be replaced with the problem of maximizing the following expressions:

(16) $$v_i(z) = c\sqrt{n}\int_0^2 G(z,\zeta)\frac{|\zeta|^{n-2}}{1-|\zeta|^{n-2}}\, t^{n-2}(1-t^2/4)^{(n-3)/2}\, dt$$

such that $|\zeta|=g_z^i(t)$. Rather than finding $g_z^i(t)$, we will obtain upper bounds on $v_i(z)$. To accomplish this we establish a number of lemmas. Let $s=1-|z|$ and $\sigma=1-|\zeta|$ in order to make future calculations less cumbersome.

Lemma 1. *If* $0\leqq c<N$, *then*

$$e^{-c/(1-c/N)} \leqq (1-c/N)^N \leqq e^{-c}. \tag{17}$$

Proof. If $0\leqq x<1$, then

$$\frac{-x}{1-x} = -\sum_{k=1}^{\infty} x^k \leqq \log(1-x) = -\sum_{k=1}^{\infty} \frac{1}{k} x^k \leqq -x.$$

This implies that

$$e^{-Nx/(1-x)} \leqq (1-x)^N \leqq e^{-Nx}.$$

Choose $x=c/N$ to conclude the proof.

Lemma 2. *The term* $\dfrac{|\zeta|^{n-2}}{1-|\zeta|^{n-2}}$ *which appears in expression* (16) *satisfies*

$$\frac{|\zeta|^{n-2}}{1-|\zeta|^{n-2}} \leqq 10|\zeta|^{n-2} \tag{18}$$

if $\sigma\geqq 1/n$ *and satisfies*

$$\frac{|\zeta|^{n-2}}{1-|\zeta|^{n-2}} \leqq \frac{10}{n\sigma} \tag{19}$$

if $\sigma\leqq 1/n$.

Proof. By Lemma 1 and the condition $\sigma\geqq 1/n$, we have

$$1-|\zeta|^{n-2} = 1-(1-\sigma)^{n-2} \geqq 1-e^{-\sigma(n-2)} \geqq 1-e^{-1/3} \geqq 1/10.$$

This verifies (18). To verify (19), we first note that

$$\frac{|\zeta|^{n-2}}{1-|\zeta|^{n-2}} \leqq \frac{1}{1-|\zeta|^{n-2}}.$$

By the mean-value theorem, we have

$$1-(1-\sigma)^{n-2} = (n-2)\sigma(1-\xi)^{n-3}$$

for some $0<\xi<\sigma$. By applying Lemma 1 and the condition $\sigma\leqq 1/n$, the following chain of inequalities is valid:

$$(n-2)(1-\xi)^{n-3} \geqq (n-2)(1-\sigma)^{n-3} \geqq (n-2)e^{\frac{-\sigma(n-3)}{1-\sigma}} \geqq (n-2)\, e^{-\frac{(n-3)}{(n-1)}} \geqq \frac{1}{3} ne^{-1}.$$

This establishes (19).

Lemma 3. *The Green's function satisfies*

$$G(z, \zeta) \leqq \frac{2n\sigma s}{|z-\zeta|^n}. \tag{20}$$

Proof. From (12) and (13), it is easy to see that

$$\left|z|\zeta|-\frac{\zeta}{|\zeta|}\right|^2 = |z-\zeta|^2+(1-|z|^2)(1-|\zeta|^2). \tag{21}$$

By equation (7), it is enough to show that

$$x^{-(n-2)/2}-(x+y)^{-(n-2)/2} \leqq \frac{ny}{2} x^{-n/2} \tag{22}$$

for any x and $y>0$. It should be noted that

$$(1-|z|^2)(1-|\zeta|^2) \leqq 4\sigma s.$$

By the mean-value theorem we have

$$x^{-(n-2)/2}-(x+y)^{-(n-2)/2} = \frac{n-2}{2} yw^{-n/2}$$

for some w such that $x<w<x+y$. Clearly

$$\frac{n-2}{2} yw^{-n/2} \leqq \frac{ny}{2} x^{-n/2}.$$

We are now ready to conclude the proof of the theorem for $n\geqq 3$. Since $V(z)\leqq v_1(z)+v_2(z)$, it is enough to prove the following lemma.

Lemma 4. *For any* $z\in B_n$, *we have* $v_1(z)\leqq c\sqrt{n}$ *and* $v_2(z)\leqq c$.

Proof. First consider $v_1(z)$. Since $|\zeta|=g_z^1(t)$ satisfies $0<g_z^1(t)\leqq 1-1/n$, this implies that

$$\frac{|\zeta|^{n-2}}{1-|\zeta|^{n-2}} \leqq 10|\zeta|^{n-2} \tag{23}$$

by Lemma 3. Furthermore we always have

$$G(z, \zeta) \leqq \frac{1}{|z-\zeta|^{n-2}} \tag{24}$$

as can be seen from (7). By (12), (16), (23), and (24), we obtain

$$v_1(z) \leqq c\sqrt{n} \int_0^2 \left[\frac{|\zeta|^2 t^2(1-t^2/4)}{(|z|-|\zeta|)^2+|z|\,|\zeta|\,t^2}\right]^{(n-2)/2} (1-t^2/4)^{-1/2}\, dt.$$

However, it is true that

$$\frac{|\zeta|^2 t^2(1-t^2/4)}{(|z|-|\zeta|)^2+|z|\,|\zeta|\,t^2} \leqq 1, \tag{25}$$

since inequality (25) reduces to simply

$$\left(|z|-|\zeta|+\frac{|\zeta|t^2}{2}\right)^2 \geqq 0,$$

which is clearly true. Hence we arrive at

$$v_1(z) \leqq c\sqrt{n}\int_0^2 (1-t^2/4)^{-1/2}\,dt \leqq c\sqrt{n},$$

as we wished to show.

Next we handle $v_2(z)$. Since $1-1/n<g_z^2(t)<1$, then $v_2(z)$ is harmonic in $\{z\in B_n\colon 0\leqq|z|\leqq 1-1/n\}$. By the maximum principle, if we show that $v_2(z)\leqq c$ for $z\in B_n\cap\{z\in B_n\colon 1-1/n\leqq|z|<1\}$, then $v_2(z)\leqq c$ for every $z\in B_n$. Thus we can assume $z\in B_n\cap\{z\in B_n\colon 1-1/n\leqq|z|<1\}$.

We break the range of integration in equation (16) for $v_2(z)$ into two parts as follows:

$$v_2(z) \equiv \int_0^{s\sqrt{n}} + \int_{s\sqrt{n}}^2 \equiv I_1+I_2.$$

We first handle I_2. By Lemmas 2 and 3,

$$G(z,\zeta)\frac{|\zeta|^{n-2}}{1-|\zeta|^{n-2}} \leqq \frac{20s}{|z-\zeta|^n}. \tag{26}$$

By (12), (16), and (26), we have

$$I_2 \leqq c\sqrt{n}\int_{s\sqrt{n}}^2 \frac{s}{[(|\zeta|-|z|)^2+|z|\,|\zeta|\,t^2]^{n/2}}\,t^{n-2}(1-t^2/4)^{(n-3)/2}\,dt$$

$$\leqq c\sqrt{n}\int_{s\sqrt{n}}^2 \frac{st^{n-2}}{[|z|\,|\zeta|\,t^2]^{n/2}}\,dt \leqq \frac{c\sqrt{n}}{(1-1/n)^n}\int_{s\sqrt{n}}^2 \frac{st^{n-2}}{t^n}\,dt,$$

since $1-1/n\leqq|z|\leqq 1$ and $1-1/n<|\zeta|<1$. We finally obtain

$$I_2 \leqq cs\sqrt{n}\int_{s\sqrt{n}}^2 t^{-2}\,dt \leqq c\left(1-s\sqrt{n}\right) \leqq c$$

as desired.

To handle I_1, suppose first $\sigma \leqq s/2$. We use estimate (26) once again. Equations (12), (16), (26), and the fact that $\sigma \leqq s/2$ imply

$$I_1 \leqq c\sqrt{n}\int_0^{s\sqrt{n}} \frac{s}{[s^2/4+|\zeta|\,|z|t^2]^{n/2}}\, t^{n-2}(1-t^2/4)^{(n-3)/2}\, dt$$

$$\leqq c\sqrt{n}\int_0^{s\sqrt{n}} \frac{st^{n-2}}{[s^2/4+(1-1/n)^2 t^2]^{n/2}}\, dt \leqq \frac{c\sqrt{n}}{(1-1/n)^n}\int_0^{s\sqrt{n}} \frac{st^{n-2}}{[s^2/4+t^2]^{n/2}}\, dt.$$

After the change of variables $t \to \dfrac{s}{2t^{1/2}}$, we obtain

$$I_1 \leqq c\sqrt{n}\int_0^{\infty} (1+t)^{-n/2} t^{-1/2}\, dt = \frac{c\sqrt{n}\,\Gamma(n/2-1/2)}{\Gamma(n/2)} \leqq c.$$

Now suppose $\sigma \geqq s/2$. We use the estimate

$$G(z,\zeta)\frac{|\zeta|^{n-2}}{1-|\zeta|^{n-2}} \leqq \frac{20}{ns|z-\zeta|^{n-2}}, \tag{27}$$

which follows by (19), (24), and the fact that $\sigma \geqq s/2$. By (12), (16), and (27), it follows that

$$I_1 \leqq c\sqrt{n}\int_0^{s\sqrt{n}} \frac{1}{ns[(|z|-|\zeta|)^2+|z|\,|\zeta|t^2]^{(n-2)/2}}\, t^{n-2}(1-t^2/4)^{(n-3)/2}\, dt$$

$$\leqq c\sqrt{n}\int_0^{s\sqrt{n}} \frac{t^{n-2}}{ns[|z|\,|\zeta|\,t^2]^{(n-2)/2}}\, dt \leqq \frac{c\sqrt{n}}{(1-1/n)^{n-2}}\int_0^{s\sqrt{n}} \frac{t^{n-2}}{nst^{n-2}}\, dt \leqq c.$$

This concludes the proof of Lemma 4 and hence also that of the theorem for $n \geqq 3$.

It remains to prove the theorem for $n=2$. Here the formulas are different as the Green's function in two dimensions has logarithmic terms. The proof is the same for $n=2$ as for $n \geqq 3$ up to expression (16), except we use

$$V(z) = \int_E G(z,\zeta)\frac{1}{\log\left(\frac{1}{|z|}\right)}\, d\Sigma(\zeta)$$

in place of (6), where

$$G(z,\zeta) = \log\left|z|\zeta| - \frac{\zeta}{|\zeta|}\right| - \log|z-\zeta| \tag{28}$$

if $\zeta \neq 0$ and $d\Sigma$ is as before. Analogously, we need an upper bound, independent of z, on

$$v_i(z) = \int_0^2 G(z,\zeta)\frac{1}{\log\left(\frac{1}{|\zeta|}\right)}(1-t^2/4)^{-1/2}\, dt \tag{29}$$

with $|\zeta|=g_z^i(t)$ a function of t such that $0<g_z^1(t)<1/2$ and $1/2\leqq g_z^2(t)<1$. It is easy to check that

$$G(z,\zeta)\leqq\frac{4\sigma s}{|z-\zeta|^2} \tag{30}$$

by the same proof as in Lemma 3 and it is easy to verify that

$$G(z,\zeta)\leqq\log\frac{100\sigma s}{|z-\zeta|^2}, \tag{31}$$

as long as $\dfrac{\sigma s}{|z-\zeta|^2}\geqq 1/20$.

Let us first handle $v_1(z)$ which means $1/2<\sigma<1$. Consider the case $0\leqq s\leqq 1/4$. We then have

$$G(z,\zeta)\leqq cs \tag{32}$$

by (12) and (30). It is obvious that

$$\frac{1}{\log\left(\frac{1}{|\zeta|}\right)}\leqq c \tag{33}$$

when $1/2<\sigma<1$. By (29), (32), and (33) we have

$$v_1(z)\leqq c\int_0^2 s(1-t^2/4)^{-1/2}\,dt\leqq c. \tag{34}$$

There is left the case $1/4\leqq s\leqq 1$. In this case it is easy to check that the log form applies. There are three subcases to consider. If $|\zeta|>2|z|$, then

$$v_1(z)\leqq c\int_0^2(1-t^2/4)^{-1/2}\log\left[\frac{400}{|\zeta|^2}\right]\Big/\log\left(\frac{1}{|\zeta|^2}\right)dt\leqq c\int_0^2(1-t^2/4)^{-1/2}\,dt\leqq c.$$

If $|\zeta|<|z|/2$, then

$$v_1(z)\leqq c\int_0^2(1-t^2/4)^{-1/2}\log\left[\frac{400}{|z|^2}\right]\Big/\log\left(\frac{1}{|\zeta|^2}\right)dt\leqq c\int_0^2(1-t^2/4)^{-1/2}\,dt\leqq c.$$

Finally if $|z|/2\leqq|\zeta|\leqq 2|z|$, then

$$v_1(z)\leqq c\int_0^2(1-t^2/4)^{-1/2}\log\left[\frac{100}{(|\zeta|-|z|)^2+|z|\,|\zeta|t^2}\right]\Big/\log\left(\frac{1}{|\zeta|^2}\right)dt$$

$$\leqq c\int_0^2(1-t^2/4)^{-1/2}\log\left[\frac{200}{|z|^2t^2}\right]\Big/\log\left(\frac{4}{|z|^2}\right)dt\leqq c\int_0^2(1-t^2/4)^{-1/2}\log\left[\frac{200}{t^2}\right]dt\leqq c.$$

Now let us handle $v_2(z)$ which means $0<\sigma\leqq 1/2$. Since $1/2\leqq g_z^2(t)<1$, then $v_2(z)$ is harmonic in $\{z\in B_2\colon 0\leqq|z|\leqq 1/2\}$. By the maximum principle, if we show that $v_2(z) \leqq c$ for $z \in B_2 \cap \{z \in B_2 \colon 1/2 \leqq |z| < 1\}$, then $v_2(z) \leqq c$ for every $z \in B_2$. Thus we can assume $0 < s \leqq 1/2$. We break the range of integration for $v_2(z)$ into two parts:

$$v_2(z) \equiv \int_0^s + \int_s^2 \equiv I_1 + I_2.$$

First consider I_2. As in the proof of Lemma 2,

$$\log(1-\sigma) \leqq -\sigma.$$

From this one immediately sees that

$$\frac{1}{\log\left(\frac{1}{|\zeta|}\right)} \leqq 1/\sigma. \tag{35}$$

By (12), (29), (30), and (35), we obtain

$$I_2 \leqq c \int_s^2 \frac{s}{t^2} (1 - t^2/4)^{-1/2} \, dt \leqq c.$$

To handle I_1 we consider two cases. First we assume $0 \leqq \sigma \leqq s/2$ or $2s \leqq \sigma \leqq 1/2$. This implies that

$$G(z, \zeta) \leqq 16\sigma/s \tag{36}$$

by (12) and (30). Hence

$$I_1 \leqq c \int_0^s \frac{1}{s} \, dt \leqq c$$

by (30), (35), and (36). Finally we consider the case $s/2 \leqq \sigma \leqq 2s$ with $0 < \sigma \leqq 1/2$. By (12) and (31), it is trivial to check that

$$G(z, \zeta) \leqq \log\left(\frac{800 s^2}{t^2}\right). \tag{37}$$

By (29), (35), and (37) it follows that

$$I_1 \leqq c \int_0^s \log\left(\frac{800 s^2}{t^2}\right) \frac{1}{s} \, dt = c \int_0^1 \log\left(\frac{800 s^2}{t^2}\right) dt \leqq c,$$

as we wished to show. This completes the proof of the theorem.

We end the paper with a new proof of the Stein—Strömberg result. Consider the Hardy—Littlewood maximal function on $\mathbf{R}^{n-1}$, which is defined to be

$$\mathcal{M}f(x) = \sup_{0 < r < \infty} \int_{|y| \leqq r} \frac{1}{|B_{n-1}| r^{n-1}} |f(x-y)| \, dy.$$

Stein and Strömberg have shown that

$$(38) \qquad \left|\{x\in\mathbf{R}^{n-1}\colon \mathcal{M}f(x) \geqq \lambda\}\right| \leqq \frac{cn}{\lambda}\|f\|_1$$

for all $\lambda>0$ and $f\in L^1(\mathbf{R}^{n-1})$. We only consider $n\geqq 3$ since the interest of this result is on the size of the constant bound in (38) in terms of n. Without loss of generality let g be a C^∞-function on ∂B_n such that $g\geqq 0$. As before, let

$$u(z) = \int_{\partial B_n} P(z,\zeta)g(\zeta)\,d\Sigma(\zeta).$$

Define

$$u_n^*(z^*) = \sup_{1-1/n\leqq r\leqq 1} u(rz^*)$$

with $z^*\in\partial B_n$. Let $E=\{rz^*\colon u(rz^*)\geqq\lambda,\ 1-1/n\leqq r\leqq 1,\ z^*\in\partial B_n\}$. Define $V(z)$ as before and observe that $V(z)\leqq c$ for $z\in B_n$ by (9), (11), (16), and Lemma 4. By the proof of the theorem and the proof of Corollary 1, we obtain

$$(39) \qquad \left|\{z^*\in\partial B_n\colon u_n^*(z^*) \geqq \lambda\}\right| \leqq \frac{c\|g\|_1}{\lambda}$$

for all $\lambda>0$. Define

$$(40) \qquad M_n g(z^*) = \sup_{0\leqq t\leqq 1/\sqrt{n}} \int_{\partial B_n} \frac{1}{|S(z^*,t)|}\chi_{S(z^*,t)}(\zeta)g(\zeta)\,d\Sigma(\zeta)$$

with $z^*\in\partial B_n$. The proof of Corollary 2, using (39) and (40), gives

$$(41) \qquad \left|\{z^*\in\partial B_n\colon M_n g(z^*) \geqq \lambda\}\right| \leqq \frac{cn}{\lambda}\|g\|_1$$

for all $\lambda>0$.

Clearly it is enough to prove (38) for C^∞-functions f with compact support and such that $f\geqq 0$. Fix f. We need only be concerned with the set $\{x\in\mathbf{R}^{n-1}\colon |x|\leqq R\}$ for some fixed R sufficiently large, since

$$f(x) \leqq \frac{c\|f\|_1}{|B_{n-1}|\,|x|^{n-1}}$$

once $|x|\geqq R$ and R is sufficiently large. By dilating the function f we can take R as small as we like and at least arrange $R\leqq 1/n$. Fix R. For any x such that $|x|\leqq R\leqq 1/n$, we now have

$$\mathcal{M}f(x) \leqq \sup_{0\leqq r\leqq 1/\sqrt{n}} \frac{1}{|B_{n-1}|r^{n-1}}\int_{|y|\leqq r}|f(x-y)|\,dy.$$

Place an $n-1$ dimensional hyperplane tangent to B_n at the point $(0,0,\dots,0,-1)$ in $\mathbf{R}^n$. Define $g(z^*)=f(x)$, where $x\in\mathbf{R}^{n-1}$ and $z^*=(x,-\sqrt{1-|x|^2})\in\partial B_n$ (see figure 2 below).

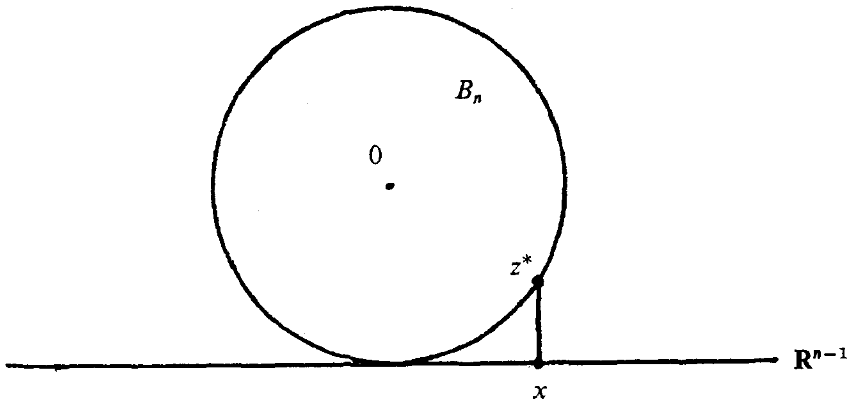

Figure 2

It is obvious that

$$cM_n g(z^*) \leqq \mathcal{M}f(x) \leqq CM_n g(z^*).$$

Inequality (38) now follows from (41).

References

1. BEURLING, A., Études sur un problème de majoration, *Thèse d'Upsal* (1933).
2. HALL, T., Sur la mesure harmonique de certains ensembles, *Arkiv för Matematik Astronomi o. Fysik, vol* 25*A, N:o 28* (1937), 1—8.
3. HAYMAN, W. K. and KENNEDY, P. B., *Subharmonic functions,* Academic Press, London, 1976.
4. LYONS, T. J., MACGIBBON, K. B. and TAYLOR, J. C., Projection theorems for hitting probabilities and a theorem of Littlewood, preprint.
5. MILNOR, J. W., *Topology from the differentiable viewpoint,* Univ. Press of Virginia, Charlottesville, 1965.
6. NEVANLINNA, R., Über eine Minimumaufgabe in der Theorie der konformen Abbildung, *Nachr. Ges. Wiss. Göttingen I,* 37 (1933), 103—115.
7. STEIN, E. M. and STRÖMBERG, J. O., Behavior of maximal functions in $\mathbf{R}^n$ for large n, *Arkiv för Matematik, vol* 21 (1983), 259—269.

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Peter M. Knopf
Department of Mathematics
Pace University
Pleasantville, N. Y. 10570
U.S.A.