The inverse Abel transform for $SU(p, q)$ *

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Introduction

Suppose that G/K is a noncompact Riemannian symmetric space and that G has an Iwasawa decomposition $G=KAN$. The Abel transform F_f of a bi-K-invariant function f on G is the Weyl group invariant function on the Lie algebra of A defined by

$$F_f(H) = e^{\varrho(H)} \int_N f(\exp(H)n)\,dn, \quad H \in \mathfrak{a}.$$

Gangolli has shown that $f \mapsto F_f$ is an isomorphism ${}^K C_c^\infty(G)^K \cong {}^{\mathfrak{W}} C_c^\infty(\mathfrak{a})$. In this paper we describe the inverse of the Abel transform in the case when $G=SU(p, q)$. The main ingredients are as follows.

Firstly, it is known that the Abel transform relates the spherical transform on G and the Euclidean space Fourier transform on A in the following manner:

$$\begin{tikzcd}
{}^K C_c^\infty(G)^K \arrow[rr, "\text{spherical transform}"] \arrow[dr, "\text{Abel transform}"'] & & \mathfrak{W}_{\mathscr{S}(\mathfrak{a}^*)} \\
& {}^W C_c^\infty(\mathfrak{a}) \arrow[ur, "\text{Fourier transform}"'] &
\end{tikzcd}$$

Hence, a function $f \in {}^K C_c^\infty(G)^K$ is equal to the inverse spherical transform of $\mathscr{F}_\mathfrak{a}(F_f)$. In the case of $SU(p, q)$ there is an explicit formula for the spherical functions, due to Berezin and Karpelevich, and so one can write out an explicit formula for the inverse spherical transform. This involves a product of inverse Jacobi transforms, one for each of the dim (A) variables describing coordinates in $\mathfrak{a}^*$, applied to the Fourier transform of a $\prod_{i<j} (\partial_j^2 - \partial_i^2) F_f$. Here ∂_i means partial differentiation with respect to the i^{th} coordinate on $\mathfrak{a}$. The final step uses a result of Koornwinder,

* Dedicated to the memory of Irving Glicksberg.

which states that an inverse Jacobi transform is a composition of fractional integral operators following the one-dimensional Fourier transform applied to even functions. Our formula states that if $f\in {}^K C_c^\infty(G)^K$ then f is a fixed function multiplied with a product of compositions of fractional integral operators (in each of dim (A) variables) applied to

$$\Pi_{i<j}(\partial_j^2-\partial_i^2)F_f.$$

We obtain a similar inversion formula for the Radon transform (in the sense of Helgason) acting on $S(U(p)\times U(q))$-invariant functions on $M_{p,q}(\mathbf{C})$. As an application of this latter formula we demonstrate a local regularity property for $S(U(p)\times U(q))$-invariant Fourier transforms on $M_{p,q}(\mathbf{C})$, when $q\geqq p>1$ or $q>p\geqq 1$, and exhibit some sets of nonsynthesis for the algebra of Fourier transforms on $M_{p,q}(\mathbf{C})$.

I. Preliminaries

Let G denote a connected noncompact semisimple Lie group with finite centre and with a fixed Iwasawa decomposition $G=KAN$. Furthermore, denote by $\mathfrak{a}$ the Lie algebra of A and $\mathbf{H}\colon G\to\mathfrak{a}$ the Iwasawa projection. Fix a Weyl chamber $\mathfrak{a}_+$ in $\mathfrak{a}$ and let R^+ be the corresponding set of positive restricted roots. The multiplicity of $\alpha\in R^+$ is written $m(\alpha)$ and we set $\varrho=(1/2)\sum_{\alpha\in R^+}m(\alpha)\alpha$. The Weyl group is denoted by $\mathfrak{w}$. The Lie algebra of G has the Cartan decomposition $\mathfrak{k}\oplus\mathfrak{p}$ and we equip $\mathfrak{p}$ with the inner product $(.|.)$ coming from the Killing form. In particular, functionals $\lambda\in\mathfrak{a}^*$ are viewed as elements of $\mathfrak{p}^*$ which take the value of zero on the orthogonal complement of $\mathfrak{a}$ in $\mathfrak{p}$. For each $\alpha\in R^+$ fix $H_\alpha\in\mathfrak{a}$ so that $(H_\alpha|H)=\alpha(H)$ for all $H\in\mathfrak{a}$. The vector field determined by H_α is denoted by ∂_α. The Lebesgue measure on $\mathfrak{a}$ and $\mathfrak{a}^*$ is normalized so that the Fourier transform

$$\mathcal{F}_\mathfrak{a}f(\lambda)=\int_\mathfrak{a}f(X)e^{-i\lambda(X)}\,dX,\quad \forall f\in\mathcal{S}(\mathfrak{a}^*),\quad \lambda\in\mathfrak{a}^*,$$

has as its inverse

$$\mathcal{F}_\mathfrak{a}^{-1}g(X)=\int_{\mathfrak{a}^*}g(\lambda)e^{i\lambda(X)}\,d\lambda,\quad \forall g\in\mathcal{S}(\mathfrak{a}^*),\quad X\in\mathfrak{a}.$$

Then normalize the Haar measures on A, N, K and G in the usual manner, see [16], section 8.1.3.

For each $\lambda\in\mathfrak{a}^*$ there is the zonal spherical function

(1) $$\varphi_\lambda(x)=\int_K e^{(i\lambda-\varrho)(\mathbf{H}(xk))}\,dk,\quad \forall x\in G,$$

and the generalized Bessel function

$$\Psi_\lambda(X) = \int_K e^{i\lambda(\mathrm{Ad}(k)X)}\,dk, \quad \forall X \in \mathfrak{p}. \tag{2}$$

The properties of these functions are described in [3, 7 and 11].

The spherical transform is defined by

$$\hat{f}(\lambda) = \int_G f(x)\varphi_\lambda(x^{-1})\,dx, \quad \forall f \in {}^K C_c(G)^K,$$

and the Abel or horospherical transform is

$$F_f(H) = e^{\varrho(H)} \int_N f(\exp(H)\,.\,n)\,dn, \quad \forall f \in {}^K C_c(G)^K, \quad H \in \mathfrak{a}.$$

It is known that $f \to F_f$ defines an isomorphism between ${}^K C_c^\infty(G)^K$ and ${}^{\mathfrak{w}} C_c^\infty(\mathfrak{a})$, and that

$$\hat{f}(\lambda) = \mathscr{F}_{\mathfrak{a}}(F_f)(\lambda), \quad \forall f \in {}^K C_c^\infty(G)^K, \quad \lambda \in \mathfrak{a}^*. \tag{3}$$

See [7] for details. For each $H \in \mathfrak{a}$ let ν_H be the probability measure on $\mathfrak{a}$ defined by

$$\int_{\mathfrak{a}} f\,d\nu_H = \int_K f(\mathbf{H}(\exp(H)\,.\,k))\,dk, \quad \forall f \in C_c(\mathfrak{a}).$$

It follows from a theorem of Kostant that the support of ν_H is the closed convex hull of $\mathfrak{w} \cdot H$. In addition, [6], if H is regular in $\mathfrak{a}$ then ν_H is absolutely continuous with respect to Lebesgue measure. Equation (1) shows that

$$\varphi_\lambda(\exp(H)) = \mathscr{F}_{\mathfrak{a}}(e^{-\varrho}\nu_H)(-\lambda), \quad \forall \lambda \in \mathfrak{a}^*. \tag{4}$$

Let $\mathbf{C}$ denote the function on $\mathfrak{a}^*$ which yields the inverse spherical transform [7],

$$f(x) = |\mathfrak{w}|^{-1} \int_{\mathfrak{a}^*} \hat{f}(\lambda)\varphi_\lambda(x)|\mathbf{C}(\lambda)|^{-2}\,d\lambda. \tag{5}$$

The function $\boldsymbol{\beta}(\lambda) = |\mathbf{C}(\lambda)|^{-2}$ is smooth and of polynomial growth on $\mathfrak{a}^*$, so that $\mathscr{F}_{\mathfrak{a}}^{-1}\boldsymbol{\beta}$ is a well-defined tempered distribution on $\mathfrak{a}$, see [5], section 3.8.

Combining equations (3), (4) and (5) we see that if $H \in \mathfrak{a}^+$ and $f \in {}^K C_c^\infty(G)^K$ then

$$\begin{aligned} f(\exp(H)) &= |\mathfrak{w}|^{-1} \int_{\mathfrak{a}^*} \mathscr{F}_{\mathfrak{a}}(F_f)(\lambda)\mathscr{F}_{\mathfrak{a}}(e^{-\varrho}\nu_H)(-\lambda)\boldsymbol{\beta}(\lambda)\,d\lambda \\ &= |\mathfrak{w}|^{-1} F_f * (e^{-\varrho}\nu_H)^{\vee} * (\mathscr{F}_{\mathfrak{a}}^{-1}\boldsymbol{\beta})(0). \end{aligned}$$

The convolution is well-defined since $(e^{-\varrho}\nu_H)$ has compact support. Note that $\boldsymbol{\beta}(-\lambda) = \boldsymbol{\beta}(\lambda)$.

6. Lemma. *The inverse of the Abel transform is given by*

$$f(\exp(H)) = |\mathfrak{w}|^{-1}\langle F_f, (e^{-\varrho}\nu_H) * (\mathscr{F}_{\mathfrak{a}}^{-1}\boldsymbol{\beta})\rangle$$

for all $F_f \in {}^{\mathfrak{w}} C_c^\infty(\mathfrak{a})$ *and* $H \in \mathfrak{a}$.

Theorem 3.5 in [10] summarizes this inversion when G has only one conjugacy class of Cartan subgroups. See also [1] for the case of $SL(3, \mathbf{R})$ and [9] for some calculations of $\mathscr{F}_{\mathfrak{a}}^{-1}\boldsymbol{\beta}$. In the next section we will explicate Lemma 6 for $G=SU(p, q)$ and $K=S(U(p)\times U(q))$.

It is possible to make a similar remark concerning the Radon transform on $\mathfrak{p}$, as defined on p. 306 of [11]. First, let G_0 denote the Cartan motion group $K\times\mathfrak{p}$ and $\mathfrak{q}$ the orthogonal complement of $\mathfrak{a}$ in $\mathfrak{p}$. Translates of $\mathfrak{q}$ by elements of G_0 are called planes in $\mathfrak{p}$. If $f\in C_c(\mathfrak{p})$ and ξ is a plane in $\mathfrak{p}$ then the value of the Radon transform of f at ξ is

$$\mathscr{R}f(\xi) := \int_{\xi} f(Y)\, dY,$$

where dY is Lebesgue measure on ξ. From [11] we know that if $f\in {}^K C_c^\infty(\mathfrak{p})$ then

$$\tilde{f}(\lambda) := \int_{\mathfrak{p}} f(X)\Psi_{-\lambda}(X)\, dX = \mathscr{F}_{\mathfrak{a}}(\mathscr{R}f(\cdot+\mathfrak{q}))(\lambda). \tag{7}$$

This is analogous to (3). Here $H\to\mathscr{R}f(H+\mathfrak{q})$ is an element of ${}^{\mathfrak{w}}C_c^\infty(\mathfrak{a})$. It is well-known that the inverse spherical transform for (G_0, K) is given by

$$f(H) = |\mathfrak{w}|^{-1}\int_{\mathfrak{a}^*} \tilde{f}(\lambda)\Psi_\lambda(H)\prod_{\alpha\in R^+}|\lambda(H_\alpha)|^{m(\alpha)}\, d\lambda, \tag{8}$$

for all $f^K\in C_c^\infty(\mathfrak{p})$ and $H\in\mathfrak{a}$.

For each $H\in\mathfrak{a}$ let η_H be the probability measure on $\mathfrak{a}$ with support $\overline{\mathrm{co}}\,(\mathfrak{w}\cdot H)$ and Fourier transform $\Psi_{-\lambda}(H)$. Furthermore, let $\mathbf{B}(\lambda)=\left|\prod_{\alpha\in R^+}\lambda(H_\alpha)^{m(\alpha)}\right|$.

9. Lemma. *If* $f\in{}^K C_c^\infty(\mathfrak{p})$ *and* $H\in\mathfrak{a}$ *then*

$$f(H) = \langle\mathscr{R}f(\cdot+\mathfrak{q}),\ \eta_H * \mathscr{F}_{\mathfrak{a}}^{-1}B\rangle.$$

In the rank 1 case the K-invariant functions on $\mathfrak{p}$ are just radial functions and this formula becomes a special case of results in 1.4 of [8]. As with the Abel transform, we will explicate Lemma 9 when $K=S(U(p)\times U(q))$ and $\mathfrak{p}=M_{p,q}(\mathbf{C})$, the space of $p\times q$ complex matrices.

II. The case $G=SU(p, q)$

In this section we fix $q\geqq p\geqq 1$ and let $G=SU(p, q)$. We use the Iwasawa decomposition described in [12], so that $K=S(U(p)\times U(q))$, $k=q-p\geqq 0$,

$$\mathfrak{a} = \left\{H_{\mathbf{t}} = \begin{pmatrix} 0_{p\times p} & \mathbf{t} & 0_{p\times k} \\ \mathbf{t} & & \\ 0_{k\times p} & & 0_{q\times q} \end{pmatrix} : \mathbf{t} = \mathrm{diag}(t_1, \ldots, t_p),\ t_1, \ldots, t_p\in\mathbf{R}\right\},$$

$a_t = \exp(H_t)$, and

$$\mathfrak{p} = \left\{\begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix}: X \in M_{p,q}(\mathbf{C})\right\}.$$

Fix the Weyl chamber $\mathfrak{a}_+ = \{H_t: t_1 > t_2 > \ldots > t_p > 0\}$ and identify $\mathfrak{a}^*$ with $\mathbf{R}^p$ via

$$\lambda(H_t) = \sum_{j=i}^p \lambda_j t_j.$$

The Weyl group $\mathfrak{w}$ is the semidirect product of $\mathfrak{S}_p$, acting as permutations $\mathbf{t} \to (t_{s(1)}, \ldots, t_{s(p)})$, and $\{\pm 1\}^p$ acting via $\mathbf{t} \to (\varepsilon_1 t_1, \ldots, \varepsilon_p t_p)$ with $\varepsilon_j = \pm 1$. Each function $f \in {}^{\mathfrak{w}}C_c^\infty(\mathfrak{a})$ is even as a function of each particular coordinate and invariant under all permutations of coordinates. The positive restricted roots, in the notation of [12], p. 71, are $\alpha_j, 2\alpha_j$ $(1 \leqq j \leqq p)$ and $\alpha_i \pm \alpha_j$ $(1 \leqq i \leqq p)$.

We abbreviate $\partial_j = \partial_{\alpha_j}$ for $1 \leqq j \leqq p$. For each $t \in \mathbf{R}$ let (see (2.2) in [13])

$$\Delta_{k,0}(t) = |(e^t - e^{-t})^{2k+1}(e^t + e^{-t})|.$$

For $\mathbf{t} \in \mathbf{R}^p$ set

$$\sigma(H_t) = \prod_{j=1}^p \Delta_{k,0}(t_j)$$

and

$$\omega(a_t) = 2^{n(n-1)/2} \prod_{i<j} (\cosh(2t_i) - \cosh(2t_j)).$$

Note that if $(s, \varepsilon) \in \mathfrak{w}$ then

$$\omega(a_{(s,\varepsilon)t}) = \operatorname{sign}(s)\omega(a_t). \tag{10}$$

The integrand in equation (8) involves $\prod_{\alpha \in R^+} \lambda(H_\alpha)^{m(\alpha)}$, which in this case is equal to

$$\operatorname{const}\left(\prod_{i<j} (\lambda_i^2 - \lambda_j^2)^2\right) \prod_{i=1}^p \lambda_i^{2k+1}. \tag{11}$$

In order to describe the zonal spherical functions and generalized Bessel functions we must recall some special functions. For each $\Lambda \in \mathbf{R}$ and $t > 0$ the Jacobi function of index $(k, 0)$ is equal to

$$\varphi_\Lambda^{(k,0)}(t) = {}_2F_1\big((k+1+i\Lambda)/2, (k+1-i\Lambda)/2;\ k+1;\ -(\sinh t)^2\big).$$

Furthermore, set

$$C_{k,0}(\Lambda) = \frac{2^{k+1}\Gamma((i\Lambda)/2)\Gamma((1+i\Lambda)/2)}{(\Gamma((k+1+i\Lambda)/2))^2},$$

as in (2.6) of [13]. It is known [15] that if $\Lambda \neq 0$ and $t > 0$ then

$$\lim_{N \to \infty} \varphi_{N\Lambda}^{(k,0)}(t/N) = \text{const.}\,(\Lambda t)^{-k} J_k(\Lambda t), \tag{12}$$

where J_k is the classical Bessel function and the constant depends only on k. Let $\mathcal{J}_k(s) = s^{-k} J_k(s)$.

The formulae of Berezin and Karpelivich [2] state that if $\lambda\in\mathfrak{a}^*$ is such that (11) is not zero then

$$\varphi_\lambda(a_t) = \text{const.} \det\left(\varphi_{\lambda_i}^{(k,0)}(t_j)\right)_{1\leqq i,j\leqq p} \Big/ \left\{\omega(a_t)\cdot \prod_{l<m}(\lambda_l^2-\lambda_m^2)\right\} \tag{13}$$

and

$$\Psi_\lambda(H_t) = \text{const.} \det\left(\mathscr{J}_k(\lambda_i t_j)\right)_{1\leqq i,j\leqq p} \Big/ \prod_{l<m}\left\{(t_l^2-t_m^2)(\lambda_l^2-\lambda_m^2)\right\} \tag{14}$$

for all $H_t\in\mathfrak{a}_+$. Equation (13) is proved in [12] and equation (14) follows from (12), (13), and [4].

Now suppose that $f\in {}^K C_c^\infty(G)^K$ and that λ is a regular element of $\mathfrak{a}^*$. Then, the spherical transform of f is given by

(15)

$$\prod_{l<m}(\lambda_l^2-\lambda_m^2)\hat{f}(\lambda) = \text{const.} \int_{\mathfrak{a}} f(a_t)\omega(a_t) \det\left(\varphi_{\lambda_i}^{(k,0)}(t_j)\right) \prod_{j=1}^p \Delta_{k,0}(t_j)\,dt_1\dots dt_p.$$

The right-hand side of (15) is the sum over all permutations $s\in\mathfrak{S}_p$ of integrals

$$\begin{aligned}\int_{\mathfrak{a}} f(a_t)\omega(a_t)\,\mathrm{sign}(s) \prod_{j=1}^p \left\{\varphi_{\lambda_j}^{(k,0)}(t_{s(j)})\Delta_{k,0}(t_{s(j)})\right\} dt_1\dots dt_p \\ = \int_{\mathfrak{a}} f(a_t)\omega(a_t) \prod_{j=1}^p \left\{\varphi_{\lambda_j}^{(k,0)}(t_j)\Delta_{k,0}(t_j)\right\} dt_1\dots dt_p,\end{aligned} \tag{16}$$

the left-hand side of (15) is equal to

$$\mathscr{F}_{\mathfrak{a}}\left(\prod_{i<j}(\partial_j^2-\partial_i^2)F_f\right)(\lambda). \tag{17}$$

In (16) the integrand is invariant under the action of $\{\pm 1\}^p$ and anti-invariant under the action of $\mathfrak{S}_p$. Let us set

$$C_+ = \{H_t\in\mathfrak{a}\colon\ t_j \geqq 0,\ j = 1, \dots, p\}.$$

Then the integral (16) is 2^p times the integral

$$\int_{C_+} f(a_t)\omega(a_t) \prod_{j=1}^p \left\{\varphi_{\lambda_j}^{(k,0)}(t_j)\Delta_{k,0}(t_j)\right\} dt_1\dots dt_p. \tag{18}$$

To proceed from here we need the results of Koornwinder [13] on the Jacobi transform. For positive real numbers μ and σ and g a compactly supported smooth function on C_+ set

$$\begin{aligned}(\mathbf{W}_\mu^\sigma(g))(H_t) = \Gamma(\mu)^{-p} \int_{t_p}^\infty \dots \int_{t_1}^\infty g(H_s)\times \\ \times \prod_{j=1}^p \left\{(\cosh(\sigma s_j)-\cosh(\sigma t_j))^{\mu-1}\sigma\sinh(\sigma s_j)\right\} ds_1\dots ds_p.\end{aligned}$$

Furthermore, for σ as above, $n=0, 1, 2, \ldots,$ and $\mu > -n$ let

$$(\mathbf{W}_\mu^\sigma(g))(H_t) = (-1)^{np}\Gamma(\mu+n)^{-p}\int_{t_p}^{\infty}\cdots\int_{t_1}^{\infty}\left(\Pi_{k=1}^{p}\frac{\partial^n}{\partial(\cosh\sigma s_k)^n}g(H_s)\right)\times$$
$$\times\Pi_{l=1}^{p}\{(\cosh(\sigma s_l)-\cosh(\sigma t_l))^{\mu+n-1}\sigma\sinh(\sigma s_l)\}ds_1\ldots ds_p.$$

These are p-fold tensor products of the operators $\mathscr{W}_\mu^\sigma$ in [13]. Combining (18) with (3.7) and (3.12) of [13] we see that if $f\in{}^K C_c^\infty(G)^K$ and $\lambda\in\mathfrak{a}^*$ then

(19) $\quad \hat{f}(\lambda) = \text{const.}\int_{C_+}\mathbf{W}_k^1\circ\mathbf{W}_{1/2}^2((f\circ\exp)\cdot(\omega\circ\exp))(H_t)\,\Pi_{j=1}^{p}\cos(\lambda_j t_j)\,dt_1\ldots dt_p.$

Rewriting equation (17), taking into account its invariance under $\{\pm 1\}^p$, $\hat{f}(\lambda)$ is also equal to

(20) $$\text{const.}\int_{C_+}(\Pi_{i<j}(\partial_j^2-\partial_i^2)F_f)(H_t)\Pi_{j=1}^{p}\cos(\lambda_j t_j)\,dt_1\ldots dt_p.$$

We can invert the cosine transform one variable at a time and similarly the transforms $\mathbf{W}_\mu^\sigma$.

21. Theorem. *For $G, K,$ and $\mathfrak{a}_+$ as above and $f\in{}^K C_c^\infty(G)^K$, the Abel transform F_f satisfies*

$$\Pi_{i<j}(\partial_j^2-\partial_i^2)F_f(H_t) = \text{const.}\,\mathbf{W}_k^1\circ\mathbf{W}_{1/2}^2((f\circ\exp)\cdot(\omega\circ\exp))(H_t)$$

for all $H_t\in\mathfrak{a}_+$. Furthermore, the inverse Abel transform is

$$f(a_t) = \text{const.}\,\omega(a_t)^{-1}\mathbf{W}_{-1/2}^2\circ\mathbf{W}_{-k}^1(\Pi_{i<j}(\partial_j^2-\partial_i^2)F_f)(H_t)$$

for all $H_t\in\mathfrak{a}_+$.

This last formula can be thought of as a higher rank version of section V.2.4 in [8]. We now turn our attention to Radon transform in Lemma 9. We know that if $f\in{}^K C_c^\infty(\mathfrak{p})$ and λ is a regular element of $\mathfrak{a}^*$ then

$$\Pi_{l<m}(\lambda_l^2-\lambda_m^2)\cdot\tilde{f}(\lambda) = \text{const.}\int_{\mathfrak{a}} f(H_t)\,\Pi_{i<j}(t_i^2-t_j^2)\det(\mathscr{J}_k(\lambda_i t_j))\times$$
$$\times\Pi_{l=1}^{p}|t_l|^{2k+1}dt_1\ldots dt_p.$$

Let us write

$$\theta(\mathbf{t}) = \Pi_{i<j}(t_i^2-t_j^2).$$

Arguing as for equation (16) above, we rewrite this as

$$\mathscr{F}_{\mathfrak{a}}(\Pi_{l<m}(\partial_m^2-\partial_l^2)\mathscr{R}f(\cdot+\mathfrak{q}))(\lambda)$$
$$= \text{const.}\int_{C_+} f(H_t)\theta(\mathbf{t})\,\Pi_{j=1}^{p}\{\mathscr{J}_k(\lambda_j t_j)|t_j|^{2k+1}\}\,dt_1\ldots dt_p.$$

Using (5.5) of [13] this becomes

$$\text{const.} \int_{C_+} \prod_{j=1}^{p} \cos(\lambda_j s_j) \int_{s_p}^{\infty} \dots \int_{s_1}^{\infty} f(H_t)\theta(\mathbf{t}) \times \prod_{l=1}^{p} \left((t_l^2 - s_l^2)^{k-(1/2)} t_l\right) \times$$
$$\times dt_1 \dots dt_p ds_1 \dots ds_p.$$

22. Theorem. *For $G, K, \mathfrak{a}_+$ and $\mathfrak{p}$ as above and for every $f \in {}^K C_c^\infty(\mathfrak{p})$ the Radon transform $\mathcal{R}f$ satisfies*

$$\prod_{i<j} (\partial_j^2 - \partial_i^2) \mathcal{R}f(H_t + \mathfrak{q})$$
$$= \text{const.} \int_{t_p}^{\infty} \dots \int_{t_1}^{\infty} f(H_s)\theta(\mathbf{s}) \prod_{l=1}^{p} \left((s_l^2 - t_l^2)^{k-(1/2)} s_l\right) ds_1 \dots ds_p,$$

for all $H_t \in \mathfrak{a}_+$

Fix f as in the statement of the theorem. We know that $\mathbf{t} \to f(H_t)$ is a smooth function of $(t_1, \dots, t_p)$ and so we can view the integral in the theorem as a p-fold tensor product of Weyl fractional integrals. The inversion formulae for these are well known, see [13].

23. Corollary. *For $G, K, \mathfrak{a}_+$ and $\mathfrak{p}$ as above and $f \in {}^K C_c^\infty(\mathfrak{p})$,*

$$f(H_t)\theta(\mathbf{t}) = \text{const.} \int_{t_p}^{\infty} \dots \int_{t_1}^{\infty} \left\{ \prod_{l=1}^{p} (x_l^{-1}\partial_l)^{k+1} \cdot \prod_{i<j} (\partial_j^2 - \partial_i^2) \mathcal{R}f(H_x + \mathfrak{q}) \right\} \times$$
$$\times \prod_{m=1}^{p} \left\{ x_m (x_m^2 - t_m^2)^{-1/2} \right\} dx_1 \dots dx_p,$$

for all $H_t \in \mathfrak{a}_+$.

3. Local regularity for K-invariant Fourier transforms

Maintain the notation of section 2 and identify $\mathfrak{p}$ and $\mathfrak{p}^*$ using the Killing form. Let $\mathcal{F}_\mathfrak{p}$ denote the Fourier transform acting on $L^1(\mathfrak{p})$, normalized by the same requirements as in section 1. We are interested in the properties of $\mathcal{F}_\mathfrak{p}({}^K L^1(\mathfrak{p}))$, the subalgebra of K-invariant elements of the Fourier algebra of $\mathfrak{p}$. For each $g \in L^1(\mathfrak{p})$ set $\|\mathcal{F}_\mathfrak{p} g\|_{A(\mathfrak{p})} = \|g\|_1$. In particular, if $f \in C_c(\mathfrak{p})$ is K-invariant and an element of $\mathcal{F}_\mathfrak{p} L^1(\mathfrak{p})$ then

$$\|f\|_{A(\mathfrak{p})} = |\mathfrak{w}|^{-1} \int_{\mathfrak{a}^*} \left|\mathcal{F}_\mathfrak{a}(\mathcal{R}f(\cdot + \mathfrak{q}))(\lambda)\right| \prod_{\alpha \in R^+} |\lambda(H_\alpha)|^{m(\alpha)} d\lambda.$$

24. Lemma. *Let $f \in C_c(\mathfrak{p}) \cap \mathcal{F}_\mathfrak{p}({}^K L^1(\mathfrak{p}))$. Then the distributional derivative $\left(\prod_{\alpha \in R^+} \partial_\alpha^{m(\alpha)}\right) \mathcal{R}f(\cdot + \mathfrak{q})$ is an element of $\mathcal{F}_\mathfrak{a}^{-1}(L^1(\mathfrak{a}^*))$ and*

$$\sup_{H \in \mathfrak{a}} \left|\prod_{\alpha \in R^+} \partial_\alpha^{m(\alpha)} \mathcal{R}f(H + \mathfrak{q})\right| \leqq \text{const.} \|f\|_{A(\mathfrak{p})}.$$

Recall that $\mathcal{R}f(\cdot + \mathfrak{q})$ is an element of ${}^\mathfrak{w} C_c(\mathfrak{a})$ whenever $f \in {}^K C_c(\mathfrak{p})$. Now fix a compact K-invariant subset $E \subset \mathfrak{p}$ and $\varphi \in {}^K C_c^\infty(\mathfrak{p})$ such that $\varphi(X) = 1$ for all $X \in E$. For every $f \in \mathscr{F}_{\mathfrak{p}}({}^K L^1(\mathfrak{p}))$ we can apply Lemma 24 to the function $\varphi.f$. Furthermore, arguing as on p. 54 of [14], we see that if n is an integer valued function on R^+ with

$$0 \leqq n(\alpha) \leqq m(\alpha), \quad \forall \alpha \in R^+$$

then there is a constant $c>0$, depending only on $\mathfrak{w}$, E, and φ such that

$$\sup_{H \in \mathfrak{a}} \left| \Pi_{\alpha \in R^+} \partial_\alpha^{n(\alpha)} \mathscr{R}(\varphi.f)(H+\mathfrak{q}) \right| \leqq c \cdot \|f\|_{A(\mathfrak{p})} \tag{25}$$

for all $f \in \mathscr{F}_{\mathfrak{p}}({}^K L^1(\mathfrak{p}))$.

We can feed these estimates into the equation in Corollary 23, and observe that

$$-(d/dy)\mathscr{W}_\mu(g)(y) = \mathscr{W}_{\mu-1}(g)(y) = \mathscr{W}_\mu(-g')(y),$$

in the notation of (3.9) of [13].

26. Theorem. *Let K, $\mathfrak{p}$, and $\mathfrak{a}_+$ be as above and suppose that $q \geqq p > 1$ or $q > p \geqq 1$. For all $f \in \mathscr{F}_{\mathfrak{p}}({}^K L^1(\mathfrak{p}))$ the distributional derivative*

$$\left(\Pi_{l=1}^p \partial_l^k \Pi_{i<j} (\partial_j^2 - \partial_i^2)\right) \cdot f(H_t)$$

is a continuous function on $\mathfrak{a}_+$. Furthermore, for each $\mathfrak{w}$-invariant compact subset E, contained properly in the set of regular elements of $\mathfrak{a}$, there is a constant $c_E > 0$ such that

$$\sup_{H_t \in E} \left| \Pi_{l=1}^p \partial_l^k \Pi_{i<j} (\partial_j^2 - \partial_i^2) f(H_t) \right| \leqq C_E \cdot \|f\|_{A(\mathfrak{p})}.$$

27. Corollary. *For $q \geqq p > 1$ or $q > p \geqq 1$ consider the action of $K = S(U(p) \times U(q))$ on $M_{p,q}(\mathbf{C})$ given by $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \cdot X = AXB^*$. Then every regular K-orbit is not a set of synthesis for the Fourier algebra of $M_{p,q}(\mathbf{C})$.*

This is proved in the same manner as Theorem 4.3 in [14]. See the references cited in [14] for details concerning spectral synthesis for $\mathscr{F}L^1(\mathbf{R}^n)$.

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Note added in proof. In a recent paper "The Fourier transform of Harish—Chandra's c-function and inversion of the Abel transform", R. J. Beerends describes $\mathscr{F}_a(\beta)$ for arbitrary noncompact G/K.