Subharmonic functions in strips

S. J. Gardiner

1. Introduction

The contents of the present paper are most easily introduced by referring to previous studies on half-spherical means of subharmonic functions s in the Euclidean half-space $D=\mathbf{R}^n\times(0, +\infty)$, $(n\geqq 1)$. Originally these means were defined as weighted integrals of s over the curved part of the boundary of a half-ball (see, for example, Huber [11], Dinghas [4, 5] and Kuran [12]). For these means to have desirable properties, it was necessary to require that

(1) $$\limsup_{M\to N} s(M) \leqq 0 \quad (N\in\mathbf{R}^n\times\{0\}).$$

Two distinct lines of development can be traced from these origins. The first, due to Ahlfors [1] $(n=1)$ and then Kuran [13] $(n\geqq 1)$, was to extend the half-spherical mean to include also a term involving the integral of s over the flat part of the half-ball boundary. This permitted the restriction (1) to be relaxed to

$$s(N) = \limsup_{M\to N} s(M) < +\infty \quad (N\in\mathbf{R}^n\times\{0\}).$$

Analogous results, using different methods, were obtained for the infinite strip $\Omega=\mathbf{R}^n\times(0, 1)$ by Armitage and Fugard [2].

The second, more recent, development due to Norstad [14] $(n=1)$ and then Wanby [15] $(n\geqq 1)$ involved instead a modification of the integral over the curved part of the surface. The requirement (1) was changed: for example, when $n=1$ it became

$$\tfrac{1}{2}\{s(-x, 0)+s(x, 0)\} \leqq \cos(\pi\lambda/2)s(0, x) \quad (x \geqq 0)$$

where $0<\lambda\leqq 1$. Analogous results for the infinite strip have recently been obtained by Wanby [16]. This current study brings together both lines of development, presenting a new mean incorporating both the modified mean over the curved surface and a term involving the flat surface. For reasons of geometric simplicity, we work in the infinite strip Ω, and make use of the material in [2]. In fact the results in [2] will be seen to be a special case $(\lambda=1)$ of the theorems stated below; we use similar notation to facilitate this comparison.

2. Notation and definitions

The closure and boundary of a subset E of $\mathbf{R}^{n+1}$ are denoted respectively by $\bar{E}$ and ∂E. An infinite strip of height α will be denoted by

$$\Omega_\alpha = \{M = (X, y) = (x_1, x_2, \ldots, x_n, y):\ 0 < y < \alpha\},$$

so that $\Omega = \Omega_1$. We also write

$$|X| = (x_1^2 + \ldots + x_n^2)^{1/2}.$$

Let $B(r)$ and $S(r)$ denote the open ball and the sphere of radius r in $\mathbf{R}^n$ centred at the origin, and define $\Gamma_\alpha(r) = S(r) \times (0, \alpha)$. The surface area measures on $S(r)$ and $\Gamma_\alpha(r)$ are denoted by σ and τ respectively (when $n=1$ we mean $\sigma(\{-r\}) = \sigma(\{r\}) = 1$). We write $\sigma_n = \sigma(S(1))$ and define a related dimensional constant

$$\gamma_1 = (2\pi)^{-1}, \quad \gamma_n = \{(n-1)\sigma_{n+1}\}^{-1} \quad (n \geqq 2).$$

Lebesgue measure on $\mathbf{R}^n$ is denoted by dX.

We shall say that a function s belongs to the class $\mathcal{S}_\alpha$ if:

(i) s is subharmonic in Ω_α, and

(ii) $\limsup_{M \to N,\, M \in \Omega_\alpha} s(M) = s(N) < +\infty \quad (N \in \partial \Omega_\alpha)$.

If we wish to include also the possibility that $s \equiv -\infty$, then we shall write $s \in \mathcal{T}_\alpha$. Clearly, if s is subharmonic in an open set containing $\bar{\Omega}_\alpha$, then $s \in \mathcal{S}_\alpha$, but the converse is false.

We will define a weighted mean of $s \in \mathcal{S}_\alpha$ in terms of the Bessel functions $I_{n/2-1}$ and $K_{n/2-1}$ which are defined in Watson [17; pp. 77, 78]. Following [2] we abbreviate these to I and K respectively and denote K/I by L. These functions have a simple form when $n=1$:

$$I(t) = (2/\pi t)^{1/2} \cosh t, \quad K(t) = (\pi/2t)^{1/2} e^{-t}, \quad L(t) = \pi/(1+e^{2t}).$$

(see [17; pp. 79, 80]).

Let f be an extended real-valued function on $\bar{\Omega}$ and $0<\lambda\leqq 2$. Provided the integrals exist, we write

$$\mathscr{M}(f, r;\ \lambda) = r^{-n/2}\{I(\pi\lambda r)\}^{-1}\int_{\Gamma_1(r)} f(M)\sin\left\{\pi\lambda\left(\tfrac{1}{2}-\left|\tfrac{1}{2}-y\right|\right)\right\} d\tau(M),$$

$$\mathscr{N}_0(f, r;\ \lambda) = 2^{-1/2}\pi^2\lambda^2\int_0^r \{\cosh(\pi\lambda t)\}^{-2}\int_{(0,t)}\cosh(\pi\lambda u)\{f(u, 0)+f(-u, 0)$$

$$+f(u, 1)+f(-u, 1)-2\cos(\pi\lambda/2)\left[f\left(u, \tfrac{1}{2}\right)+f\left(-u, \tfrac{1}{2}\right)\right]\right\} du\, dt \quad (n=1),$$

$$\mathscr{N}_0(f, r;\ \lambda) = \pi\lambda\int_1^r t^{-1}\{I(\pi\lambda t)\}^{-2}\int_{B(t)}|X|^{1-n/2}I(\pi\lambda|X|)$$

$$\times\left\{f(X, 0)+f(X, 1)-2\cos(\pi\lambda/2)f\left(X, \tfrac{1}{2}\right)\right\} dX\, dt \quad (n\geqq 2),$$

and

$$\mathscr{P}_0(f, r;\ \lambda) = \mathscr{M}(f, r;\ \lambda)+\mathscr{N}_0(f, r;\ \lambda).$$

We also define

$$\mathscr{M}_E(f, r;\ \lambda) = \log\mathscr{M}(h_\lambda\exp(f/h_\lambda), r;\ \lambda)$$

where

$$h_\lambda(M) = |X|^{1-n/2}I(\pi\lambda|X|)\cos\left\{\pi\lambda\left(y-\tfrac{1}{2}\right)\right\} \quad (M\in\Omega),$$

$$\mathscr{M}_\infty(f, r;\ \lambda) = \sup\{f(M)/h_\lambda(M):\ M\in\Gamma_1(r)\},$$

and, if f is non-negative,

$$\mathscr{M}_p(f, r;\ \lambda) = r^{n/2-1}\{I(\pi\lambda r)\}^{-1}$$

$$\times\left\{r^{1-n}\int_{\Gamma_1(r)}[f(M)]^p\cos^{1-p}\left[\pi\lambda\left(y-\tfrac{1}{2}\right)\right]\sin\left[\pi\lambda\left(\tfrac{1}{2}-\left|\tfrac{1}{2}-y\right|\right)\right] d\tau(M)\right\}^{1/p}.$$

3. Results

The central result of the paper is Theorem 1 below, most other results being deduced from it. When a function is described as "increasing", the term should be understood in its wide sense, i.e. non-decreasing. We list the following boundary conditions which will be referred to in statements of results:

(2) $$\int_{S(r)}\tfrac{1}{2}[s(X, 0)+s(X, 1)]\, d\sigma(X) \leqq \cos(\pi\lambda/2)\int_{S(r)} s\left(X, \tfrac{1}{2}\right) d\sigma(X)$$

(3) $$\sup\{s(M): M\in S(r)\times\{0, 1\}\} \leqq \cos(\pi\lambda/2)\sup\left\{s(M):\ M\in S(r)\times\left\{\tfrac{1}{2}\right\}\right\}$$

(4) $$\left\{\int_{S(r)}\tfrac{1}{2}[s^p(X, 0)+s^p(X, 1)]\, d\sigma(X)\right\}^{1/p} \leqq \cos(\pi\lambda/2)\left\{\int_{S(r)} s^p\left(X, \tfrac{1}{2}\right) d\sigma(X)\right\}^{1/p}$$

(5) $$\int_{S(r)}\tfrac{1}{2}[\exp\{s(X, 0)/h_\lambda(X, 0)\}+\exp\{s(X, 1)/h_\lambda(X, 1)\}]\, d\sigma(X)$$

$$\leqq \int_{S(r)}\exp\left\{s\left(X, \tfrac{1}{2}\right)/h_\lambda\left(X, \tfrac{1}{2}\right)\right\} d\sigma(X).$$

Clearly conditions (2)—(5) are satisfied if

$$\max\{s(X,0),\, s(X,1)\} \leqq \cos(\pi\lambda/2)\, s\left(X, \tfrac{1}{2}\right) \qquad (X \in \mathbf{R}^n).$$

Theorem 1. *If* $0<\lambda\leqq 2$ *and* $s\in\mathcal{S}_1$, *then* $\mathcal{P}_0(s, r; \lambda)$ *is real-valued and increasing as a function of* $r\in(0, +\infty)$ *and convex as a function of* $L(\pi\lambda r)$.

Corollary 1. *If* $0<\lambda\leqq 2$, $s\in\mathcal{S}_1$ *and* (2) *holds for a.e. (Lebesgue)* $r>0$, *then* $\mathcal{M}(s, r; \lambda)$ *is increasing as a function of* $r\in(0, +\infty)$ *and convex as a function of* $L(\pi\lambda r)$.

Corollary 1 $(0<\lambda<1)$, together with Theorems 6 and 7 below, are to appear also in [16] where different methods are used (I am grateful to Professor Wanby for sending me a preprint of his paper). The greater generality of Theorem 1 appears to be new.

The conclusions of the above results could alternatively be stated in terms of convexity with respect to the family $\{AI(\pi\lambda r)+BK(\pi\lambda r)\colon A, B\in\mathbf{R}\}$. The case $n=1$ warrants special mention, the following result being a generalization of a theorem due to Heins [9], who gave the $\lambda=1$ case.

Corollary 2. *If* $0<\lambda\leqq 2$ *and s is subharmonic in a rectangle* $(a, b)\times(0, 1)$ *and (extending s to* $(a, b)\times[0, 1]$ *by its* lim sup*)*

$$\tfrac{1}{2}[s(x,0)+s(x,1)] \leqq \cos(\pi\lambda/2)\, s\left(x, \tfrac{1}{2}\right) \quad (x\in(a, b)),$$

then the function

$$x \mapsto \int_{[0,1]} s(x, y) \sin\left[\pi\lambda\left(\tfrac{1}{2}-\left|\tfrac{1}{2}-y\right|\right)\right] dy$$

is convex on (a, b) *with respect to the family* $\{Ae^{\pi\lambda x}+Be^{-\pi\lambda x}\colon A, B\in\mathbf{R}\}$.

The following result is implicit in [2; Theorem 2]. By taking $n=1$ and $s=\log|f|$ (where f is holomorphic in Ω) and using an appropriate conformal mapping, it can be seen to be a generalization of a well-known uniqueness theorem of F. Carlson: namely that a holomorphic function of exponential growth in the half-plane cannot approach zero exponentially along the boundary unless it vanishes identically.

Theorem A. *If* $s\in\mathcal{T}_1$,

(i) $$\liminf_{r\to\infty} r^{(1-n)/2} e^{-\pi r} \int_{\Gamma_1(r)} s(M) \sin(\pi y)\, d\tau(M) < +\infty$$

and

(ii) $$\int_{\mathbf{R}^n} (1+|X|)^{(1-n)/2} e^{-\pi|X|} \sum_{k=0}^{1} s^+(X, k)\, dX < \int_{\mathbf{R}^n} (1+|X|)^{(1-n)/2} e^{-\pi|X|} \sum_{k=0}^{1} s^-(X, k)\, dX = +\infty,$$

then $s\equiv-\infty$.

It is natural to ask if other results of this type can be obtained by weakening the growth restriction (i) and compensating with a stronger "decay" in (ii). We answer in the affirmative by stating the following easy application of Theorem 1.

Theorem 2. *Let* $1<\lambda\leqq 2$. *If* $s\in\mathcal{T}_1$,

(i) $$\liminf_{r\to\infty} r^{(1-n)/2}e^{-\pi\lambda r}\int_{\Gamma_1(r)} s(M)\sin\left\{\pi\lambda\left(\tfrac{1}{2}-\left|\tfrac{1}{2}-y\right|\right)\right\}d\tau(M)<+\infty$$

and

(ii) $$\int_{\mathbf{R}^n}(1+|X|)^{(1-n)/2}e^{-\pi\lambda|X|}\sum_{k=0}^{2}s^+(X,k/2)\,dX$$

$$<\int_{\mathbf{R}^n}(1+|X|)^{(1-n)/2}e^{-\pi\lambda|X|}\sum_{k=0}^{2}s^-(X,k/2)\,dX=+\infty,$$

then $s\equiv-\infty$.

The new feature of Theorem 2 is that condition (ii) involves three hyperplanes instead of two. To see that this is essential, consider the harmonic function $(n=1)$

$$s(x,y)=e^{\pi\lambda x}\cos\pi\lambda\left(y-\tfrac{1}{2}\right)\qquad(y\in[0,1]).$$

It is clear that s satisfies (i) of Theorem 2. It would also satisfy (ii) if the term $k=1$ were omitted from the sums, since $\cos(\pi\lambda/2)<0$.

Theorem 3. *Let* $0<\lambda\leqq 1$ *and let* $s\in\mathcal{S}_1$ *be non-negative. If* (2) *holds for a.e.* $r>0$ *and*

(6) $$\liminf_{r\to\infty}\mathcal{M}(s,r;\lambda)=0,$$

then $s\equiv 0$ *in* Ω.

The case $\lambda=1$ of Theorem 3 is the usual type of Phragmén—Lindelöf result where $s\leqq 0$ on $\partial\Omega$. When $0<\lambda<1$ we are weakening the boundary requirement at the expense of a stronger growth restriction in (6). Similar observations can be made concerning the following criteria for harmonic majorization in Ω, which generalize [2; Theorems 5, 6].

Theorem 4. *Let* $0<\lambda\leqq 1$. *If* $s\in\mathcal{S}_1$ *and* $\mathcal{P}_0(s,\cdot\,;\lambda)$ *is bounded above on* $(0,+\infty)$, *then* s *has a harmonic majorant in* Ω.

Theorem 5. *Let* $1\leqq\lambda\leqq 2$ *and suppose that* s *is subharmonic in an open set* W *containing* $\bar\Omega$. *If* s *has a harmonic majorant in* Ω, *then* $\mathcal{P}_0(s,\cdot\,;\lambda)$ *is bounded above on* $(0,+\infty)$.

Finally we give results on other types of means.

Theorem 6. *Let* $0<\lambda\leqq 1$ *and* $s\in\mathcal{S}_1$ *be non-negative. If* $p>1$ *and* (4) *holds for a.e.* $r>0$, *then* $\mathcal{M}_p(s,r;\lambda)$ *is increasing as a function of* $r\in(0,+\infty)$ *and convex as a function of* $L(\pi\lambda r)$.

Theorem 7. *Let* $0<\lambda\leqq 1$ *and* $s\in\mathcal{S}_1$. *If* (3) *holds for all* $r\geqq 0$, *then* $\mathcal{M}_\infty(s, r; \lambda)$ *is increasing as a function of* $r\in(0, +\infty)$ *and convex as a function of* $L(\pi\lambda r)$.

Theorem 8. *Let* $0<\lambda\leqq 1$ *and* $s\in\mathcal{S}_1$. *If* (5) *holds for a.e.* $r>0$, *then* $\mathcal{M}_E(s, r; \lambda)$ *is increasing as a function of* $r\in(0, +\infty)$ *and convex as a function of* $L(\pi\lambda r)$.

4. Preparatory material

4.1. For the purposes of proving Theorem 1 we need to widen some of our definitions. We will keep close to the notation of [2]. We put $A(\varrho, R)=B(R)\setminus\bar{B}(\varrho)$ (so that $A(0, R)=B(R)$) and define

$$\Omega_\alpha(\varrho, R) = A(\varrho, R)\times(0, \alpha), \quad A_\alpha^\sim(\varrho, R) = A(\varrho, R)\times\{0, \alpha\}.$$

If $t\leqq\varrho$, then $A(\varrho, t)$ should be understood as the empty set. We shall say that $s\in\mathcal{S}_\alpha(\varrho, R)$ if

(i) s is defined at least in $\bar{\Omega}_\alpha(\varrho, R)$,

(ii) s is subharmonic in $\Omega_\alpha(\varrho, R)$, and

(iii) $\limsup_{\substack{M\to N\\ M\in\Omega_\alpha(\varrho, R)}} s(M)=s(N)<+\infty$ $\left(N\in A_\alpha^\sim(\varrho, R)\right)$.

Suppose that $\varrho<r<R$ and let f be an extended real-valued function defined at least on $\Omega_\alpha(\varrho, R)$. Then, provided the integrals exist, we write

$$\mathcal{M}(f, r;\ \lambda, \alpha) = r^{-n/2}\{I(\pi\lambda r/\alpha)\}^{-1}\int_{\Gamma_\alpha(r)} f(M)\sin\left[\pi\lambda\left(\tfrac{1}{2}\alpha-\left|\tfrac{1}{2}\alpha-y\right|\right)/\alpha\right]d\tau(M),$$

$$\mathcal{N}_\varrho(f, r;\ \lambda, \alpha) = 2^{-1/2}\pi^2\lambda^2\alpha^{-2}\int_0^r\{\cosh(\pi\lambda t/\alpha)\}^{-2}\int_{(\varrho,t)}\cosh(\pi\lambda u/\alpha)\{f(u,0)+f(-u, 0)$$
$$+f(u, \alpha)+f(-u, \alpha)-2\cos(\pi\lambda/2)\left[f\left(u, \tfrac{1}{2}\alpha\right)+f\left(-u, \tfrac{1}{2}\alpha\right)\right]\}\,du\,dt \qquad (n=1),$$

$$\mathcal{N}_\varrho(f, r;\ \lambda, \alpha) = \pi\lambda\alpha^{-1}\int_{\min\{1, R\}}^r t^{-1}\{I(\pi\lambda t/\alpha)\}^{-2}\int_{A(\varrho, t)}|X|^{1-n/2}I(\pi\lambda|X|/\alpha)$$
$$\times\left\{f(X, 0)+f(X, \alpha)-2\cos(\pi\lambda/2)f\left(X, \tfrac{1}{2}\alpha\right)\right\}dX\,dt \qquad (n\geqq 2),$$

$$\mathcal{P}_\varrho(f, r;\ \lambda, \alpha) = \mathcal{M}(f, r;\ \lambda, \alpha)+\mathcal{N}_\varrho(f, r;\ \lambda, \alpha).$$

Whenever $\alpha=1$ and $\varrho=0$ these definitions coincide with those given earlier.

4.2. We now recall some results about the special case $\lambda=1$ which was studied in [2] and implicitly in [8].

Theorem B. (i) *Suppose that either* $0\leqq\varrho_0<\varrho<R$ *or* $0=\varrho_0=\varrho<R$. *If* $s\in\mathcal{S}_\alpha(\varrho_0, R)$, *then* $\mathcal{P}_\varrho(s, r; 1, \alpha)$ *is real-valued on* (ϱ, R) *and is a convex function of*

$L(\pi r/\alpha)$ on (ϱ, R). If $s\in\mathcal{S}_\alpha(0, R)$, then $\mathcal{P}_0(s, r; 1, \alpha)$ is also an increasing function on $(0, R)$.

(ii) *If h is real-valued and continuous in $\bar{\Omega}_\alpha(\varrho, R)$ and harmonic in $\Omega_\alpha(\varrho, R)$, then in the case where $\varrho>0$, $\mathcal{P}_\varrho(h, r; 1, \alpha)$ is a linear function (i.e. polynomial of degree at most 1) of $L(\pi r/\alpha)$ on $[\varrho, R]$ and, in the case where $\varrho=0$, $\mathcal{P}_0(h, r; 1, \alpha)$ is constant on $(0, R]$.*

This result is [2; Theorem 1], trivially modified to deal with strips of arbitrary width α. Clearly it covers the case $\lambda=1$ of Theorem 1. Further, the $\lambda=2$ case follows easily by applying the above in $\Omega_{1/2}$ to the functions s and s_1, where $s_1(M)=s(X, y+\frac{1}{2})$, and then considering $\mathcal{P}_0(s, r; 1, \frac{1}{2})+\mathcal{P}_0(s_1, r; 1, \frac{1}{2})$.

However there is more work to be done when $\lambda\in(0, 1)\cup(1, 2)$, and we require the following related result.

Theorem C. *If $s\in\mathcal{S}_\alpha$, then there exists a unique measure Λ_s on $\partial\Omega_\alpha$ and a positive constant c such that*

$$\mathcal{P}_0(s, r;\ 1, \alpha) = \mathcal{M}(s, 1;\ 1, \alpha)+c\int_1^r t^{-1}\{I(\pi t/\alpha)\}^{-2}[\Lambda_s(A_\alpha^-(0, t))$$
$$+\int_{\Omega_\alpha(0,t)} |X|^{1-n/2} I(\pi|X|/\alpha)\sin(\pi y/\alpha)\, d\mu_s(M)]dt,$$

where μ_s is given by $\gamma_n\Delta s$ in the distributional Laplacian sense. Further, if s is continuous on $\bar{\Omega}_\alpha$, then Λ_s is the zero measure on $\partial\Omega_\alpha$.

The first assertion of the above theorem is a special case of [8; Theorem 2]-for further details, see [6; Chapter 4, § 11]. The second assertion is implicit in the proof of [8; Theorem 1].

5. Proof of Theorem 1

5.1. It is perhaps worth remarking that Theorem 1 is straightforward to establish for $s\in C^2(\Omega)\cap C(\bar{\Omega})$; the more sophisticated argument below is necessitated by the much weaker assumptions we make concerning s. We will in fact prove the following more general result which is directly analogous to Theorem B. The greater generality is needed to prove some of the other results of this paper.

Theorem 1′. (i) *Suppose that either $0\leq\varrho_0<\varrho<R$ or $0=\varrho_0=\varrho<R$. If $s\in\mathcal{S}_1(\varrho_0, R)$, then $\mathcal{P}_\varrho(s, r; \lambda, 1)$ is real-valued on (ϱ, R) and is a convex function of $L(\pi\lambda r)$ on (ϱ, R). If $s\in\mathcal{S}_1(0, R)$, then $\mathcal{P}_0(s, r; \lambda, 1)$ is also an increasing function on $(0, R)$.*

(ii) *If h is real-valued and continuous in $\bar{\Omega}(\varrho, R)$ and harmonic in $\Omega(\varrho, R)$,* *then in the case where* $\varrho > 0$, $\mathscr{P}_\varrho(h, r; \lambda, 1)$ *is a linear function of* $L(\pi\lambda r)$ *on* $[\varrho, R]$ *and in the case where* $\varrho = 0$, $\mathscr{P}_0(h, r; \lambda, 1)$ *is constant on* $(0, R]$.

In view of what was said in § 4.2, Theorem 1′ is already established when $\lambda = 2$, so from now on we will assume $\lambda \in (0, 2)$. The proof of the finiteness of $\mathscr{P}_\varrho(s, r; \lambda, 1)$ is almost identical to the $\lambda = 1$ case, so we refer the reader to [2; § 7.1].

As for the rest of Theorem 1′ (i), we need only prove it under the additional hypothesis that s is harmonic in $A(\varrho_0, R) \times \left(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right)$ for some $\varepsilon \in \left(0, \frac{1}{2}\right)$. To see this, we consider the case $\varrho_0 = 0$ (the argument when $\varrho_0 > 0$ being similar), and suppose that this special case of Theorem 1′ has been proved.

Let $0 < R' < R$. For each integer $m > 2$, let

$$E(m) = B(R') \times \left(\tfrac{1}{2} - m^{-1}, \tfrac{1}{2} + m^{-1}\right),$$

and define

$$s_m(M) = \begin{cases} H_s^{E(m)}(M) & \text{if } M \in E(m) \\ s(M) & \text{elsewhere in } \bar{\Omega}_1(0, R), \end{cases}$$

where H_s^E denotes the Perron—Wiener—Brelot generalized solution of the Dirichlet problem in E with boundary data $s(M)$ (see [10; Chapter 8]). Then, by the special case of Theorem 1′, $\mathscr{P}_\varrho(s_m, r; \lambda, 1)$ is increasing as a function of r and convex as a function of $L(\pi\lambda r)$ for $r \in [0, R')$. It is easy to see that $s_m \downarrow s$ in $\Omega_1(0, R)$, whence by monotone convergence $\mathscr{P}_\varrho(s, r; \lambda, 1)$ has the same properties on $[0, R')$, and hence on $[0, R)$ since $R' \in (0, R)$ was arbitrary.

5.2. It remains to prove Theorem 1′ (i) when s is harmonic in $A(\varrho_0, R) \times \left(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right)$. If $\varrho = 0$ let $\varrho' = 0$, otherwise let $\varrho' \in (\varrho_0, \varrho)$; also let $R' \in (\varrho, R)$. We define $\varkappa = \lambda(2+\lambda)/(2-\lambda)$,

$$s^*(M) = \begin{cases} 2\cos\left[\pi\varkappa\left(y - \frac{1}{2}\right)\right] s\left(X, \frac{1}{2}\right) & \text{if } M \in A(\varrho_0, R) \times \left(\frac{1}{2}, 1/\lambda\right] \\ s(X, y) + s(X, 1-y) & \text{if } M \in A(\varrho_0, R) \times \left[0, \frac{1}{2}\right], \end{cases}$$

and

$$g(M) = \begin{cases} 2\cos\left[\pi\varkappa\left(y - \frac{1}{2}\right)\right] \left\{(\partial^2 s/\partial y^2)\left(X, \frac{1}{2}\right) + \pi^2\varkappa^2 s\left(X, \frac{1}{2}\right)\right\} & \text{if } M \in A(\varrho', R') \times \left(\frac{1}{2}, 1/\lambda\right) \\ 0 & \text{elsewhere in } \bar{\Omega}_{1/\lambda}(\varrho_0, R). \end{cases}$$

We note that

$$\int_{1/2}^{1/\lambda} \sin(\pi\lambda y) \cos\left[\pi\varkappa\left(y - \tfrac{1}{2}\right)\right] dy = 0. \tag{7}$$

Next we need to evaluate the distributional Laplacian of s^* in $A(\varrho', R') \times \left(\frac{1}{2} - \varepsilon, 1/\lambda\right)$. Routine calculations yield that, in $A(\varrho', R') \times \left(\frac{1}{2}, 1/\lambda\right)$, the ordinary Laplacian of s^* is given by $\Delta s^* = -g$. Let Ψ be a C^∞ function with compact support in $A(\varrho', R')\times(\frac{1}{2}-\varepsilon, 1/\lambda)$. From Green's theorem it now follows that

$$(8)\qquad (\Delta s^*)(\Psi) = \int_{A(\varrho',R')\times(1/2-\varepsilon,1/\lambda)} s^*(M)\Delta\Psi(M)\,dX\,dy$$
$$= \int_{A(\varrho',R')\times(1/2,1/\lambda)} \Psi(M)\Delta s^*(M)\,dX\,dy$$
$$+\lim_{\delta\to 0+}\int_{A(\varrho',R')}\left\{\Psi\left(X,\frac{1}{2}+\delta\right)\frac{\partial s^*}{\partial y}\left(X,\frac{1}{2}+\delta\right)\right.$$
$$-s^*\left(X,\frac{1}{2}+\delta\right)\frac{\partial\Psi}{\partial y}\left(X,\frac{1}{2}+\delta\right)$$
$$+s^*\left(X,\frac{1}{2}-\delta\right)\frac{\partial\Psi}{\partial y}\left(X,\frac{1}{2}-\delta\right)$$
$$\left.-\Psi\left(X,\frac{1}{2}-\delta\right)\frac{\partial s^*}{\partial y}\left(X,\frac{1}{2}-\delta\right)\right\}dX$$
$$= -\int_{A(\varrho',R')\times(1/2,1/\lambda)} \Psi(M)g(M)\,dX\,dy,$$

using the facts that s^* is continuous in $A(\varrho', R')\times(\frac{1}{2}-\varepsilon, 1/\lambda)$ and that

$$(\partial s^*/\partial y)\left(X, \tfrac{1}{2}+\delta\right) = -2\pi\varkappa\sin(\pi\varkappa\delta)s\left(X, \tfrac{1}{2}\right)\to 0,$$
$$(\partial s^*/\partial y)\left(X, \tfrac{1}{2}-\delta\right) = (\partial s/\partial y)\left(X, \tfrac{1}{2}-\delta\right)-(\partial s/\partial y)\left(X, \tfrac{1}{2}+\delta\right)\to 0$$

as $\delta\to 0+$.

Let $G(\cdot,\cdot)$ denote the Green kernel for $\Omega_{1/\lambda}$, and define measures μ_1 and μ_2 on $\Omega_{1/\lambda}$ by $d\mu_1(M)=\gamma_n g^+(M)\,dX\,dy$ and $d\mu_2(M)=\gamma_n g^-(M)\,dX\,dy$. Since g is continuous and bounded on $A(\varrho', R')\times(\frac{1}{2}, 1/\lambda)$, it is not hard to see that the potentials $G\mu_1$ and $G\mu_2$ are continuous on $\Omega_{1/\lambda}$ and vanish continuously at $\partial\Omega_{1/\lambda}$ (cf. [10; Theorem 6.22 and Lemma 6.24]). In view of (8), the function $s_0=s^*-G\mu_1+G\mu_2$ has zero Laplacian (i.e. is harmonic) in $A(\varrho', R')\times(\frac{1}{2}-\varepsilon, 1/\lambda)$. Further, it is clearly subharmonic in $A(\varrho', R')\times(0, \frac{1}{2})$. Thus, if we could show that

$$(9)\qquad \mathscr{P}_\varrho(s, r;\ \lambda, 1) = \mathscr{P}_\varrho(s_0, r;\ 1, 1/\lambda)+\text{constant},$$

for $r\in(\varrho', R')$, then from Theorem B, $\mathscr{P}_\varrho(s, r; \lambda, 1)$ is convex as a function of $L(\pi\lambda r)$ on (ϱ', R'). In view of the arbitrariness of ϱ' and R', this must be true on (ϱ, R). The rest of Theorem 1′ (i) and (ii) would also follow directly from Theorem B.

It therefore remains to establish (9). From Theorem C,

$$\mathscr{M}(-G\mu_i, r;\ 1, 1/\lambda) = \mathscr{M}(-G\mu_i, 1;\ 1, 1/\lambda)+c\int_1^r t^{-1}\{I(\pi\lambda t)\}^{-2}$$
$$\times\int_{\Omega_{1/\lambda}(0,t)} |X|^{1-n/2}I(\pi\lambda|X|)\sin(\pi\lambda y)\,d\mu_i(M)\,dt$$

($i=1, 2$) and so using (7),

$$\mathcal{M}(G\mu_2-G\mu_1, r;\ 1, 1/\lambda) = \mathcal{M}(G\mu_2-G\mu_1, 1;\ 1, 1/\lambda)+c'\int_1^r t^{-1}\{I(\pi\lambda t)\}^{-2}$$

$$\times\int_{\Omega_{1/\lambda}(0,t)} |X|^{1-n/2}I(\pi\lambda|X|)\sin(\pi\lambda y)g(M)\,dX\,dy\,dt$$

$$= \mathcal{M}(G\mu_2-G\mu_1, 1;\ 1, 1/\lambda) = d,$$

say. Thus

$$\mathcal{P}_\varrho(s_0, r;\ 1, 1/\lambda) = d+\mathcal{P}_\varrho(s^*, r;\ 1, 1/\lambda)$$

$$= d+\mathcal{P}_\varrho(s, r;\ \lambda, 1)+r^{-n/2}\{I(\pi\lambda r)\}^{-1}\int_{S(r)\times(1/2, 1/\lambda)} s^*(M)\sin(\pi\lambda y)\,d\tau(M)$$

$$= d+\mathcal{P}_\varrho(s, r;\ \lambda, 1)$$

using (7) again. This proves (9) as required.

6. The corollaries of Theorem 1

6.1. Again we prove a more detailed result, as follows:

Corollary 1'. *If* $0<\lambda\leq 2$, $s\in\mathcal{S}_1(\varrho, R)$ *and* (2) *holds for a.e.* $r\in(\varrho, R)$, *then* $\mathcal{M}(s, r; \lambda)$ *is a convex function of* $L(\pi\lambda r)$ *on* (ϱ, R). *In the case where* $\varrho=0$, $\mathcal{M}(s, r; \lambda)$ *is also an increasing function of* $r\in(0, R)$.

Let $\varrho'\in(\varrho, R)$. We recall that the Wronskian determinant of I and K evaluated at t is t^{-1} [17; p. 80], and so

$$(d/dt)L(\pi\lambda t) = -t^{-1}\{I(\pi\lambda t)\}^{-2} < 0 \quad (t>0).$$

Hence

$$d\mathcal{N}_{\varrho'}(s, r;\ \lambda, 1)/dL(\pi\lambda r) = -\pi\lambda\int_{A(\varrho', r)} |X|^{1-n/2}I(\pi\lambda|X|) \times\left\{s(X, 0)+s(X, 1)-2\cos(\pi\lambda/2)\,s\left(X, \tfrac{1}{2}\right)\right\}dX. \tag{10}$$

The boundary condition (2) ensures that the right hand side of (10) decreases as $L(\pi\lambda r)$ increases, so $\mathcal{N}_{\varrho'}(s, r; \lambda, 1)$ is concave as a function of $L(\pi\lambda r)$ on (ϱ', R). Since

$$\mathcal{M}(s, r;\ \lambda) = \mathcal{P}_{\varrho'}(s, r;\ \lambda, 1)-\mathcal{N}_{\varrho'}(s, r;\ \lambda, 1) \tag{11}$$

and $\varrho'\in(\varrho, R)$ was arbitrary, $\mathcal{M}(s, r; \lambda)$ is convex as a function of $L(\pi\lambda r)$ on (ϱ, R) as required.

If $\varrho=0$, the monotonicity of $\mathcal{M}(s, r; \lambda)$ follows from (10) and (11) since the right hand side of (10) is non-negative.

6.2. In proving Corollary 2, we assume without loss of generality that $a>0$, and we define s_0 on $\Omega_1(a, b)\cup A_1^-(a, b)$ by

$$s_0(M)=\begin{cases} s(M) & \text{if } M\in(a, b)\times[0, 1] \\ 0 & \text{if } M\in(-b, -a)\times[0, 1]. \end{cases}$$

Let $a<x_1<x_2<b$ and choose c, d such that

$$\mathcal{M}(s_0, x_i; \lambda)=c+dL(\pi\lambda x_i) \quad (i=1, 2).$$

From Corollary 1′,

$$\mathcal{M}(s_0, x; \lambda)\leqq c+dL(\pi\lambda x) \quad (x_1\leqq x\leqq x_2).$$

Since

$$\mathcal{M}(s_0, x; \lambda)=x^{-1/2}\{I(\pi\lambda x)\}^{-1}\int_{[0,1]}s(x, y)\sin\left[\pi\lambda\left(\tfrac{1}{2}-\left|\tfrac{1}{2}-y\right|\right)\right]dy$$

and $(n=1)$

$$I(t)=(2/\pi t)^{1/2}\cosh t, \quad K(t)=(\pi/2t)^{1/2}e^{-t},$$

it follows that

$$\int_{[0,1]}s(x, y)\sin\left[\pi\lambda\left(\tfrac{1}{2}-\left|\tfrac{1}{2}-y\right|\right)\right]dy\leqq c(2/\lambda)^{1/2}\pi^{-1}\cosh(\pi\lambda x)+d(2\lambda)^{-1/2}e^{-\pi\lambda x}$$

with equality when $x=x_1, x_2$. Hence we have the claimed convexity with respect to the family $\{Ae^{\pi\lambda x}+Be^{-\pi\lambda x}\colon A, B\in R\}$.

7. Proofs of Theorems 2 and 3

7.1.

Lemma A. *Let f be non-negative and locally integrable in $\mathbf{R}^n$, let $\lambda>0$*

$$A_1(r)=\int_1^r t^{-1}\{I(\pi\lambda t)\}^{-2}\int_{B(t)}|X|^{1-n/2}I(\pi\lambda|X|)f(X)\,dX\,dt,$$

$$A_2(r)=\int_{B(r)\setminus B(1)}|X|^{(1-n)/2}e^{-\pi\lambda|X|}f(X)\,dX,$$

and let $A_1(\infty)$ and $A_2(\infty)$ be the limits of $A_1(r)$ and $A_2(r)$ as $r\to\infty$. Then $A_1(\infty)$ is finite if and only if $A_2(\infty)$ is finite.

The proof of Lemma A involves integration by parts. In fact, the $\lambda=1$ case is proved in [2; Lemma 6] and other values of λ require only trivial modification to the argument. We therefore omit the details.

Theorem 2 will now be proved. Suppose $s\not\equiv-\infty$. Then hypothesis (i) together with the fact (see [17; p. 202]) that

$$(2\pi t)^{1/2}e^{-t}I(t)\to 1 \quad (t\to+\infty)$$

imply

$$\liminf_{r\to\infty} \mathcal{M}(s, r;\ \lambda) < +\infty.$$

Also, hypothesis (ii) together with Lemma A yield

$$\lim_{r\to\infty} \mathcal{N}_0(s^+, r;\ \lambda) < \lim_{r\to\infty} \mathcal{N}_0(s^-, r;\ \lambda) = +\infty,$$

since $\cos(\pi\lambda/2) < 0$. Hence

$$\liminf_{r\to\infty} \mathcal{P}_0(s, r;\ \lambda) = \liminf_{r\to\infty} \mathcal{M}(s, r;\ \lambda) + \lim_{r\to\infty} \mathcal{N}_0(s^+, r;\ \lambda) - \lim_{r\to\infty} \mathcal{N}_0(s^-, r;\ \lambda) = -\infty,$$

yielding a contradiction to Theorem 1. Thus $s \equiv -\infty$ as required.

7.2. To prove Theorem 3, we apply Corollary 1 to show that $\mathcal{M}(s, \cdot\,; \lambda)$ is increasing on $(0, +\infty)$. This, together with (6) and the fact that $s \geqq 0$ implies that $\mathcal{M}(s, \cdot\,; \lambda) \equiv 0$. It follows that $s=0$ a.e. (Lebesgue) in Ω, and so $s \equiv 0$ in Ω by the volume mean-value inequality.

8. Proofs of Theorems 4 and 5

8.1. We prove Theorem 4 when $n \geqq 2$, the proof for $n=1$ requiring minor modification due to the fact that the first integral in the definition of $\mathcal{N}_0(f, r; \lambda)$ is over $(0, r)$ rather than $(1, r)$. Let s_m be the function equal to H_s^E in $E = \Omega_1(0, m)$ and equal to s elsewhere in $\bar{\Omega}$. Then $s_m \geqq s$ and $s_m \in \mathcal{S}_1$ ([2; Lemma 2]). Since $\lambda \in (0, 1]$, it follows that $\mathcal{N}_0(s_m, r; \lambda) \leqq \mathcal{N}_0(s, r; \lambda)$ for all $r > 0$, and so $\mathcal{P}_0(s_m, r; \lambda) \leqq \mathcal{P}_0(s, r; \lambda)$ for $r \geqq m$. Hence, using Theorem 1,

$$\begin{aligned}(12)\qquad \mathcal{M}(s_m, 1;\ \lambda) = \mathcal{P}_0(s_m, 1;\ \lambda) &\leqq \mathcal{P}_0(s_m, m;\ \lambda) \leqq \mathcal{P}_0(s, m;\ \lambda)\\ &\leqq \sup_{r>0} \mathcal{P}_0(s, r;\ \lambda) < +\infty.\end{aligned}$$

It is easy to see that $(s_m)_{m \geqq R}$ is an increasing sequence of harmonic functions in $\Omega_1(0, R)$, and so $\lim s_m$ is either identically $+\infty$ or harmonic in Ω. From (12) and the monotone convergence theorem the former is impossible, so the result follows.

8.2. Let s_0 be in Ω equal to the least harmonic majorant of s in Ω, and in $W \setminus \Omega$ equal to s. Also, let $R > 0$. Then s_0 is subharmonic in W ([2; Lemma 3]), and s_0 equals $H_{s_0}^E$ in $E = \Omega_1(0, R)$, since the latter is regular. Since the restriction of s_0 to ∂E is u.s.c. and bounded above, there is a decreasing sequence of continuous functions (f_m) on ∂E such that $f_m \to s_0$. Let

$$h_m(M) = \begin{cases} H_{f_m}^E(M) & \text{if } M \in E \\ f_m(M) & \text{if } M \in \partial E. \end{cases}$$

Then $\mathscr{P}_0(h_m, r; \lambda)$ is constant on $(0, R]$ for all m, using Theorem 1′ (ii). Since $h_m \downarrow s_0$ on $\bar{E}$, it follows that $\mathscr{P}_0(s_0, r; \lambda)$ is constant on $(0, R]$. Since $R>0$ is arbitrary, $\mathscr{P}_0(s_0, r; \lambda)$ is constant on $(0, +\infty)$, and since $\lambda\in[1, 2]$,

$$\mathscr{P}_0(s, r;\ \lambda) \leqq \mathscr{P}_0(s_0, r;\ \lambda) = \mathscr{P}_0(s_0, 1;\ \lambda) \quad (r\in(0, +\infty)),$$

proving the theorem.

9. Proof of Theorem 6

9.1. We require the following two lemmas.

Lemma B. *If $\lambda>0$, then the functions*

$$|X|^{1-n/2}I(\pi\lambda|X|)\sin(\pi\lambda y), \quad |X|^{1-n/2}I(\pi\lambda|X|)\cos(\pi\lambda y)$$

are harmonic in $\mathbf{R}^{n+1}$, and the functions

$$|X|^{1-n/2}K(\pi\lambda|X|)\sin(\pi\lambda y), \quad |X|^{1-n/2}K(\pi\lambda|X|)\cos(\pi\lambda y)$$

are harmonic in $\{(X, y)\in R^{n+1}\colon |X|>0\}$.

Lemma C. *Let W be an open subset of $\mathbf{R}^{n+1}$, let s be subharmonic in W and let h be positive and harmonic in W. If $f\colon \mathbf{R}\to\mathbf{R}$ is convex and increasing, then $hf(s/h)$ is subharmonic in W.*

The proof of Lemma B is only a trivial modification of the proof of [3; Lemma 1]. Lemma C is given in [7; Theorem 1 — see the end of § 3].

9.2. The $\lambda=1$ case of Theorem 6 is [2; Theorem 7], so we will assume in what follows that $\lambda\in(0, 1)$. Let h_λ be the function defined in § 2, which is harmonic in $\mathbf{R}^{n+1}$ by Lemma B, and clearly positive in $\bar{\Omega}$ since $\lambda\in(0, 1)$. It follows from Lemma C (with $f(x)=[\max\{x, 0\}]^p$) that the function $s_1=h_\lambda^{1-p}s^p$ is non-negative and subharmonic in Ω.

The condition (4) on s can be rearranged to give

$$\int_{S(r)}\tfrac{1}{2}\cos^{1-p}(\pi\lambda/2)[s^p(X,0)+s^p(X,1)]\,d\sigma(X) \leqq \cos(\pi\lambda/2)\int_{S(r)} s^p\left(X, \tfrac{1}{2}\right)d\sigma(X)$$

which, on multiplying across by $\{|X|^{1-n/2}I(\pi\lambda|X|)\}^{1-p}$, shows that condition (2) is satisfied by the function s_1. By Corollary 1 of Theorem 1, $\mathscr{M}(s_1, r; \lambda)$ is real-valued and increasing on $(0, +\infty)$. Since $\mathscr{M}_p(s, r; \lambda)=\{\mathscr{M}(s_1, r; \lambda)\}^{1/p}$, the same properties hold for $\mathscr{M}_p(s, r; \lambda)$ on $(0, +\infty)$.

Further, $\mathscr{M}(s_1, r; \lambda)$ is convex as a function of $L(\pi\lambda r)$, and from this the convexity of $\mathscr{M}_p(s, r; \lambda)$ may be deduced. The details are as in the $\lambda=1$ case, so we refer the reader to [2; § 11.2].

10. Proof of Theorem 7

The $\lambda=1$ case of Theorem 7 is contained in [8; Theorem 9], so we assume $\lambda\in(0,1)$. The monotonicity of $\mathcal{M}_\infty(s,r;\lambda)$ will first be proved. Let $0<r_1<r_2$. The function s/h_λ is u.s.c. and so attains its supremum, a say, on the compact set $\bar{\Omega}_1(0,r_2)$. Let $N\in\bar{\Omega}_1(0,r_2)$ be such that $s(N)=ah_\lambda(N)$. If $N\in\Omega_1(0,r_2)$, then the subharmonic function $s-ah_\lambda$ attains its supremum in $\Omega_1(0,r_2)$ and so is constant there by the maximum principle. Hence

$$\mathcal{M}_\infty(s,r_1;\ \lambda)=\mathcal{M}_\infty(s,r_2;\ \lambda). \tag{13}$$

If $N\in A_1^{\sim}(0,r_2)$, then it follows from (3) that there exists $N'\in A(0,r_2)\times\{\frac{1}{2}\}$ for which $s(N')=ah_\lambda(N')$ and (13) again follows. The only remaining possibility is that $a\in\bar{\Gamma}_1(r_2)$, which clearly implies that $\mathcal{M}_\infty(s,r_1;\lambda)\leqq\mathcal{M}_\infty(s,r_2;\lambda)$. Hence the monotonicity is proved.

To prove the convexity, choose a, b such that

$$\mathcal{M}_\infty(s,r_i;\ \lambda)=a+bL(\pi\lambda r_i)\quad(i=1,2). \tag{14}$$

Letting

$$u_\lambda(M)=|X|^{1-n/2}K(\pi\lambda|X|)\cos\left[\pi\lambda\left(y-\tfrac{1}{2}\right)\right],$$

which by Lemma B is harmonic for $|X|>0$, it follows from (14) that $s\leqq ah_\lambda+bu_\lambda$ on $\Gamma_1(r_1)\cup\Gamma_1(r_2)$. Applying the maximum principle to the subharmonic function $s-ah_\lambda-bu_\lambda$ in the open set $\Omega_1(r_1,r_2)$ shows that (following the same type of argument as in the previous paragraph) $s\leqq ah_\lambda+bu_\lambda$ in $\Omega_1(r_1,r_2)$, whence

$$\mathcal{M}_\infty(s,r;\ \lambda)\leqq a+bL(\pi\lambda r)\quad(r_1\leqq r\leqq r_2)$$

as required.

11. Proof of Theorem 8

We again assume $\lambda\in(0,1)$ since the $\lambda=1$ case is covered by [8; Theorem 10]. From Lemma C and (5) the function $s_1=h_\lambda\exp(s/h_\lambda)$ belongs to $\mathcal{S}_1$ and satisfies (2). It follows from Corollary 1 of Theorem 1 that $\mathcal{M}\left(h_\lambda\exp(s/h_\lambda),r;\lambda\right)$ is real-valued and increasing as a function of r, whence the same must be true of $\mathcal{M}_E(s,r;\lambda)$ since h_λ is positive in Ω.

To show the convexity property, let $0<r_1<r_2$, choose $a\in(0,r_1)$, and observe from Lemmas B, C that the function

$$s_1=h_\lambda\exp[(ku_\lambda+s)/h_\lambda]$$

where

$$k=\{\mathcal{M}_E(s,r_2)-\mathcal{M}_E(s,r_1)\}/\{L(\pi\lambda r_1)-L(\pi\lambda r_2)\} \tag{15}$$

belongs to $\mathscr{S}_1(a, +\infty)$. Further, it is easy to see from (5) that (2) holds for s_1 and a.e. $r>0$. From Corollary 1' in § 6 it follows that $\mathscr{M}(s_1, r; \lambda)$ is real-valued and convex as a function of $L(\pi\lambda r)$ on $(a, +\infty)$. Thus, if $r\in(r_1, r_2)$, then

$$\exp\{kL(\pi\lambda r)\}\exp\{\mathscr{M}_E(s, r; \lambda)\}$$
$$\leqq \left\{\frac{L(\pi\lambda r_2)-L(\pi\lambda r)}{L(\pi\lambda r_2)-L(\pi\lambda r_1)}\right\}\exp\{kL(\pi\lambda r_1)\}\exp\{\mathscr{M}_E(s, r_1; \lambda)\}$$
$$+\left\{\frac{L(\pi\lambda r)-L(\pi\lambda r_1)}{L(\pi\lambda r_2)-L(\pi\lambda r_1)}\right\}\exp\{kL(\pi\lambda r_2)\}\exp\{\mathscr{M}_E(s, r_2; \lambda)\}$$

which, upon rearranging, using (15) and taking logs, yields

$$\mathscr{M}_E(s, r; \lambda) \leqq \left\{\frac{L(\pi\lambda r_2)-L(\pi\lambda r)}{L(\pi\lambda r_2)-L(\pi\lambda r_1)}\right\}\mathscr{M}_E(s, r_1; \lambda)+\left\{\frac{L(\pi\lambda r)-L(\pi\lambda r_1)}{L(\pi\lambda r_2)-L(\pi\lambda r_1)}\right\}\mathscr{M}_E(s, r_2; \lambda)$$

as required.

References

1. Ahlfors, L. V., Remarks on Carleman's formula for functions in a halfplane, *SIAM J. Numer. Anal.* **3** (1966), 183—187.
2. Armitage, D. H. and Fugard, T. B., Subharmonic functions in strips, *J. Math. Anal. Appl.* **89** (1982), 1—27.
3. Brawn, F. T., Mean value and Phragmén—Lindelöf theorems for subharmonic functions in strips, *J. London Math. Soc.* **3** (1971), 689—698.
4. Dinghas, A., Über das Anwachsen einiger Klassen von subharmonischen und verwandten Funktionen, *Ann. Acad. Sci. Fenn. Ser. A. I. Math.* **336/1** (1963).
5. Dinghas, A., Über einige Konvexitätsätze für die Mittelwerte von subharmonischen Funktionen, *J. Math. Pures Appl.* **44** (1965), 223—247.
6. Gardiner, S. J., Generalized means of subharmonic functions, Doctoral thesis, the Queen's University of Belfast, 1982.
7. Gardiner, S. J., A maximum principle and results on potentials, *J. Math. Anal. Appl.* **108** (1985), 507—514.
8. Gardiner, S. J., Generalized means of subharmonic functions, *Ann. Acad. Sci. Fenn. Ser. A. I. Math.* **11** (1986), 3—27.
9. Heins, M., On some theorems associated with the Phragmén—Lindelöf principle, *Ann. Acad. Sci. Fenn. Ser. A. I. Math.-Phys.* **46** (1948).
10. Helms, L. L., *Introduction to potential theory*, Wiley-Interscience, New York, 1969.
11. Huber, A., On functions subharmonic in a half-space, *Trans. Amer. Math. Soc.* **82** (1956), 147—159.
12. Kuran, Ü., Study of superharmonic functions in $\mathbf{R}^n\times(0, +\infty)$ by a passage to $\mathbf{R}^{n+3}$, *Proc. London Math. Soc.* **20** (1970), 276—302.
13. Kuran, Ü., On half-spherical means of subharmonic functions in half-spaces, *J. London Math. Soc.* **2** (1970), 305—317.

14. NORSTAD, F., Convexity of means and growth of certain subharmonic functions, *Ark. Mat.* **16** (1978), 141—152.
15. WANBY, G., Convexity of means and growth of certain subharmonic functions in an n-dimensional cone, *Ark. Mat.* **21** (1983), 29—43.
16. WANBY, G., Convexity of means and growth of subharmonic functions in strips, *J. Math. Anal. Appl.*, to appear.
17. WATSON, G. N., *A treatise on the theory of Bessel functions*, Cambridge Univ. Press, London, 1922.

Received May 9, 1985

S. J. Gardiner
Department of Mathematics
University College
Belfield, Dublin 4
Ireland