Conformal mapping and Hausdorff measures

N. G. Makarov

Introduction

Let \( f: D \rightarrow \Omega \) be a conformal mapping of the unit disc \( D \) onto a Jordan domain \( \Omega \) and let \( E \) be a Borel subset of the unit circle \( T = \partial D \). Suppose we know that \( E \) is of positive \( p \)-dimensional Hausdorff measure. What can be said about metric properties of the set \( fE \), the image of \( E \) under the boundary correspondence induced by \( f \)? In particular, what are the restrictions on its Hausdorff dimension \( \dim fE \)?

In the case \( p = 1 \) this problem corresponds to a problem on the metric properties of harmonic measure. It had for a long time been known (see [25]) that if \( |E| > 0 \), then

\[
\dim fE \equiv \frac{1}{2}
\]

but some efforts were required to improve this bound to the sharp result

\[
\dim fE \equiv 1
\]

see [21]. The basic step was taken by Lennart Carleson [9] who proved that

\[
\dim fE \equiv \beta < \frac{1}{2}
\]

for an (unspecified) absolute constant \( \beta \).

In fact, it is easy to establish the inequality

\[
\dim fE \equiv \frac{1}{2} \dim E,
\]

generalizing (0.1), for all \( p < 1 \) (see Section 1.2). The natural question whether the estimates (0.2) and (0.3) can also be extended to \( p < 1 \) is the starting-point in our study. The answer is as follows. An estimate of Carleson type exists for all \( p > 0 \), whereas an estimate of the type (0.2) holds only in case \( p = 1 \). The present paper provides some quantitative amplifications of this answer. Some other relevant problems, including those concerning the relationship between boundary distortion and the behaviour of the derivative in conformal mappings, are also considered.
It should be noted that no nontrivial upper bound of \( \dim fE \) in terms of \( \dim E \) is possible in the whole class of conformal mappings onto Jordan domains. In fact, as was shown in [25], the image of a set of arbitrarily small Hausdorff dimension may have a positive area.

**0.1. Notation related to Hausdorff measures.** For \( p > 0 \) the \( p \)-dimensional Hausdorff measure is denoted by \( \Lambda_p \). We shall always consider \( \Lambda_p \) only on Borel subsets of \( \mathbb{T} \) or of \( \mathbb{C} \) (the complex plane). The measures \( \Lambda_p \) are particular cases (with \( \varphi(t) = t^p \)) of general Hausdorff measures \( \Lambda_\varphi \) corresponding to measure functions \( \varphi \) (i.e. continuous increasing functions on \([0, +\infty)\) satisfying \( \varphi(0) = 0 \)). For the definition and properties of \( \Lambda_\varphi \), see [8], [13]. However, it will often be more convenient to use the set functions \( H_\varphi \), which are defined by

\[
H_\varphi(e) = \inf \sum \varphi(r_j)
\]

the infimum being taken over all coverings of a plane set \( e \) with discs of radii \( r_j \). The quantities \( H_\varphi \) enjoy all properties of general capacities and Borel sets are capacitable with respect to \( H_\varphi \) (see [8], Chapter 1 and 2). Obviously, \( H_\varphi \) are finite on bounded sets and

\[
H_\varphi(e) = 0 \Leftrightarrow \Lambda_\varphi(e) = 0.
\]

Similarly to the definition of the harmonic measure, we introduce the following notion.

**Definition.** Let \( f: \mathbb{D} \to \Omega \) be a conformal mapping onto a Jordan domain \( \Omega \) and \( p > 0 \). The set function \( h_p^f \) is defined on Borel subsets \( e \) of \( \partial \Omega \) by

\[
h_p^f(e) = H_p(f^{-1}e).
\]

When \( f \) is clear from the context, we simplify the notation to \( h_p \). The set function \( h_p^f \) is said to be absolutely continuous with respect to \( H_\varphi \) (notation: \( h_p^f \ll H_\varphi \)) if

\[
H_\varphi(e) = 0 \Rightarrow h_p^f(e) = 0.
\]

Observe that

\[
E \subset \mathbb{T}, \quad H_p(E) > 0 \Rightarrow \dim fE \equiv q
\]

is equivalent to

\[
\forall q' < q: h_p^f \ll H_{q'}.
\]

Sometimes it is convenient to consider (0.5) as the definition of \( h_p^f(e) \) for arbitrary plane Borel sets \( e \). Clearly, in this case

\[
h_p^f(e) = h_p^f(e \cap \partial \Omega).
\]

We also define a set function \( \kappa = \kappa^f \) by

\[
\kappa^f(e) = \text{cap} f^{-1}e
\]
where \( \text{cap} \) is the logarithmic capacity. As follows from the well-known relationship between the capacities \( H_p \) and \( \text{cap} \) (see [18], p. 253),

\[
\text{h}_{p}(e) \leq C[\kappa'(e)]^p
\]

where \( C \) is a universal constant.

0.2. Main results on boundary distortion. In the statements, for brevity, we introduce the following function.

**Definition.** Let \( p \in (0, 1] \). By \( d(p) \) we denote the supremum of the set of numbers \( q > 0 \) satisfying

\[
E \subset T, \quad H_p(E) > 0 \Rightarrow \dim fE \equiv q
\]

for all \( f \) mapping \( D \) onto a Jordan domain. In other words

\[
d(p) = \sup \{ q : \text{h}_{p} < H_p \}.
\]

In terms of \( d(p) \), (0.2) means that

\[
d(1) = 1.
\]

I do not know the exact value of \( d(p) \) for any other \( p \). As it was noted, our purpose is to improve upon trivial bounds

\[
\frac{1}{2} p \leq d(p) \leq p
\]

(the left-hand inequality is just (0.4)).

**Theorem 0.1.** If \( p > 1 \) then

\[
\frac{1}{2} p < d(p) < p.
\]

Moreover

\[
\lim_{p \to \infty} \frac{d(p)}{p} = \frac{1}{2},
\]

\[
\lim_{p \to 1} \frac{d(p)}{p} = 1.
\]

Remark that (0.9) provides a generalization of (0.2):

\[
\dim E = 1 \Rightarrow \dim fE \equiv 1.
\]

Upper bounds of \( d(p) \), including the right-hand inequality in (0.7), require constructions of the corresponding examples. Such examples already happen to exist in the class of starlike domains and they even provide the stronger bound

\[
d(p) \leq \frac{p}{2 - p}
\]

which implies (0.8).
Theorem 0.2. For any $p \in (0, 1]$, there exist a conformal mapping $f$ onto a Jordan starlike domain with rectifiable boundary and a subset $E \subset \mathbb{T}$ satisfying

$$H_p(E) > 0, \quad \dim fE \leq \frac{p}{2-p}.$$ 

It is interesting to note that this bound is sharp even in the wider class, that of close-to-convex domains.

Theorem 0.3. Let $f$ be a close-to-convex function and $E \subset \mathbb{T}$. Then

$$\dim fE \leq \frac{\dim E}{2-\dim E}.$$ 

On the other hand, (0.10) is not sharp for arbitrary Jordan domains, at least when $p$ is close to one. An example will be constructed to show that the order of $p - d(p)$ is at most $\frac{1}{2}$ as $p \to 1-$, whereas $p - p(2-p)^{-1}$ is infinitesimal of first order. More precisely, the following is valid.

Theorem 0.4. If $p > \frac{19}{20}$, then

$$\frac{1}{30} \sqrt{1-p} \leq p - d(p) \leq \sqrt{\frac{1}{12} - p}.$$ 

The lower bound of $d(p)$ contained in Theorem 0.4 easily follows from some known properties of integral means of the derivative of a univalent function. This estimate implies (0.9) as well as the inequality $d(p) \geq \frac{p}{2}$ in (0.7) for $p$ sufficiently close to one.

For arbitrary $p > 0$, I do not dispose of such an elementary proof of $d(p) \geq \frac{p}{2}$. The proof (rather crude) is obtained by a famous device due to L. Carleson [9]. Our argument differs from that in [9] only in technical details. The main difference is that we avoid any modification of $\Omega$, for it is difficult to track down the effect of Hausdorff measures under such modifications. At the same time, the approach of L. Carleson seems to be much deeper than the result we derive with its help. It would not be a surprise if the inequality $d(p) \geq \frac{p}{2}$ admits a more direct proof.

0.3. Connection with the behaviour of the derivative. The greater part of the results stated relies on the relationship between boundary distortion and the behaviour of the derivative in conformal mappings. The following two assertions occur to be most convenient in further applications.
Theorem 0.5. Let \( \alpha > 0 \) and
\[
\liminf_{r \to 1} \frac{|f'(r\zeta)|}{(1-r)^2} > 0 \text{ for } \Lambda_p \text{-almost all } \zeta \in \mathcal{T}.
\]
Then for any \( q < \frac{p}{1+\alpha} \),
\[
h^f_p \ll H_q.
\]

Theorem 0.6. If
\[
h^f_p \ll H^{(1+\alpha)^{-1}},
\]
then for all \( \zeta \) outside a possible exceptional set of \( \Lambda_p \)-measure zero,
\[
\limsup_{r \to 1} \frac{|f'(r\zeta)|}{(1-r)^2} = 0.
\]

These assertions are certainly very far from being invertible, for one of them contains \( \liminf \) while the other \( \limsup \). No criterion expressed in terms of \( f' \) is known to me for the validity of
\[
h^f_p \ll H_q.
\]
However, such a criterion can easily be obtained in the particular case when \( \Omega \) is a quasidisc.

If \( I \) is a subarc of \( \mathcal{T} \) with center at \( \zeta \), then by \( a_I \) we denote the point \((1-|I|)\zeta; \cdot \cdot \cdot \) denotes the normalized Lebesgue measure on \( \mathcal{T} \).

Definition. Let \( f \) be a univalent function on \( \mathbb{D} \) and \( \varphi \) be a measure function. Define the set function \( D^\varphi \) by
\[
D^\varphi(E) = D^{\varphi}_p(E) = \inf \sum \varphi(|I_v||f'(q_v)|)
\]
where \( E \subset \mathcal{T} \), and the infimum is taken over all coverings of \( E \) with subarcs \( \{I_v\} \) of \( \mathcal{T}, a_v = a_{I_v} \). If \( \varphi(t) = t^q \), for some \( q > 0 \), \( D^\varphi \) is denoted by \( D_q^\varphi \). As above, the notation
\[
H_p \ll D_q^\varphi
\]
means that
\[
E \subset \mathcal{T}, \quad D^\varphi(E) = 0 \Rightarrow H_p(E) = 0.
\]

If \( \Omega \) is a quasidisc and \( E \subset \mathcal{T} \), then (see Section 2.1)
\[
D_q^\varphi(E) \asymp H_q(fE)
\]
and hence
\[
h^f_p \ll H_q \Leftrightarrow H_p \ll D_q^\varphi.
\]

It is probable that also in the general case a criterion could be expressed in similar terms. I can prove only a weaker version of the necessity.
Theorem 0.7. Let $f$ be a conformal mapping onto a Jordan domain. If
\begin{equation}
H_p \not\ll D_q',
\end{equation}
then
\[ \forall q' < q: h_f^q \not\ll H_{q'}.
\]

Observe that $D_q^q(E)=0$ if and only if there exists a subset $A \subset D$ such that $E$ lies in the cluster set of $A$, and
\begin{equation}
\sum_{\lambda \in A} \varphi \left[ \text{dist} \left( f(\lambda), \partial \Omega \right) \right] < \infty.
\end{equation}

Sums of the form (0.13) were studied in [20], [23] in the context of dominating subsets. It was proved ([23], Lemma 2.3) that, for regular $\varphi$, the following two assertions are equivalent:
\[ H_1 \ll D^f_{\varphi}
\]
and
\[ \lim \inf_{r \to 1^-} \frac{|f'(r\zeta)|}{\psi(1-r)} > 0 \quad \text{for a.e. } \zeta \in \Omega
\]
where $\psi(t) = t^{-1} \varphi^{-1}(t)$. Thus, for $p=1$, the conditions (0.11) and (0.12) in Theorems 0.5 and 0.7 are equivalent. This is no longer true for $p < 1$.

Theorem 0.8. For any $p \in (0, 1)$, there exist $\alpha > 0$ and a conformal mapping onto a Jordan domain such that, for some $q > \frac{p}{1+\alpha}$,
\[ h_f^q \not\ll H_q,
\]
but
\[ \lim \inf_{r \to 1^-} \frac{|f'(r\zeta)|}{(1-r)^{\alpha}} = 0, \quad \zeta \in E
\]
on some subset $E \subset \Omega$ of positive $\Lambda_p$-measure.

Consequently, Theorem 0.6 will be false for $p < 1$ if we substitute $\lim \sup$ by $\lim \inf$.

0.4. Strong absolute continuity. If $p=1$, the set function $h_f^1$ coincides with the harmonic measure of $\Omega$ evaluated at $f(0)$. In this case $h_f^1 \not\ll H_\varphi$ is equivalent to the condition
\begin{equation}
\forall \varepsilon > 0 \quad \exists \delta > 0: H_\varphi(\varepsilon) < \delta \Rightarrow h_f^1(\varepsilon) < \varepsilon.
\end{equation}
This motivates the following

Definition. The set function $h_f^1$ is said to be strongly absolutely continuous with respect to $H_\varphi$ (notation: $h_f^1 \ll H_\varphi$) if it satisfies the condition (0.14). The notation $H_p \ll D^f_q$ has a similar meaning.
It is clear that
\[ h_p' < H_\varphi \Rightarrow h_p' \ll H_\varphi, \]
but the converse is false for \( p < 1 \) (see Section 4.3). Therefore, it is interesting to look at the boundary distortion also from the viewpoint of the new notion. For the strong absolute continuity we are able to trace the relationship with the behaviour of the derivative even more precisely (compare with Theorem 0.7).

**Theorem 0.9.**
1) \[ H_p < D_q' \Rightarrow \forall q' < q: h_p' < H_{q'}. \]
2) \[ h_p' < H_q \Rightarrow H_p < D_q'. \]

Another important distinction between the two notions follows from the fact that the simple condition (0.11) in Theorem 0.5 is no longer sufficient for the strong absolute continuity (see Section 4.3).

On the other hand, we shall see that all estimates stated in Section 0.2 (including that of Carleson type) stay in force also for the strong absolute continuity.

**0.5. A problem on dominating subsets.** The methods employed in the paper to study the set functions \( D_\varphi' \) enable us to make an advance in a problem on dominating subsets stated in [22], Section 3.1. Let \( \Omega \) be a Jordan domain and \( \varphi \) be a measure function. It is known (see [22], Lemma 3.1) that if there exists a dominating subset \( A \) of \( \Omega \) satisfying
\[ (0.15) \sum_{\lambda \in A} \varphi[\text{dist}(\lambda, \partial \Omega)] \ll \infty, \]
then the harmonic measure of \( \Omega \) is singular with respect to \( H_\varphi \). (The latter means that there exists a Borel subset \( e \subset \partial \Omega \) of full harmonic but zero \( A_\varphi \)-measure.) Is the converse true? We answer in the affirmative for measure functions of special type.

**Theorem 0.10.** Let \( \varphi \) be a logarithmico-exponential function. The harmonic measure of \( \Omega \) is singular with respect to the Hausdorff measure \( A_\varphi \) if and only if there exists a dominating subset of \( \Omega \) satisfying (0.15).

**0.6. Organization of the paper.** The paper consists of seven sections.

In Section 1 we derive several basic results on the boundary distortion (most of them are known) from a theorem due to A. Pfuger [27].

In Section 2 we study relations between boundary distortion and the derivative of the conformal mapping, and prove Theorems 0.5, 0.6, 0.7 and 0.9.

In Section 3 we study distortion properties of close-to-convex functions and prove Theorems 0.2 and 0.3.

In Section 4 two examples are provided. The first corresponds to Theorem 0.8. The second exhibits distinction between absolute and the strong absolute continuity.
In Section 5 we study the boundary distortion in dimensions close to one and prove Theorem 0.4.

As was noted, the results of Sections 3 and 5 imply all assertions of Theorem 0.1 except for the inequality $d(p) > \frac{p}{2}$. The proof of the latter is the subject of Section 6.

Section 7 is devoted to concluding remarks. First, we list the facts on the radial growth of the reciprocal of the derivative obtained in the previous sections and also state the counterparts of these results concerning the growth of the derivative itself. Secondly, we prove Theorem 0.10 on dominating subsets.

**Some more notation.** $\mathbb{N}$ is the set of positive integers; $A(z_0, r)$ is the disc \{\(z: |z-z_0| < r\)\}, and $\overline{A}(z_0, r)$ is its closure; $R(z_0; r_1, r_2)$ is the annulus $A(z_0, r_2) \setminus \overline{A}(z_0, r_1)$.

The letters $c$ and $C$ are used to denote various constants.

### 1. Some consequences of Pfluger's Theorem

In this section some auxiliary results on boundary distortion are provided. All of them admit simple proofs based on the technique of extremal lengths, mainly invoking a theorem due to A. Pfluger. For the most part, these results are well-known, but our approach may possibly deserve some interest.

#### 1.1. Facts on extremal length

For the convenience of the reader, we recall the definition and some basic properties of extremal lengths. See [3], [2] and [26] for a more comprehensive account.

**Definition.** Let $\Gamma$ be a family of locally rectifiable curves in $\mathbb{C}$. Consider all non-negative Borel measurable functions $\varphi$ on $\mathbb{C}$, integrable with respect to the area measure $m_2$, and for each such $\varphi$ define

$$L(\varphi) = \inf_{\gamma \in \Gamma} \int \varphi(z) |dz|.$$ 

The supremum

$$\lambda(\Gamma) = \sup_{\varphi} \frac{[L(\varphi)]^a}{\iint \varphi^2 dm_2}$$

is called the extremal length of the family $\Gamma$.

**Properties.**
1) Extremal length is conformally invariant.
2) If each curve $\gamma_2 \in \Gamma_2$ contains some curve $\gamma_1 \in \Gamma_1$, then $\lambda(\Gamma_1) \geq \lambda(\Gamma_2)$.
3) If the families $\{\Gamma_j\}$ lie in disjoint Borel sets and if $\Gamma = \bigcup \Gamma_j$, then

$$[\lambda(\Gamma)]^{-1} \equiv \sum [\lambda(\Gamma_j)]^{-1}.$$
4) If the families \( \{\Gamma_j\} \) are as above and \( \Gamma \) is a family such that each \( \gamma \in \Gamma \) contains at least one \( \gamma_j \in \Gamma_j \) for any \( j \), then

\[
\lambda(\Gamma) \geq \sum \lambda(\Gamma_j).
\]

**Example.** Let \( \Gamma \) be the family of all curves in the annulus \( R(z_0; r_1, r_2) \) that join the boundary circumferences. Then

\[
\lambda(\Gamma) = \frac{1}{2\pi} \log \frac{r_2}{r_1}.
\]

**Pfluger's theorem** ([27]). Let \( K \) be a Jordan curve in \( A(0, \frac{1}{3}) \) surrounding the origin. Let \( E \) be a Borel subset of \( \mathbb{T} \) and \( \Gamma \) be the family of all curves in \( D \) joining \( E \) with \( K \). Then

\[
c_K \exp \{-\pi \lambda(\Gamma)\} \leq \text{cap } E \leq C_K \exp \{-\pi \lambda(\Gamma)\}
\]

where

\[
c_K = \inf \frac{1-|z|}{|z|}, \quad C_K = \sup \frac{1+|z|}{|z|}.
\]

In the sequel we shall use only the right-hand inequality in (1.1). Taking into account the properties of the extremal length listed above, this inequality may be rewritten in the following conformally invariant form.

**Corollary.** Let \( f: D \to \Omega \) be a conformal mapping onto a Jordan domain \( \Omega \) and \( K \) be a continuum lying in \( \Omega \). Let \( E \) be a Borel subset of \( \partial \Omega \) and \( \Gamma \) be the family of all curves in \( \Omega \) joining \( K \) with \( E \). Then

\[
\kappa^f(e) \leq C \exp \{-\pi \lambda(\Gamma)\}
\]

where \( C \) does not depend on \( e \).

By (1.3) and (0.6), we also have

\[
h^f_p(e) \leq C \exp \{-p \pi \lambda(\Gamma)\}.
\]

**1.2. Proposition.** Let \( f \) be a conformal mapping onto a Jordan domain \( \Omega \). There exists \( C > 0 \) such that for all \( e \subset \partial \Omega \),

\[
\kappa^f(e) \leq C [\text{diam } e]^{1/2}.
\]

**Proof.** Fix a continuum \( K \) in \( \Omega \). Let \( R = \text{dist} (K, \partial \Omega) \). Clearly, it is enough to prove (1.5) only for \( e \) of diameter less than \( R \). Let \( \Gamma \) be a family of all curves joining \( e \) and \( K \). Then

\[
\lambda(\Gamma) \geq \frac{1}{2\pi} \log \frac{R}{\text{diam } e}.
\]

By (1.3),

\[
\kappa(e) \leq C \exp \left\{ -\frac{\pi}{2\pi} \log \frac{R}{\text{diam } e} \right\} = C[R^{-1} \text{diam } e]^{1/2}. \quad \Box
\]
50  N. G. Makarov

**Corollary 1.** For any \( p > 0 \),
\[
    h_p > H_{p/2}.
\]

**Proof.** Let \( e \subset \partial \Omega \) and \( H_{p/2}(e) < \delta \). Then \( e \subset \bigcup \Delta_v \) where \( \{ \Delta_v \} \) are discs of radii \( r_v \), and
\[
    \sum r_v^{p/2} < \delta.
\]
By (0.6) and (1.5),
\[
    h_p(e) \leq \sum h_p(\Delta_v) = C \sum [\kappa(\Delta_v)]^p
    \leq C \sum (\text{diam } \Delta_v)^{p/2} \leq C \delta.
\]

**Corollary 2.**
\[
    d(p) \equiv \frac{1}{2} p. \quad \Box
\]

**Remark.** The inequality (1.5) is immediate from a result of Ch. Pommerenke [28] which asserts, in particular, the following. If \( g \) is a conformal mapping of \( \mathbb{C} \setminus \mathbb{D} \) onto the exterior of a Jordan domain and satisfies \( g(\infty) = \infty, \ g'(\infty) = 1 \), then
\[
    [\text{cap } E]^p \equiv \text{cap } gE, \quad E \subset \mathbb{T}.
\]
As to the latter, it may also be derived from Pfluger’s theorem. One has to apply the formula (which follows from (1.1) and (1.2))
\[
    \text{cap } E = \lim_{R \to \infty} \sqrt{R} \exp \{-\pi \lambda(\Gamma_R)\},
\]
where \( \Gamma_R \) is the family of all arcs joining \( E \) and \( \partial \mathbb{D}(0, R) \), and the estimate
\[
    \text{cap } e \equiv \lim_{R \to \infty} R \exp \{-\pi \lambda(\Gamma'_R)\}
\]
valid for any bounded plane set \( e \) (\( \Gamma'_R \) is the family of all arcs joining \( e \) and \( \partial \mathbb{D}(0, R) \)). The inequality (1.6) readily follows from the estimate of \( \lambda(\Gamma'_R) \) arising by the choice \( q = |\text{grad } u| \) in the definition of the extremal length, where \( u \) is the equilibrium potential of \( e \) (cf. [26], § 2.23).

**1.3. Proposition.** Let \( f \) be a conformal mapping onto a Jordan domain, let \( I \) be a subarc of \( \mathbb{T} \) and onto \( a = a_I \). Let \( R > 1 \) and set
\[
    \Delta_R = \Delta(f(a), R|I|/|f'(a)|).
\]
Then
\[
    \text{cap } (I \setminus f^{-1} \Delta_R) \leq C R^{-1/2} |I|
\]
where \( C \) is a universal constant.

**Proof.** Applying an appropriate Möbius transformation, we can reduce the problem to the case \( a = 0, \ f(0) = 0, \ f'(0) = 1 \). Let \( e = \partial \Omega \setminus \Delta(0, R) \). We must verify that
\[
    \kappa(e) \equiv CR^{-1/2}.
\]
Let $\Gamma$ denote the family of all curves in $\Omega$ joining $\partial A(0, \frac{1}{2})$ with $e$. Then

$$\lambda(\Gamma) \equiv \frac{1}{2\pi} \log 8R$$

and, by (1.3),

$$\kappa(e) \equiv C \exp\{-\pi \lambda(\Gamma)\} \leq C \exp\{-\frac{1}{2} \log 8R\} = CR^{-1/2}$$

where, by (1.2) and the distortion theorem, $C$ can be chosen absolute. □

The inequality (1.7) is a particular case of another result due to Ch. Pommerenke [29], see also [30], Chapter 11.

1.4. Theorem. Let $f: D \to \Omega$ be a conformal mapping onto a Jordan domain $\Omega$. For any $p \in (0, 1]$ and any $M > 0$ there are numbers $r_0 > 0$ and $k_0 \in \mathbb{N}$ satisfying the following. If $A$ is a disc of radius $r$, $r \geq r_0$, and $A'$ is a disc of radius $2r$ concentric with $A$, then there exist $N$ subarcs $\sigma_1, \ldots, \sigma_N$ of $\partial A'$,

$$N \leq k_0 \log \frac{1}{r},$$

which are crosscuts of $\Omega$ and separate from $f(0)$ the subarcs $\beta_1, \ldots, \beta_N$ of $\partial \Omega$ such that

$$h^f[\Delta \setminus \bigcup_{j=1}^N \beta_j] \equiv r^M.$$

Proof. Fix a small circle $K$ centered at $f(0)$. We can assume that $\text{dist}(K, \partial \Omega) > 2r_0$. Let $A$ be a disc of radius $r \geq r_0$ such that $\partial \Omega \cap A \neq \emptyset$. We carry out the following construction. See Figure 1.
Let $\Omega_0$ denote the component of $\Omega \setminus \overline{A}$ containing $f(0)$. Let $\{U_j\}$ be the set of those components of $\Omega_0 \cap A'$ whose boundary has an arc on $\partial A$. It is clear that this set is not empty, that the components $U_j$ are disjoint and lie in the annulus $A \setminus \overline{A}$.

The set $(\partial U_j \cap \partial A') \setminus \partial \Omega$ is relatively open with respect to $\partial A'$. Let $\sigma_j$ denote the component of this set that separates $U_j$ from $f(0)$, and $\beta_j$ denote the subarc of $\partial \Omega$ that is separated from $f(0)$ by the crosscut $\sigma_j$. Denote $e_j = \beta_j \cap A$. Then

$$\partial \Omega \cap A = \bigcup e_j. \quad (1.9)$$

Let $\Gamma_j$ be the family of all arcs in $\Omega$ joining $K$ with $e_j$, and $\tilde{\Gamma}_j$ be the family of all arcs in $U_j$ which join the boundary circumferences of the annulus $A \setminus \overline{A}$. By the properties of extremal lengths,

$$\lambda(\Gamma_j) \equiv \lambda(\tilde{\Gamma}_j)$$

and

$$\sum [\lambda(\Gamma_j)]^{-1} \equiv \left[ \frac{1}{2\pi} \log 2 \right]^{-1}. \quad (1.9)$$

By (1.4)

$$h_p(e_j) \equiv A \exp \{-\pi p \lambda(\Gamma_j)\}$$

with a constant $A$ depending only on $f$, $K$ and $p$. The two last inequalities imply that for all $k \in \mathbb{N}$

$$\text{card} \{j: h_p(e_j) \equiv A r^{npk} \} \equiv \frac{2\pi}{\log 2} \frac{k \log \frac{1}{r}}{r}. \quad (1.10)$$

Choose $k_0 \in \mathbb{N}$ large enough to satisfy

$$\pi pk_0 > M.$$ 

Then

$$\sum_{(j: h_p(e_j) < A r^{npk_0})} h_p(e_j) \equiv \sum_{k \geq k_0} \sum_{(j: A r^{np(k+1)} \leq h_p(e_j) \leq A r^{npk})} h_p(e_j)$$

$$\equiv \frac{2\pi A}{\log 2} \frac{1}{r} \sum_{k \geq k_0} (k+1) r^{npk} \equiv r^M$$

provided $r \geq r_0$ and $r_0$ is sufficiently small.

We have actually proved the theorem. In fact, put $N$ equal to

$$\text{card} \{j: h_p(e_j) \equiv A r^{npk} \}.$$ 

By (1.10), $N \leq k_0 \log \frac{1}{r}$. Suppose that the $U_j$ are arranged in such a way that the numbers $h_p(e_j)$ come in decreasing order. By (1.9),

$$h_p(A \setminus \bigcup_{j=1}^{N} \beta_j) \equiv \sum_{(j: h_p(e_j) < A r^{npk_0})} h_p(e_j) \equiv r^M. \quad \square$$

**Remark.** In the case $p = 1$ the theorem was established in [22], Lemma 2.3, and the present proof is quite similar to that. The method of proof is essentially due to L. Carleson [9].
Corollary. Let $f: D \rightarrow \Omega$ be a conformal mapping onto a Jordan domain $\Omega$ and $K$ be a continuum in $\Omega$. For any $p, q \in (0, 1]$ and any $q' > q$ there exists $r_0 > 0$ satisfying the following. If $\Delta$ is a disc of radius $r = r_0$ with

$$h_p(\Delta) \geq r^{pq},$$

then there exists a subarc $\sigma$ of $\partial \Delta'$ which constitutes a crosscut of $\Omega$ and satisfies

$$(1.11) \quad \lambda(\Gamma_\sigma) \geq \frac{q'}{\pi} \log \frac{1}{r}$$

where $\Gamma_\sigma$ is the family of all curves in $\Omega$ joining $\sigma$ with $K$.

Proof. Applying the last theorem with a sufficiently large $M > 0$, we obtain $N$ subarcs $\beta_j$ of $\partial \Omega$ such that

$$N \leq k_0 \log \frac{1}{r}$$

and

$$h_p(\Delta \setminus \bigcup_{j=1}^{N} \beta_j) \geq \frac{1}{2} r^{pq}.$$ 

Hence

$$\sum_{j=1}^{N} h_p(\beta_j) \geq \frac{1}{2} r^{pq}.$$ 

Consequently, there exists $j_0$ such that for $\beta = \beta_{j_0}$ we have

$$h_p(\beta) \geq c |\log r|^{-1} r^{pq}.$$

By (1.4)

$$h_p(\beta) \geq A \exp \left\{-\pi p \lambda(\Gamma_\sigma)\right\}$$

where $\sigma = \sigma_{j_0}$. Thus (if $r_0$ is small),

$$\exp \left\{-\pi p \lambda(\Gamma_\sigma)\right\} \geq c |\log r|^{-1} r^{pq} \geq r^{pq} = \exp \left\{-pq' \log \frac{1}{r}\right\}$$

which implies (1.11). $\square$

1.5. Proposition. Let $f: D \rightarrow \Omega$ be a conformal mapping onto a Jordan domain $\Omega$, $I$ be a subarc of $T$ and $\sigma$ be a crosscut of $\Omega$ joining the endpoints of $f(I)$. Then

$$\text{diam } \sigma \geq c |I||f'(a_I)|$$

where $c > 0$ is a universal constant.

Proof. By means of an appropriate Möbius transformation we can reduce the problem to the case $f(0) = 0$, $f'(0) = 1$, $I = T_+$ (the upper semicircle) and $a_I = 0$. We should verify that in this case

$$\text{diam } \sigma \equiv c.$$ 

Let $\Delta = \Delta \left(0, \frac{1}{2}\right)$. Then $\text{dist}(f\Delta, \partial \Omega) \equiv \frac{1}{54}$ by the distortion theorem. If $\sigma \cap f\Delta \neq \emptyset$, 

then diam $\sigma \equiv \frac{1}{54}$. Also if $\sigma \cap f\Delta = \emptyset$, then the crosscut $f^{-1}\sigma$ separates a semicircle, say $T_+$, from $\Delta$. Denote the family of all arcs in $D$ joining $T_+$ with $\partial \Delta$ by $\Gamma$. On the one hand,

$$\lambda(\Gamma) \geq \frac{1}{\pi} \log 2.$$ 

On the other hand (see Figure 2),

$$\lambda(\Gamma) \geq \frac{1}{2\pi} \| \log (54 \text{diam } \sigma) \|.$$

Hence diam $\sigma \equiv \frac{1}{216}$. \hfill \Box

The last result is certainly well-known (cf. [30], Exercise 2, p. 318). Lemma 2.2 in [22] is a consequence of Proposition 1.5.

2. Boundary distortion and the behaviour of the derivative

2.1. It is instructive first to consider the case of quasidiscs. Recall that a Jordan domain $\Omega$ is said to be a quasidisc if there exists a number $M > 0$ such that for any two points $w_1$ and $w_2$ on $\partial \Omega$,

$$\min (\text{diam } \beta_1, \text{diam } \beta_2) \equiv M |w_1 - w_2|$$
where \( \beta_1 \) and \( \beta_2 \) are the components of \( \partial \Omega \setminus \{ w_1, w_2 \} \). See [1] for the relation with quasiconformal mappings.

**Lemma.** Let \( f \) be a conformal mapping of \( D \) onto a quasidisc \( \Omega \). There exists \( C > 0 \) such that if \( I \) is a subarc of \( T \), then

\[
C^{-1} |I| |f'(a_i)| \leq \text{diam} \, f \, I \leq C |I| |f'(a_i)|.
\]

**Proof.** The left-hand inequality (with a universal \( C \)) follows from Proposition 1.5. To prove the second inequality, let \( I \) be a subarc of \( T \) with \( |I| \equiv \frac{1}{2} \). From both endpoints of \( I \) we draw the subarcs \( I_1 \) and \( I_2 \) congruent to \( I \). By Proposition 1.3 there are points \( z_1 \in I_1 \) and \( z_2 \in I_2 \) such that

\[
|f(a_i) - f(z_1)|, \quad |f(a_i) - f(z_2)| \leq C |I| |f'(a_i)|
\]

with a universal \( C \). Hence

\[
|f(z_1) - f(z_2)| \leq C_1 |I| |f'(a_i)|.
\]

By \( \tilde{I} \) we denote the subarc containing \( I \) with endpoints \( z_1 \) and \( z_2 \). Then either

\[
\text{diam} \, f \, \tilde{I} \leq MC_1 |I| |f'(a_i)|
\]

and (2.1) is true, or

\[
\text{diam} \, f \, (T \setminus \tilde{I}) \leq MC_1 |I| |f'(a_i)|.
\]

Since \( T \setminus \tilde{I} \) contains a semicircle, by Proposition 1.5,

\[
\text{diam} \, f \, (T \setminus \tilde{I}) \leq c |f'(0)|
\]

where \( c \) is a universal constant. Therefore, by (2.2),

\[
|I| |f'(a_i)| \leq c_1 > 0
\]

where \( c_1 \) does not depend on \( I \). Hence

\[
\text{diam} \, f \, \tilde{I} \equiv \text{diam} \, \Omega \equiv (c_1^{-1} \text{diam} \, \Omega) |I| |f'(a_i)|,
\]

and the right-hand inequality in (1.1) follows. \( \square \)

**Proposition.** Let \( f \) be a conformal mapping onto a quasidisc and \( q > 0 \). There exists \( C > 0 \) such that for any \( E \subset T \),

\[
C^{-1} D_q^f(E) \equiv H_q(fE) \equiv CD_q(E).
\]

**Proof.** First we establish the right-hand inequality. Cover \( E \) by arcs \( \{ I_i \} \) with

\[
\sum [ |I_i| |f'(a_i)|]'^t < D_q(E) = \varepsilon.
\]

Then the right-hand inequality in (2.1) implies that

\[
H_q(fE) \leq \sum [C |I_i| |f'(a_i)|]^q \leq CD_q(E) + C\varepsilon.
\]
To prove the second inequality in (2.3), we consider a covering of \( fE \) with discs \( \{A_v\} \) of radii \( r_v \) such that
\[
\sum r_v^q < H_q(fE) + \varepsilon.
\]
Let \( e_v = fE \cap A_v \). By the definition of a quasidisc, either \( r_v > c > 0 \) with \( c \) not depending on \( E \) (in this case, (2.3) already follows) or there is a subarc \( I_v \) of \( T \) such that \( e_v \subset fI_v \) and \( fI_v \) is contained in a disc of radius \( Mr_v \). If the latter holds for all \( v \), then \( E \subset \bigcup I_v \) and, by (2.1),
\[
\sum [\|f'(a_v)\|] q = \sum \sum [CMr_v] q \leq CH_q(fE) + C\varepsilon. \quad \Box
\]

**Corollary.** If \( f \) is a conformal mapping onto a quasidisc and \( p, q \) are positive numbers, then
\[
\begin{align*}
(2.4) & \quad h_p^f < H_q \iff H_p < D_q^f; \\
(2.5) & \quad h_p^f < H_q \iff H_p < D_q^f.
\end{align*}
\]

**2.2. Conjecture.** The equivalences (2.4) and (2.5) are valid for arbitrary Jordan domains.

If this conjecture were true, the problem of the boundary distortion would be completely reduced to a question concerning derivatives of conformal mappings. Unfortunately, I can prove only partial results in this direction.

**2.3. Theorem.** Let \( f: D \to \Omega \) be a conformal mapping onto a Jordan domain \( \Omega \). Let \( p, q \in (0, 1] \) and
\[
\phi(t) = t^q|\log t|^{1-q}.
\]
Then
\[
\begin{align*}
(2.6) & \quad H_p < D_q^f \Rightarrow h_p^f < H_q; \\
(2.7) & \quad H_p < D_q^f \Rightarrow h_p^f < H_q.
\end{align*}
\]

**Proof.** First we prove (2.7). Suppose that \( H_p < D_q \), and let \( e = fE \subset \partial \Omega \) satisfy \( H_q(e) < \delta \) where \( \delta \) is a small number. We have to prove that \( H_q(E) \) is also small.
Consider a covering of \( e \) with discs \( \{A_v\} \) of radii \( r_v \) satisfying
\[
\sum \phi(r_v) < \delta.
\]
To each disc we apply Theorem 1.4 with a fixed constant \( M > q \). There exist crosscuts \( \sigma_j^{(v)}, \quad 1 \equiv j \equiv N(v) \),
\[
N(v) \equiv k_0 \log \frac{1}{r_v},
\]
which lie on \( \partial A_v \) and separate the subarcs \( \beta_j^{(v)} \) of \( \partial \Omega \) from \( f(0) \) such that
\[
h_p(A_v \setminus \bigcup_{j=1}^{N(v)} \beta_j^{(v)}) \equiv r_v^M.
\]
Denote
\[ e_\delta = fE_\delta = e \cap \bigcup_v N_v \cup \bigcup_{j=1}^{N(v)} \beta_j. \]
Then
\[ e \setminus e_\delta \subset \bigcup_v \left[ A_v \setminus \bigcup_{j=1}^{N(v)} \beta_j \right] \]
and
\[ h_p(e \setminus e_\delta) \leq \sum r_v^M \leq \delta. \]
Also denote \( I_j^{(v)} = f^{-1}\beta_j \) and \( a_j^{(v)} = a_j^{(v)} \). Applying Hölder's inequality, (2.8) and Proposition 1.5, we have
\[ \sum_{j=1}^{N(v)} |I_j^{(v)}| |f'(a_j^{(v)})|^q \equiv [N(v)]^{1-q} \left[ \sum_{j=1}^{N(v)} |I_j^{(v)}| |f'(a_j^{(v)})|^q \right]^q \]
\[ \leq C \left[ \log \frac{1}{r_v} \right]^{1-q} \left[ \sum_{j=1}^{N(v)} \text{diam } \sigma_j^{(v)} \right]^q. \]
Since \( \sigma_j^{(v)} \) are disjoint, the latter does not exceed
\[ C |\log r_v|^{1-q} r_v^q = C \varphi(r_v). \]
Since \( E_\delta \subset \bigcup_v I_j^{(v)} \),
\[ D_q(E_\delta) \leq C \sum_v \varphi(r_v) \leq C \delta. \]
Because of \( H_p < D_q \), \( H_p(E_\delta) \) tends to zero as \( \delta \to 0 \). Hence
\[ h_p(e) = H_p(E) \leq H_p(E_\delta) + H_p(E \setminus E_\delta) \leq \delta + o(1) \quad \text{as } \delta \to 0, \]
and \( h_p < H_p \).

Next we verify (2.6). Let \( H_p < D_q \), and suppose that \( E \subset T \) satisfies \( H_p(fE) = 0 \). We have to prove that \( H_p(E) = 0 \). Reasoning as above, for each \( \delta > 0 \), we obtain a subset \( E_\delta \subset E \) such that
\[ H_p(E \setminus E_\delta) < \delta, \quad D_q(E_\delta) < \delta. \]
Define
\[ E_0 = \bigcap_{n \geq 1} \bigcup_{k \geq n} E_{2^{-k}}. \]
Then, for all \( n \in \mathbb{N} \),
\[ D_q(E_0) \equiv D_q(\bigcup_{k \geq n} E_{2^{-k}}) \equiv \sum_{k \geq n} D_q(E_{2^{-k}}) \equiv 2^{-n+1}. \]
Hence \( D_q(E_0) = 0 \) and
\[ H_p(E_0) = 0. \]
For any \( n \in \mathbb{N} \) we also have
\[ E \setminus E_0 \subset \bigcup_{k \geq n} E \setminus E_{2^{-k}}. \]
Therefore
\[ H_p(E \setminus E_0) \equiv \sum_{k \geq n} H_p(E \setminus E_{2^{-k}}) \equiv 2^{-n+1}, \]
and \( H_p(E \setminus E_0) = 0 \). Combined with (2.9), this gives \( H_p(E) = 0 \). \( \square \)
The result obtained implies Theorem 0.7 and the first assertion in Theorem 0.9. The second assertion will be proved in Section 2.4.

**Remark.** For \( p=q=1 \), Theorem 2.3 was established in [20], Theorem 4, where it was also noted that the implications (2.6) and (2.7) (coinciding when \( p=1 \)) are reversible. In Section 7 we shall proceed with the discussion of the case \( p=1 \), that of the harmonic measure.

### 2.4. Theorem

**Theorem.** Let \( f \) be a conformal mapping onto a Jordan domain. Then

\[
h_p' < H_q \Rightarrow H_p < D_q.
\]

**Proof.** Suppose that \( h_p < H_q \). To prove the theorem, it is sufficient to verify that for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) satisfying:

\[
E \text{ is compact, } D_q(E) < \delta \Rightarrow H_p(E) \leq \varepsilon.
\]

In fact, if \( E \) is an arbitrary Borel subset of \( T \), then, by the properties of the "capacities" \( H_p \) (see [8], Ch. 2), there exists a sequence of compact subsets \( E_n \subset E \) such that \( H_p(E) = \lim H_p(E_n) \). If \( D_q(E) < \delta \), then \( D_q(E_n) < \delta \) and, by (2.10), \( H_q(E_n) \leq \varepsilon \). Hence \( H_q(E) \leq \varepsilon \), which provides \( H_p < D_q \).

Thus, let \( E \) be a compact subset of \( T \) and \( D_q(E) < \delta \). We can cover \( E \) by a finite number of arcs \( \{I_v\} \) with

\[
\sum |I_v||f'(a_v)|^q < \delta,
\]

so that the multiplicity of the covering is at most two. Fix a large number \( R > 0 \) and apply Proposition 1.3 to each arc \( I_v \). For each \( v \) there exists a compact subset \( F_v \subset I_v \) such that

\[
|F_v| \equiv \frac{1}{2} |I_v|,
\]

and \( ff_v \) lies in a disc of radius \( R|I_v||f'(a_v)| \). Denote the compact set \( \bigcup F_v \) by \( F \). Then \( H_q(ff) \equiv R^q \delta \) and, since \( h_p < H_q \), \( H_p(F) = o(1) \) as \( \delta \to 1 \).

Now we prove that (2.11) implies that

\[
H_p(F) \equiv cH_p(E)
\]

with a universal \( c \). This will yield (2.10). By the Frostman theorem (see, e.g., [8], Ch. 2, Theorem I), there exists a nonnegative measure \( \mu \) supported by \( E \) such that

\[
\int d\mu \equiv cH_p(E),
\]

and for any subarc \( I \) of \( T \)

\[
\mu |I| \equiv |I|^p.
\]
By means of $\mu$ we shall construct a measure $\eta$ supported by $F$ and satisfying
\begin{equation}
\int d\eta \equiv cH_p(E); \quad \forall I: \eta(I) \equiv 10|I|^p.
\end{equation}
Then (2.12) will trivially follow.

Let $\eta$ be defined by
\[ \eta = \sum \eta_v \]
where the measures $\eta_v$ are supported by $F_v$ and have the constant density $|F_v|^{-1}\mu(I_v)$ with respect to Lebesgue measure. Then
\[ \int d\eta_v = \mu(I_v); \quad \int d\eta = \sum \mu(I_v) \equiv \int d\mu \equiv cH_p(E). \]

To prove (2.14), let $I$ denote a subarc of $T$ with endpoints $\zeta_1$ and $\zeta_2$.
\[ \eta(I) = \sum_{v: \zeta_v \in I_v} \eta(I_v) \equiv \sum_{v: \zeta_v \in I_v,} + \sum_{v: \zeta_v \in F_v,} + \sum_{v: \zeta_v \in I_v \setminus I} \]

The last sum does not exceed
\[ \sum_{v: \zeta_v \in F_v} \mu(I_v) \equiv 2\mu(I) \equiv 2|I|^p. \]
The first and the second sums contain at most two terms. Let, for instance, $\zeta_1 \in I_v$.
If $|I_v| \leq |I|$, then
\[ \eta_v(I) = |F_v|^{-1}|F_v \cap I|\mu(I_v) \equiv \mu(I_v) \equiv |I_v|^p \equiv |I|^p. \]
If $|I_v| \geq |I|$, then
\[ \eta_v(I) \leq 2|I_v|^{-1}|I||I_v|^p \leq 2|I|^p \]
because of $p \equiv 1$. Hence (2.14) follows. \[ \Box \]

I do not know whether Theorem 2.4 remains true with $\leq$ replaced by $\prec$. Sometimes the following partial result turns out to be useful.

2.5. Proposition. Let $f$ be a conformal mapping onto a Jordan domain and $h^f \ll H_q$. Then for any subset $E \subset T$ of positive $A_p$-measure and for any $C > 0$
\begin{equation}
\inf \left\{ \sum \frac{|I_v(1)|^p}{|f'(a_v)|}: E \subset \bigcup I_v, \sum |I_v|^p \equiv C \right\} > 0.
\end{equation}

Proof. Assume that the infimum in (2.15) is zero for some $C > 0$ and $E \subset T$ with $H_p(E) > 0$. Fix $\delta > 0$ and consider a covering $E \subset \bigcup I_v$ satisfying
\[ \sum |I_v|^p \equiv C, \quad \sum \frac{|I_v|}{|f'(a_v)|}^q < \delta. \]
Applying Proposition 1.3 (with $R = \delta^{-1/q}$) to each $I_v$, we obtain subsets $F_v \subset I_v$ such that
\[ H_p(I_v \setminus F_v) \equiv C\delta^{-1/q}|I_v|^p, \]
and $fF_v$ lies in a disc of radius $\delta^{-1/q}|I_v||f'(a_v)|$. Define
\[ F = F^{(0)} = \bigcup F_v. \]
Then
\[ H_q(f^{\mathcal{F}(\xi)}) \equiv e^{-q/2} \sum |I_\nu| |f'(a_\nu)|^q \equiv e^{1-q/2} \]
and
\[ H_q(E \setminus F^{\mathcal{F}(\xi)}) \equiv \sum H_p(I_\nu \setminus F_\nu) \equiv C_0^{p/4} \sum |I_\nu|^p \equiv C_0^{p/4} \]
Let
\[ E_0 = \cap_{n \geq 1} \cup_{k \geq n} F^{(2^{-k})} \]
For any \( n \in \mathbb{N} \)
\[ H_q(f E_0) \equiv \sum_{k \geq n} H_q(f F^{(2^{-k})}) \equiv \sum_{k \geq n} 2^{-k(1-\alpha/2)} \]
Hence
\[ (2.16) \]
\[ H_q(f E_0) = 0. \]
Also, for any \( n \in \mathbb{N} \),
\[ E \setminus E_0 \subset \cup_{k \geq n} (E \setminus F^{(2^{-k})}) \]
and
\[ H_p(E \setminus E_0) \equiv C \sum_{k \geq n} 2^{-k p/4} \]
which implies \( H_p(E_0) > 0 \). Combined with (2.16), this contradicts the assumption \( h_p < H_q \). \( \square \)

As consequences of Theorem 2.3 and Proposition 2.5, we shall prove Theorems 0.5 and 0.6 (see Introduction).

2.6. Proof of Theorem 0.5. Because of (2.6), it is sufficient to prove \( H_p \leq D_{p(1+\alpha)^{-1}} \).
Assume the contrary — that there is a subset \( E \subset T \) satisfying \( H_p(E) > 0 \) and \( D_{p(1+\alpha)^{-1}}(E) = 0 \). By hypothesis, there exists a subset \( E_0 \subset E \) of positive \( A_p \)-measure such that, for some \( c > 0 \),
\[ (2.17) \]
\[ |f'(r \xi)| \equiv c(1-r)^\alpha \]
for all \( \xi \in E_0 \) and \( r \in (0, 1) \). For any \( \varepsilon > 0 \), there exists a covering \( E_0 \subset \bigcup I_\nu \) with
\[ \sum |I_\nu| |f'(a_\nu)|^{p/1+\alpha} < \varepsilon. \]
Clearly, we can assume that all \( I_\nu \) meet \( E_0 \). Then, by (2.17) and the distortion theorem,
\[ |f'(a_\nu)| \equiv c |I_\nu|^\alpha. \]
Consequently,
\[ \sum |I_\nu|^p \equiv C \sum |I_\nu| |f'(a_\nu)|^{p/1+\alpha} \equiv C\varepsilon, \]
and hence \( H_p(E_0) = 0. \) \( \square \)

Remark. The hypothesis of Theorem 0.5 does not, in general, imply the strong absolute continuity, see Section 4.3. Reasoning as above, one can easily verify the following sufficient condition: If
\[ \liminf_{n \to \infty} H_p(\xi \colon n |f'(r \xi)| \equiv (1-r)^\alpha \text{ for some } r \in (0, 1)) = 0, \]
then
\[ h_p^f < H_p(1+\alpha)^{-1}. \]
2.7. Proof of Theorem 0.6. Assume the contrary — that there is a subset $E_0 \subset T$ of positive $A_p$-measure satisfying

$$\lim_{r \to 1^-} \frac{|f'(r \zeta)|}{(1-r)^2} = 0, \quad \zeta \in E_0.$$  

Then there exists a subset $E \subset E_0$ of finite positive $A_p$-measure with the uniform estimate

$$(2.18) \quad |f'(r \zeta)| \equiv \delta(1-r)(1-r)^2, \quad \zeta \in E,$$

where $\delta(t) = o(1)$ as $t \to 0$, holding on it. Fix $\varepsilon > 0$ and consider a covering of $E$ with arcs $I_v$ such that

$$|I_v| < \varepsilon; \quad H_p(E) \equiv \sum |I_v|^p \leq A_p(E) + \varepsilon.$$

By (2.18) and the distortion theorem

$$|f'(a_v)| \leq \delta(a)I_v^a.$$

Therefore

$$\sum ||I_v||f'(a_v)||^{p+\varepsilon} \leq \delta(a)^q[A_p(E) + \varepsilon] \to 0.$$  

Hence the infimum in (2.15) is zero for $q = p(1+\varepsilon)^{-1}$ and $C = H_p(E)$. By Proposition 2.5 this contradicts the assumption $h_p \ll H_p$. $\square$

2.8. Remark. As it has been noted, for $p=1$, $h_p \ll H_{p(1+\varepsilon)^{-1}}$ implies that

$$\lim_{r \to 1^-} \frac{|f'(r \zeta)|}{(1-r)^2} > 0$$

for a.e. $\zeta \in T$. For $p < 1$, this is no longer true, see Section 4.1. The reason is that the lower density (with respect to $H_p$, $p < 1$) of a set of positive $A_p$-measure may be zero everywhere. But if the set is subject to some density condition, we can claim more

**Proposition.** Let the subset $E \subset T$ satisfy $0 < A_p(E) < \infty$ and

$$\lim_{t \to 0} \frac{H_p(E \cap A(\zeta, t))}{t^p} \equiv c > 0, \quad \zeta \in E,$$

and let

$$\lim_{r \to 1^-} \frac{|f'(r \zeta)|}{(1-r)^2} = 0, \quad \zeta \in E.$$  

Then $h_p \ll H_{p(1+\varepsilon)^{-1}}$.

**Proof.** Fix $\varepsilon > 0$. For any $\zeta \in E$ there is a subarc $I_\zeta$ of $T$ with center at $\zeta$ such that

$$|f'(a_{I_\zeta})| < \varepsilon |I_\zeta|^a;$$  

$$H_p(E \cap I_\zeta) \equiv c |I_\zeta|^p.$$
Applying the covering lemma to \( \{I_\varepsilon\}_{\varepsilon \in E} \), we obtain a covering \( \{I_\varepsilon\} \) of finite multiplicity. For brevity, we shall write \( I_\varepsilon \) instead of \( I_{\varepsilon} \). Then

\[ \sum |I_\varepsilon|^p \leq C \sum \lambda_p(E \cap I_\varepsilon) \leq CA_p(E) = C. \]

On the other hand

\[ \sum [|I_\varepsilon| f'(a_\varepsilon)]^{p/(1+\varepsilon)} \leq C \sum |I_\varepsilon|^p \to 0 \quad \text{as} \quad \varepsilon \to 0. \]

Therefore, the infimum in (2.15) is zero for \( q = p(1+\varepsilon)^{-1} \). By Proposition 2.5

\[ h_p \notin H_q. \]

The argument is applicable, for example, to standard Cantor sets of constant ratio.

3. Boundary distortion of close-to-convex functions

To illustrate the results obtained in the previous section, we consider a question on the boundary distortion in the class of close-to-convex functions. This class plays an important role in the theory of conformal mappings. On the one hand, it is large enough to contain many interesting examples of univalent functions. On the other, this class is often much easier to deal with, and many problems, open for arbitrary univalent functions, admit a complete solution for close-to-convex functions.

Recall that a function \( f \) analytic in the unit disc is said to be close-to-convex if there is a starlike function \( g \) such that

\[ \text{Re} \frac{zf'(z)}{g(z)} > 0 \quad \text{for all} \quad z \in \mathbb{D}. \]

Also recall that an analytic function \( g \) satisfying \( g(0) = 0, \ g'(0) > 0 \) is a starlike function if

\[ \text{Re} \frac{zg'(z)}{g(z)} > 0, \quad z \in \mathbb{D}. \]

Starlike functions are obviously close-to-convex. Close-to-convex functions are univalent (see [12] §2.5 and 2.6).

Our main result is as follows. If \( f \) is a close-to-convex function, then

\[ \dim_f E \leq \frac{\dim E}{2 - \dim E}, \quad E \subset \mathbb{T}, \]

and this bound is sharp, see Theorem 0.2 and 0.3.

3.1. Proof of Theorem 0.2. Let \( 0 < p < 1 \). Take an appropriate Cantor set \( E \subset \mathbb{T} \) satisfying

\[ 0 < A_p(E) < \infty. \]
and denote by \( \mu \) the probability measure which is a multiple of the restriction \( \lambda_p|E \).
Let \( u \) denote the Poisson integral of \( \mu \) and \( \bar{u} \) the conjugate function with \( \bar{u}(0)=0 \).
Define
\[
 w = \frac{2}{1+u+i\bar{u}}.
\]
Then \( w(0)=1 \) and
\[
(3.3) \quad \text{Re } w = 2 \frac{1+u}{(1+u)^2 + \bar{u}^2} > 0.
\]
If \( f \) is defined by
\[
 f(z) = z \exp \left\{ \int_0^z \frac{w(z)-1}{z} \, dz \right\}, \quad z \in \mathbb{D},
\]
then \( \frac{zf'}{f} = w \) and (3.2) is valid. Hence \( f \) is univalent and maps \( \mathbb{D} \) onto a domain \( \Omega \) starlike with respect to the origin.

We shall verify that \( \partial \Omega \) is a rectifiable Jordan curve. Since
\[
|f'(z)| \leq 2|f(z)||w(z)| \leq 4|f(z)|, \quad |z| > \frac{1}{2},
\]
the derivative \( f' \) is bounded on \( \mathbb{D} \). Consequently, \( f \) is continuous up to the boundary and \( \partial \Omega \) has a finite length. The injectivity of \( f| \mathbb{T} \) follows from the well-known identity
\[
 \frac{\partial}{\partial \theta} \arg w(re^{i\theta}) = \text{Re } w(re^{i\theta})
\]
and also from (3.3) and the fact that \( \text{Re } w \neq 0 \) a.e. on \( \mathbb{T} \).

To check the distortion properties of \( f \), we make use of Theorem 0.6. Because of the homogeneity of the Cantor set \( E \),
\[
 \liminf_{t \to 0} \frac{\mu A(\zeta, t)}{t^p} \equiv c > 0
\]
for all \( \zeta \in E \). By a simple estimate of the Poisson integral, this implies that
\[
 u(r_\zeta) \equiv c(1-r)^{p-1}
\]
for all \( \zeta \in E \) and \( r \in (0, 1) \). Hence, for \( \zeta \in E \),
\[
|f'(r_\zeta)| \equiv C[u(r_\zeta)]^{-1} \equiv C(1-r)^{1-p}.
\]
Applying Theorem 0.6 with \( \alpha < 1-p \), we have
\[
 h_p \nmid H_{p(1+\alpha)^{-1}}.
\]
The assertion now easily follows.

Remark. The idea to use the Poisson integral of a singular measure for the construction of an example in boundary distortion is due to A. Lohwater and G. Piranian [19].
3.2. Proposition. Let \( f \) be a close-to-convex function mapping \( D \) onto a Jordan domain. If \( 0 < p < 1 \) and \( 0 < q < \frac{p}{2 - p} \), then

\[
h_p' < H_q.
\]

Proof. Having in view to apply the assertion contained in Section 2.6, we prove that if

\[
F_n = \{ \zeta \in \mathbb{T} : n|f'(r\zeta)| \leq (1-r)^{1-p} \text{ for some } r(0, 1) \},
\]

then

\[
(3.4) \quad H_p(F_n) \to 0 \text{ as } n \to \infty.
\]

Let \( w = \frac{zf'}{g} \), where \( g \) is a starlike function satisfying (3.1). Since (see [30], Ch. 11 Theorem 9)

\[
\text{cap} \{ \zeta \in \mathbb{T} : \int_0^1 |g'(r\zeta)|dr > R \} = o(1) \text{ as } R \to \infty,
\]

for the validity of (3.4) it is enough to prove the following. If \( \Phi \) is an analytic function in \( D \) with positive real part and

\[
E_n = \{ \zeta \in \mathbb{T} : |\Phi(r\zeta)| \geq n(1-r)^{p-1} \text{ for some } r(0, 1) \},
\]

then

\[
(3.5) \quad H_p(E_n) \to 0 \text{ as } n \to \infty.
\]

Let \( \mu \) be a positive measure on \( \mathbb{T} \) such that

\[
\Phi(z) = \int \frac{z + \zeta}{z - \zeta} d\mu(\zeta), \quad z \in D.
\]

If \( \zeta \in E_n \), then

\[
(3.6) \quad \sup_{t > 0} \frac{\mu A(\zeta, t)}{t^p} \equiv cn
\]

with \( c \) depending only on \( p \). In fact, if \( \mu A(\zeta, t) \equiv At^p \) for all \( t > 0 \), then

\[
|\Phi(r\zeta)| \equiv CA \int_0^1 \frac{tp}{|r - e^{it}|^2} \, dt \equiv CA (1-r)^{p-1}
\]

where \( C \) depends only on \( p \).

By (3.6), for each \( \zeta \in E_n \), there is an interval \( I_\zeta \subset \mathbb{T} \) with center at \( \zeta \) satisfying

\[
\mu I_\zeta \equiv cn |I_\zeta|^p.
\]
Applying the covering lemma to \( \{I_{\zeta}\}_{\zeta \in E_n} \), we can select a subcovering \( \{I_{\zeta}\} \) of multiplicity at most two. Then

\[
H_p(E_n) \leq \sum_{\zeta} |I_{\zeta}|^{p} \leq (cn)^{-1} \sum_{\zeta} \mu I_{\zeta} \leq 2(cn)^{-1} \mu E
\]

which implies (3.5). \( \square \)

Theorem 0.3 is a consequence of the last result.

4. Two examples

We shall apply the method of the previous section to construct two examples which shed more light on the relationship between boundary distortion and the behaviour of the derivative. The first example (it corresponds to Theorem 0.8) shows that, for \( p < 1 \), in contrast with \( p = 1 \), the condition

\[
H_p \ll D_q^{f}
\]

is strictly weaker than the condition

\[
\liminf_{r \to 0} \frac{|f'(r\zeta)|}{(1-r)^{\alpha}} > 0 \quad \text{for } A_p\text{-a.e. } \zeta
\]

where \( q = p(1 + \alpha)^{-1} \). The second example shows that, for \( p < 1 \),

\[
(4.1) \quad h_p^f \ll H_q \Leftrightarrow h_p^f < H_q.
\]

4.1. First example. Choose a rapidly increasing sequence \( \{v_k\} \) of positive integers and set

\[
N_n = \prod_{k=1}^{n} v_k, \quad l_n = N_n^{-1/2}.
\]

Consider \( v_1 \) closed subarcs \( I^{(1)} \) of \( T \) which are of length \( l_1 \) and equidistributed on \( T \). At the \( n \)-th stage of the construction, we place \( v_n \) closed arcs \( I^{(n)} \) of length \( l_n \) into each arc \( I^{(n-1)} \) obtained at the previous stage, in such a manner that the distance between any two neighbouring arcs \( I^{(n)} \) is at least

\[
(4.2) \quad \frac{1}{2} \frac{l_{n-1}}{v_n} = \frac{1}{2} (l_{n-1} l_n)^{1/2}.
\]

The union of all \( N_n \) arcs \( I^{(n)} \) constitutes the closed subset \( E^{(n)} \subset T \). Define the compact set \( E \) by

\[
E = \bigcap_{n \geq 1} E^{(n)}.
\]

A standard argument shows that

\[
(4.3) \quad 0 < A_{1/2}(E) \leq 1.
\]
For each $I^{(n)}$, by $\zeta(I^{(n)})$ we denote a point at the distance $l_n^{3/4}$ from $I^{(n)}$ and consider the corresponding point measure of magnitude $l_n^{5/8}$. Let $\mu_n$ be the sum of all these point measures. Clearly,

$$\int d\mu_n = l_n^{1/8}. $$

Define the probability measure $\mu$ by

$$\mu = C \sum \mu_n $$

with an appropriate constant $C$. As in Section 3.1, let $u$ denote the Poisson integral of the measure $\mu$,

$$w = 2(1 + u + i\delta)^{-1}$$

(4.4)

$$f(z) = z \exp \left\{ \int_0^z \frac{w(z) - 1}{z} \, dz \right\}. $$

Then $f$ maps $D$ onto a starlike Jordan domain $\Omega$.

**Theorem.** The conformal mapping $f$ and the set $E$ constructed above satisfy the conditions:

(4.6) $H_{1/2} \lesssim D_{4/9}$;

(4.7) $\forall \zeta \in \bar{E}: \liminf_{r \to 1-} \frac{|f'(r_\zeta)|}{(1-r)^{1/6}} < \infty$.

**Remarks.** 1) From (4.3) and the inequality

$$\frac{4}{9} \geq \frac{1/2}{1 + 1/6} = \frac{3}{7}, $$(4.8)

we obtain (4.1) for $p=1/2$. Observing that, by Theorem 0.7, (4.6) implies that

$$\forall q < 4/9: \ h_{r/2} \lesssim H_q, $$

we also obtain Theorem 0.8 for $p=1/2$. Similar arguments are applicable for any $p < 1$. We omit the details.

2) In addition to (4.8), we will see that

(4.9) $\forall q > \frac{4}{9}: \ h_{r/2} \lesssim H_q$.

**4.2. Proof of Theorem.** Let $\zeta \in I^{(n)}$ and $r = 1 - l_n^{3/4}$. Then

$$u(r_\zeta) \equiv c(1-r)^{-1} \mu_n(\zeta(I^{(n)})) = c(1-r)^{-1/6}. $$

Consequently,

$$|f''(r_\zeta)| \equiv C(1-r)^{1/6}, $$

and (4.7) follows.
The rest of the proof is devoted to the verification of (4.6). We need a lemma.

**Lemma.** Let I be a subarc of T. Then

\[ H_{1/2}(E \cap I) \equiv \max \{|I|^{8/9}, C|f'(a_I)|^4\} \]

where C does not depend on I.

**Proof of Lemma.** Let \( H_{1/2}(E \cap I) = I^* \) with \( \alpha \in [\frac{1}{9}, \frac{8}{9}] \). It is sufficient to check the inequality

\[ |(u + i\bar{u})(a_I)| \equiv C|I|^{-2/3}. \]

Choose n so that \( l_n \leq |I| < l_{n-1} \). We consider two cases.

- If \( |I| < \frac{1}{a} (l_{n-1} l_n)^{1/2} \), then, by (4.2), \( H_{1/2}(E \cap I) \equiv H_{1/2}(I^{(o)}) = l_n^{1/2} \), and

\[ |I| \leq l_n^{1/2a}. \]

We estimate

\[ |(u + i\bar{u})(a_I)| \equiv C \sum_{k \leq 0} \int \frac{d\mu_k(\xi)}{\xi - a_I}. \]

Observe that

\[ \sum_{k < n} \equiv C \sum_{k < n} l_k^{(1/8) - (3/4)} \equiv C|I|^{-a/4}, \]

\[ \sum_{k > n} \equiv \frac{1}{|I|} \sum_{k > n} \|\mu_k\| \equiv l_n^{-1} \sum_{k > n} l_k^{1/8} \equiv C. \]

Finally, by (4.2)

\[ \int \frac{d\mu_k(\xi)}{\xi - a_I} \equiv C l_n^{5/8} \left[ l_n^{-5/4} + (l_{n-1} l_n)^{-1/2} \left( 1 + \frac{1}{2} + \ldots + \frac{1}{N_n} \right) \right] \]

\[ \equiv C l_n^{-1/8} \equiv C|I|^{-a/4}. \]

If \( |I| > \frac{1}{a} (l_{n-1} l_n)^{1/2} \) then, by (4.2), I meets at most \( 1 + 2|I|(l_{n-1} l_n)^{-1/3} \) intervals \( I^{(o)} \). Hence

\[ H_{1/2}(E \cap I) \equiv C|I|(l_{n-1} l_n)^{-1/2} l_n^{1/2} = C|I| |l_{n-1}^{1/2} \]

and

\[ |I| \geq c l_n^{1/2(1-s)}. \]

This case is further analysed along similar lines. \( \square \)

**Corollary.** If I is a subarc of T then

\[ (4.10) \quad H_{1/2}(E \cap I) \equiv C|I||f'(a_I)|^{4/9}. \]

**Proof of Corollary.** If \( H_{1/2}(E \cap I) \equiv |I|^{8/9} \) then

\[ [H_{1/2}(E \cap I)]^{9/4} \equiv |I|^2 \equiv |I||f'(a_I)|. \]

If \( H_{1/2}(E \cap I) \equiv C|f'(a_I)|^4 \) then

\[ [H_{1/2}(E \cap I)]^{9/4} \equiv |I||H_{1/8}(E \cap I)|^{1/4} \equiv C|I||f'(a_I)|. \] \( \square \)
Proof of (4.6). Let \( F \subset T \). Since the derivative \( f' \) has nonzero radial limits on the complement to the support of the measure \( \mu \) and since the set \( \text{supp} \mu \setminus E \) is countable,
\[
H_{1/2}(F \setminus E) > 0 \Rightarrow D_{1/2}(F \setminus E) > 0;
\]
see the proof of Theorem 0.5 in Section 2.6. Therefore, for the validity of (4.6) it is sufficient to show that
\[
F \subset E, \quad D_{4/9}(F) = 0 \Rightarrow H_{1/2}(F) = 0.
\]
If \( D_{4/9}(F) = 0 \), then, for any \( \varepsilon > 0 \), there exists a covering of \( F \) with intervals \( I_v \) such that
\[
\sum |I_v||f'(a_v)|^{1/8} < \varepsilon.
\]
By (4.10) we have
\[
H_{1/2}(F) \leq \sum [H_{1/2}(E \cap I_v)] < C\varepsilon.
\]
This concludes the proof of the theorem. \( \Box \)

Proof of (4.9). Observe that
\[
u(a_{I(n)}) \geq c|n|^{(3/8)-(3/4)} = c1 \cdot \delta_n,
\]
which implies
\[
|f'(a_{I(n)})| \preceq C|n|^{1/8}.
\]
If \( q > \frac{4}{9} \), then
\[
\sum [|I^{(n)}||f'(a_{I(n)})|^p \preceq N|n|^{9/8q} = |n|^{(9/8q-1/2)} \to 0 \quad \text{as} \quad n \to \infty,
\]
the summation being carried out over all intervals \( I^{(n)} \). Since
\[
\sum |I^{(n)}|^{1/2} = 1,
\]
the infimum in (2.15) is zero for \( p = \frac{1}{2} \) and \( C = 1 \). Now (4.9) follows from Proposition 2.5. \( \Box \)

4.3. Second example. We fix (4.11)
\[
p = (\log 9)^{-1} \log 2
\]
until the end of the section. Let \( E_0 \) be the standard ternary Cantor set on \( T \). In the construction of \( E_0 \), \( 2^n \) closed intervals \( I_0^{(0)} \) of length \( 3^{-n} \) arise at the \( n \)-th step. Let \( I^{(n)} \) denote the interval of length \( 9^{-n} \) concentric with \( I_0^{(n)} \), and let \( E^{(n)} \) be the union of all such \( I^{(n)} \). It is easy to see that for all \( n \)
\[
H_p(E^{(n)}) \geq c > 0.
\]
Let \( \mu_n \) be the measure supported on \( E^{(n)} \) with constant density such that \( \int d\mu_n = n^{-2} \), and let \( \mu = \sum \mu_n \) be a probability measure. We shall consider the conformal mapping \( f \) defined by the formulae (4.4) and (4.5).
Theorem. 1) If \( x > 1 - (\log 3)^{-1} \log 2 \) then

\[
(4.13) \quad \lim \inf_{r \to 1^-} \frac{|f'(r\zeta)|}{(1-r)^2} > 0, \quad \zeta \in \mathbb{D}.
\]

2) If \( q > \left( \log \frac{81}{2} \right)^{-1} \log 2 \), then

\[
(4.14) \quad D_q(E^{(n)}) \to 0, \quad n \to \infty.
\]

Corollary. If

\[
\left( \log \frac{81}{2} \right)^{-1} \log 2 < q < \left( \log \frac{81}{4} \right)^{-1} \log 2
\]

then

\[
h_p^f \ll H_q, \quad h_p^f \ll H_q.
\]

Proof of Corollary. The absolute continuity follows from Theorem 0.5. The lack of strong absolute continuity follows from Theorem 0.9 and the fact that (4.12) and (4.14) imply \( H_p \ll D_q \). \( \square \)

Remark. One can easily obtain similar results for any \( p < 1 \). We omit the details.

Proof of Theorem. 1) Obviously, the inequality (4.13) requires a proof only for \( \zeta \) in \( E_0 \). Let \( \zeta \in E_0 \) and

\[1-r = 3^{-k}.\]

We estimate

\[|(u+i\eta)(r\zeta)| \leq C \sum_{n \geq 1} \int \frac{d\mu_n(\eta)}{|\eta-r\zeta|}.\]

To this end we show that

\[
(4.15) \quad \int \frac{d\mu_n(\eta)}{|\eta-r\zeta|} \equiv C n^{-3} k(3/2)^{k}.
\]

Then

\[|(u+i\eta)(r\zeta)| \leq C (1-r)^{-x}\]

for \( x = 1 - \frac{\log 2}{\log 3} \) and (4.13) follows.

To prove (4.15) consider two cases.

Suppose that \( n \leq k \). Taking into account that dist \( (\zeta, \text{supp} \mu_n) \leq 3^{-(n+1)} \) and that the distance between any two intervals \( I^{(n)} \) is at least \( 3^{-(n+1)} \), we have

\[
\int \frac{d\mu_n(\eta)}{|\eta-r\zeta|} \equiv C n^{-2} 2^{-n} 3^{n+1} \left( 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2^n} \right) \equiv C n^{-2} n(3/2)^n \equiv C n^k k(3/2)^k
\]

Suppose that \( n > k \). If \( I \) is an interval of length \( 3^{-k} \), then it meets at most two intervals \( I_k^{(n)} \). Therefore

\[\mu_n(I) \leq n^{-2} 2^{-k}.
\]
Hence
\[ \int \frac{d\mu_\eta(\eta)}{|\eta - r\zeta|} \leq n^{-2} 2^{-k} 3^k \left(1 + \frac{1}{2} + \ldots + \frac{1}{3^k}\right) \leq Cn^{-2} k(3/2)^k. \]

2) The set \( E^{(n)} \) is covered by \( 2^n \) intervals \( I^{(n)} \). Observe that
\[ u(a^{(n)}) \approx n^{-2} 2^{-n/2}. \]
Therefore
\[ D_q(E^{(n)}) \leq 2^n [9^{-n/4} 2^{n/2} 9^{-n}] \]
and (4.14) follows. \( \Box \)

5. Boundary distortion in dimensions close to one

In this section we prove Theorem 0.4 which asserts, in particular, that
\[ p - d(p) \approx (1 - p)^{1/2} \quad \text{as} \quad p \to 1-. \]
First we derive the lower estimate of \( d(p) \).

5.1. Lemma. Let \( \gamma, \sigma \) and \( \alpha \) be positive numbers. If a univalent function \( f \) satisfies the condition
\[ \int |f'(r\zeta)|^{-\gamma} |d\zeta| = O((1-r)^{-\sigma}), \quad r \to 1-, \]
then
\[ \dim \left\{ \zeta \in T: \liminf_{r \to 1-} \frac{|f'(r\zeta)|}{(1-r)^{\sigma}} < \infty \right\} \leq \max \{0, 1 + \sigma - \alpha \gamma\}. \]

Proof. For any \( M > 0 \) and \( v \in \mathbb{N} \) we define
\[ E_v(M) = \{ \zeta \in T: |f'(1-e^{-v}\zeta)| \leq Me^{-av}\}; \]
\[ E(M) = \{ \zeta \in T: \liminf_{r \to 1-} \frac{|f'(r\zeta)|}{(1-r)^{\sigma}} < M \}. \]
By the distortion theorem
\[ E(2M) \subset \cap_{n \geq 1} \cup_{v \geq n} E_v(M). \]
We have to estimate the Hausdorff dimension of the set
\[ E = \bigcup_{M > 0} E(M). \]
Assume that \( p > 0 \) and \( H_p(E) > 0 \). Then
\[ H_p(E(2M)) > 0 \]
for some \( M > 0 \). Fixing this value of \( M \) we will write, for brevity, \( E_v \) instead of \( E_v(M) \). By (5.3),
\[ \sum_{v \geq 1} H_p(E_v) = \infty. \]
Hence there exists an increasing sequence \( \{v_j\} \) of positive integers such that for any \( j \),
\[
H_p(E_{v_j}) \equiv [v_j]^{-2}.
\]
Now we fix \( v=v_j \). By \( G \) we denote the \( e^{-v} \)-neighbourhood of \( E_v \). The set \( G \) consists of disjoint open intervals of length at least \( e^{-v} \). Subdivide \( G \) into a union of \( N \) disjoint intervals, not necessarily open, of length between \( \frac{1}{2} e^{-v} \) and \( e^{-v} \). Then
\[
Ne^{-\nu} \equiv H_p(G) \equiv H_p(E_v) > v^{-2}
\]
and
\[
|G| \geq \frac{1}{2} e^{-v} N \geq \frac{1}{2} e^{-v} e^{\nu} v^{-3}.
\]
Observe now that if \( r=1-e^{-v} \) and \( \zeta \in G \), then
\[
|f'(r \zeta)| \leq CMe^{-v}. \tag{5.4}
\]
Combined with (5.1) and (5.4), this yields
\[
Ce^\sigma \equiv \int_G |f'(r \zeta)|^{-\gamma} |d \zeta| \equiv ce^{-v} e^{\nu} v^{-3} e^{\nu} v^{-\gamma}.
\]
For large \( v \), the last estimate is possible only if
\[
\sigma = -1+p+\alpha \gamma,
\]
so (5.2) follows. \( \Box \)

The application of the lemma is based on a recent result due to Ch. Pommerenke [31] which improves an earlier estimate of J. Clunie and Ch. Pommerenke [11].

**Pommerenke Theorem.** Let \( f \) be a function univalent in \( D \) and \( \lambda \in \mathbb{R} \). Then
\[
\int |f'(r \zeta)|^\gamma |d \zeta| = o((1-r)^{-\sigma}), \quad r \to 1-,
\]
for any \( \sigma \) satisfying
\[
\sigma > -\frac{1}{2} + \lambda + \left(\frac{1}{2} - \lambda + 4\lambda^2\right)^{1/2}. \tag{5.5}
\]
Notice that the expression on the right is \( \sim 3\lambda^2 \) as \( \lambda \to 0 \) and it does not exceed \( 3\lambda^2 \) provided \( \lambda < 0 \). Thus, for any \( \gamma > 0 \),
\[
\int |f'(r \zeta)|^{-\gamma} |d \zeta| = o((1-r)^{-\gamma}). \tag{5.6}
\]

**Corollary 1.** If \( f \) is a univalent function and \( 0 < \alpha \leq 1 \) then
\[
\dim \left\{ \zeta \in T : \liminf_{r \to 1-} \frac{|f'(r \zeta)|}{(1-r)^{2}} < \infty \right\} \leq 1 - \frac{\alpha^2}{12}.
\]

**Proof.** By (5.6) and the established lemma, for any \( \gamma > 0 \) we have
\[
\dim \{ \ldots \} \leq 1 + 3\gamma^2 - \alpha \gamma.
\]
The minimum of the right-hand side is obtained by the choice \( y = \frac{1}{6} \alpha \) and is equal to \( 1 - \frac{1}{12} \alpha^2 \). □

**Corollary 2.**

\[
 d(p) = \frac{p}{1 + \sqrt{12}(1-p)^{1/2}}.
\]

**Proof.** By the previous corollary we have

\[
 \lim_{r \to 1-} \inf_\zeta \frac{|f'(r\zeta)|}{(1-r)^2} > 0 \quad \text{for } A_p\text{-a.e. } \zeta \in T
\]

for any \( p \in (0, 1) \) and any \( \alpha > \sqrt{12}(1-p)^{1/2} \). It remains to apply Theorem 0.5. □

**Remark 1.** By Corollary 2, for any \( p \in (0, 1) \),

\[
 p - d(p) = \frac{\sqrt{12}p(1-p)^{1/2}}{1 + \sqrt{12}(1-p)^{1/2}} \leq \sqrt{12}(1-p)^{1/2}.
\]

This yields the right-hand inequality in Theorem 0.4.

**Remark 2.** Also by Corollary 2 we obtain the inequality \( d(p) > \frac{p}{2} \) for

\[
 p \in \left(\frac{11}{12}, 1\right).
\]

This result could easily be improved to

\[
 p \in \left(\frac{3}{4}, 1\right]
\]

by referring to (5.5) in place of (5.6). Moreover, by a similar argument one can obtain the bound

\[
 p \in (0.601, 1]
\]

if one applies the second theorem of Ch. Pommerenke in [31]: if \( f \) is a univalent function then

\[
 \int |f'(r\zeta)|^{-1}|d\zeta| = o((1-r)^{-0.801}), \quad r \to 1-.
\]

On the other hand, I cannot extend this method to prove \( d(p) > p/2 \) for all \( p \in (0, 1) \). See the next section for a proof based on the method due to L. Carleson.

**5.2.** Now we turn to the proof of the upper estimate of \( d(p) \) for \( p \) close to one. To obtain an upper bound of \( d(p) \) one should provide an example. We construct the corresponding example with the help of the lacunary power series

\[
 b(z) = \sum_{\nu \geq 0} z^{\nu^2}, \quad z \in \mathbb{D},
\]
which gives rise to the univalent function

\[ f(z) = \int_0^z \exp \left\{ \frac{i}{5} b(z) \right\} dz, \quad z \in \mathbb{D}, \]

mapping \( \mathbb{D} \) onto a Jordan domain (see, e.g., [30], Ch. 10, § 2).

**Theorem.** Let \( f \) be a function defined by (5.8) and (5.7), and let \( p \in \left( \frac{19}{20}, 1 \right) \) and 
\[ \alpha = \frac{1}{25} (1 - p)^{1/2}. \]
There exists a subset \( E \subset \mathbb{T} \) of positive \( \Lambda_p \)-measure such that

\[ \lim_{r \to 1^-} \frac{|f'(r \zeta)|}{(1-r)^2} = 0, \quad \zeta \in E. \]

**Corollary.** If \( p > \frac{19}{20} \) then

\[ d(p) = \frac{p}{1 + \frac{1}{25} (1-p)^{1/2}}. \]

**Proof of Corollary.** Apply Theorem 0.6. \( \Box \)

**Remark.** If \( p \in \left( \frac{19}{20}, 1 \right) \) then, by (5.10),

\[ p - d(p) = \frac{p}{1 + \frac{1}{25} \sqrt{1-p}} = \frac{1}{30} (1-p)^{1/2}, \]

which proves the left-hand inequality in Theorem 0.4.

For the proof of the theorem, we need a result on the boundary behaviour of the lacunary series (5.7) which was established in [24] as a slight amplification of a theorem due to J. Hawkes [16]. For the sake of completeness the proof is included.

**Lemma.** Let \( \delta < \frac{1}{10} \). There exists a subset \( E \subset \mathbb{T} \) of Hausdorff dimension greater than \( 1 - 5\delta^2 \) such that

\[ \liminf_{r \to 1^-} \frac{\text{Im} b(r \zeta)}{\log \frac{1}{1-r}} > \frac{\delta}{3}, \quad \zeta \in E. \]

**Proof of lemma.** On the segment \([0, 1]\) we define the functions

\[ S_n(t) = \sum_{\nu=1}^{n} \sin (2^\nu \cdot 2\pi t). \]
It is easy to verify that if \( n = \lceil \log_2 (1 - r) \rceil \) then
\[
| \text{Im} \left( b(r e^{i n \theta}) - S_n(i) \right) | \leq C.
\]

On \([0, 1]\) we also consider the probability measure \( \mu \) with respect to which the functions \( t \mapsto t_v \) (\( = \) the \( v \)-th figure in the dyadic expansion of \( t \)) are independent random variables with distribution
\[
\mu \{ t : t_v = 0 \} = \frac{1}{2} + \delta, \quad \mu \{ t : t_v = 1 \} = \frac{1}{2} - \delta.
\]

The measure \( \mu \) is invariant under the dyadic transformation
\[
T(t) = 2t \pmod{1}
\]
and ergodic with respect to it (see [5], Example 3.5). By the ergodic theorem, for \( \mu \)-a.e.
\[
\frac{1}{n} S_n(t) = \frac{1}{n} \sum_{v=1}^{n} \sin(2\pi T^v t) - \int_0^1 \sin(2\pi t) d\mu(t).
\]

By the Eagleston—Billingsley theorem ([5], §14), the measure \( \mu \) is absolutely continuous with respect to the Hausdorff measure \( A_\delta \) provided that
\[
\alpha \leq \frac{\text{Ent} \, T}{\log 2},
\]
where
\[
\text{Ent} \, T = \left( \frac{1}{2} + \delta \right) \left| \log \left( \frac{1}{2} + \delta \right) \right| + \left( \frac{1}{2} - \delta \right) \left| \log \left( \frac{1}{2} - \delta \right) \right|
\]
is the entropy of \( T \). It remains only to note that
\[
1 - \frac{\text{Ent} \, T}{\log 2} < 5\delta^a
\]
provided \( \delta < \frac{1}{10} \), and that
\[
\int_0^1 \sin(2\pi t) d\mu(t) = 2\delta \int_0^{\pi/2} \sin(2\pi t) d\mu(t)
\]
\[
> 2\delta \frac{\sqrt{2}}{2} \mu \left[ \frac{1}{8}, \frac{3}{8} \right] = \frac{\sqrt{2} \delta}{4} (1 + 2\delta)^2 (1 - 2\delta) > \frac{\delta}{3}. \quad \Box
\]

**Proof of theorem.** Fix \( p \in \left[ \frac{19}{20}, 1 \right] \) and define \( \delta = \frac{1}{\sqrt{5}} (1 - p)^{1/3} \). Since \( \delta < \frac{1}{10} \)
we can apply the lemma. Because of the identity
\[
|f'(z)| = \exp \left\{ -\frac{1}{5} \text{Im} \, b(z) \right\},
\]
we have
\[ |f'(r\zeta)| \leq (1-r)^{\delta/(15 \log 2)} \]
for all \( \zeta \) in the set \( E \) obtained in the lemma and all \( r \) sufficiently close to one. Hence (5.8) is valid for all \( \alpha < \delta(15 \log 2)^{-1} \), in particular, for
\[ \alpha = \frac{1}{25} (1-p)^{1/2}. \]

6. A version of a theorem by Carleson

This section is devoted to the proof of the inequality
\[(6.1) \quad d(p) > \frac{p}{2}, \quad p \in (0, 1].\]
In fact, we shall establish two stronger results, any of which implies (6.1).

6.1. Theorem. For any \( p \in (0, 1] \) there is a number \( q > \frac{p}{2} \) such that if \( f \) is a conformal mapping of the unit disc onto a Jordan domain, then
\[(6.2) \quad h_f^p < H_q.\]

6.2. Theorem. For any \( p \in (0, 1] \) there is a number \( \alpha < 1 \) such that if \( f \) is a conformal mapping of the unit disc onto a Jordan domain, then the inequality
\[(6.3) \quad \lim \inf_{r \downarrow 1} \frac{|f'(r\zeta)|}{(1-r)^{\delta}} > 0 \]
holds for all \( \zeta \in \mathcal{T} \) outside an exceptional set of \( \Lambda_p \)-measure zero.

Both assertions are consequences of the following basic result essentially due to Lennart Carleson.

6.3. Theorem. Let \( p \in (0, 1] \). For any \( \varepsilon > 0 \) there is a \( \delta > 0 \) satisfying the following. If \( f \) is a conformal mapping onto a Jordan domain, then there exists a positive number \( r_0 = r_0(\varepsilon, f) \) such that for any \( r, \ 0 < r < r_0 \), the maximal number of discs \( \Delta \) of radius less than \( r \) with
\[ h_f^p(\Delta) \leq [H_p(\Delta)]^{1/2 + \delta} \]
and with centers separated by \( 2r \) does not exceed \( r^{-\varepsilon} \).

First we derive Theorems 6.1 and 6.2 from the latter result.
6.4. **Proof of Theorem 6.1.** Fix $p \in (0, 1]$. Define $\varepsilon = p/4$ and let $\delta$ be a corresponding number in Theorem 6.3. We shall establish (6.2) with

$$q = \frac{p}{2} (1 + \delta).$$

Let $E \subset T$ and $H_q(f_E) < \eta$ where $\eta$ is a sufficiently small number. There exists a covering of $f_E$ by discs $A_j$ of radii $r_j$ satisfying

$$\sum_j r_j < \eta, \quad \max_j r_j < 2^{-N} < r_0 = r_0(\varepsilon, f),$$

where $N \in \mathbb{N}$ and $N = N(\eta) \to \infty$ as $\eta \to 0$. Introduce the notation:

$$J_0 = \{j: h_p(A_j) < r_j\},$$

$$J(r) = \{j: h_p(A_j) \geq r_j, \frac{r}{2} < r_j \leq r\}, \quad r > 0.$$

Then

$$H_p(E) \equiv \sum_{j \in J_0} h_p(A_j) + \sum_{k \geq N} h_p(\bigcup_{j \in J(2^{-k})} A_j).$$

By (6.4), the first sum does not exceed $\eta$. To estimate the second, observe that if $0 < r < r_0$ then

$$h_p(\bigcup_{j \in J(\varepsilon)} A_j) \equiv C r^{p/4}.$$

To this end, we apply the covering lemma to the collection of discs

$$\{2A_j: j \in J(r)\}$$

and choose $m$ disjoint discs $2A_{(v)}$, $1 \leq v \leq m$, such that

$$\bigcup_{j \in J(\varepsilon)} A_j \subset \bigcup_{v=1}^m 10A_{(v)}.$$

(For a disc $A$ and $k > 0$, by $kA$ we denote the concentric disc of radius $k$ times the radius of $A$.) Since the centers of $A_{(v)}$ are separated by $2r$ and

$$h_p(A_{(v)}) \equiv 2^{-q} r^q \equiv r^{p(1/2 + \delta)},$$

we have by Theorem 6.3

$$m \equiv r^{-p/4}.$$

Hence

$$h_p(\bigcup_{j \in J(\varepsilon)} A_j) \equiv \sum_{v=1}^m h_p(10A_{(v)}) \equiv C r^{p/4} r^{-p/4} = C r^{p/4}$$

(in the last inequality we have applied (1.5) and (0.6)). Returning to (6.5), we have

$$H_p(E) \equiv \eta + C \sum_{k \in \mathbb{N}} (2^{-k})^{p/4} \to 0 \quad \text{as} \quad \eta \to 0. \quad \square$$
Remark. Our deduction of Theorem 6.1 from Theorem 6.3 coincides essentially with the corresponding part of Carleson’s proof ([9], § 8).

6.5. Proof of Theorem 6.2. Let \( p \in (0, 1], \varepsilon = \frac{1}{4} p, \) and \( \delta \) be the corresponding number in Theorem 6.3. The assertion will be proved with

\[
\alpha = \frac{1 - 2\delta}{1 + 2\delta}.
\]

Assume that there is a compact subset \( E \subset T \) of positive \( \Lambda_p \)-measure such that (6.3) is false everywhere on \( E \). For an arbitrary \( N \in \mathbb{N} \) we carry out the following construction. Let \( \zeta \in E \). Then, by assumption, there exists an interval \( I_\zeta \) with center at \( \zeta \) such that

\[
|I_\zeta| \leq 2^{-N}, \quad |f'(a_{I_\zeta})| \leq |I_\zeta|^\alpha.
\]

Applying the covering lemma, we choose a finite subcovering \( \{I_{\zeta_j}\} \) of \( E \) of multiplicity at most two. In the sequel we write \( I_j \) instead of \( I_{\zeta_j} \). By Proposition 1.3, applied with a sufficiently large \( R \), for any \( j \) there is a closed subset \( F_j \subset I_j \) satisfying

\[
H_p(F_j) \equiv \frac{1}{2} |I_j|^p,
\]

where

\[
fF_j \subset A_j \overset{\text{def}}{=} A(f(a_j), r_j),
\]

(6.6)

(6.7)

\[
r_j = R |I_j| |f'(a_j)| \sim R |I_j|^{1+\alpha}.
\]

Define

\[
F^{(N)} = \bigcup F_j.
\]

As in the proof of Theorem 2.4, (6.6) implies

\[
H_p(F^{(N)}) > cH_p(E).
\]

On the other hand the inclusion \( fF^{(N)} \subset \bigcup A_j \) implies the inequality

(6.8)

From (6.6) and (6.7) we have successively

\[
h_p(A_j) \equiv H_p(F_j) \equiv \frac{1}{2} |I_j|^p \equiv \frac{1}{2} (R^{-1} r_j)^{p/1+\alpha} \equiv r_j^q
\]

where

\[
q = \frac{p}{2} \left( \frac{1}{2} + \delta \right).
\]

Similarly to the proof of Theorem 6.1, the latter implies that the expression on the right in (6.8) tends to zero as \( N \to \infty \). This contradicts the assumption \( H_p(E) > 0 \).
Remark. The idea to apply Carleson’s technique to questions concerning the derivative of a conformal mapping is due to J. Brennan [7].

6.6. The beginning of the proof of Theorem 6.3. Fix $p \in (0, 1]$ and $\varepsilon > 0$. Let $A$ be the least positive number greater than $20 \varepsilon^{-1}$ with $\exp A \in \mathbb{N}$. We shall verify the assertion for

$$\delta = \tau A^{-2}$$

where $\tau = \tau(A)$ is the number appearing in the following lemma borrowed from Carleson’s proof ([9], § 6).

Lemma. For any $A > 0$ there is $\tau > 0$ satisfying the following. Let $U$ be a Jordan domain with two distinguished boundary arcs lying on different components of $\partial R(0; 1, e^A)$. Suppose that

$$\lambda(\Gamma) < (2\pi)^{-1} A + \tau$$

where $\Gamma$ is the family of all curves joining these arcs in $U$. Then $U$ contains a sector of the annulus $R(0; 1, e^A)$ with central angle greater than $7\pi/4$.

Carleson’s elegant proof is based on a normal families argument. We shall not reproduce it here.

Now we fix a conformal mapping $f$. For convenience, we assume that $f$ maps $D$ onto the exterior of a Jordan domain. Denote $\Omega = f(D)$. Without loss of generality, we may also assume that $f(0) = \infty$ and

$$\text{diam } \partial \Omega = \frac{1}{4}.$$ 

Fix a number $r \equiv r_0$ with $r_0$ small. In the course of the proof we shall obtain a finite number of restrictions on the magnitude of $r_0$.

Definition. Let $z_0 \in \mathbb{C}$ and $v \in \mathbb{N}$. The pair $(z_0, v)$ is said to be exceptional if $\Omega$ contains no sector of the annulus $R(z_0; re^{vA}, re^{(v+1)A})$ of apperture $7\pi/4$. If $z_0 \in \mathbb{C}$, $n(z_0)$ denotes the number of $v$’s for which the pair $(z_0, v)$ is exceptional.

6.7. Lemma. If

$$h_p'(A(z_0, r)) \equiv r^{p(1/2 + \delta)}$$

then

$$n(z_0) \equiv A^{-2} \log \frac{1}{r}.$$
Proof. By (6.10) we can assume that 
\[ \partial \Omega \subset A (0, \frac{1}{\delta}) . \]
Suppose that (6.11) is valid. We apply Corollary to Theorem 1.4 with \( K = T, \ q = \frac{1}{2} + \delta \) and \( q' = \frac{1}{2} + 2\delta \). If \( r_0 \) is small, then there exists a subarc \( \sigma_0 \) of \( \partial A (z_0, 2r) \) which is a crosscut of \( \Omega \) and satisfies 
\[ \lambda (\Gamma) \equiv \pi^{-1} \left( \frac{1}{2} + 2\delta \right) \log \frac{1}{r} \]
where \( \Gamma \) is the family of all curves in \( \Omega \) joining \( T \) with \( \sigma_0 \). Let 
\[ N = \left[ \frac{1}{A} \log \frac{1}{r} \right] . \]
On each circle \( \partial A (z_0, re^{\pi i}) \), \( v = 1, \ldots, N \), we choose a subarc \( \sigma_v \), which is a crosscut of \( \Omega \), in such a way that for \( v = 0, \ldots, N - 1 \), \( \sigma_v + 1 \) separates \( \sigma_v \) from infinity. Let \( \Gamma_v \) denote the family of all curves in \( \Omega \) joining \( \sigma_v \) with \( \sigma_v + 1 \). By properties of extremal length, 
\[ \lambda (\Gamma) \equiv \sum \lambda (\Gamma_v) . \]
For every \( \nu \) we have the trivial estimate 
\[ \lambda (\Gamma_v) \equiv (2\pi)^{-1} A , \]
whereas, by Lemma in Section 6.6, 
\[ \lambda (\Gamma_v) \equiv (2\pi)^{-1} A + \tau \]
provided that \((z_0, \nu)\) is an exceptional pair. Taking these estimates into account, from (6.12), (6.13) and (6.14), we have 
\[ \frac{1}{\pi} \left( \frac{1}{2} + 2\delta \right) \log \frac{1}{r} \geq [N - n(z_0)] \frac{A}{2\pi} + n(z_0) \left( \frac{A}{2\pi} + \tau \right) \]
\[ = \frac{NA}{2\pi} + n(z_0) \tau \geq \frac{1}{2\pi} \log \frac{1}{r} + n(z_0) \tau . \]
Hence 
\[ \delta \log \frac{1}{r} \geq n(z_0) \tau , \]
and by (6.8) 
\[ n(z_0) \equiv \frac{\delta}{\tau} \log \frac{1}{r} = A^{-2} \log \frac{1}{r} . \]

6.8. End of the proof of Theorem 6.3. The argument used at the same stage in [9] (see § 10) applies verbatim. We shall therefore only supply few additional elucidations.
Suppose there are $m$ discs $A_j = A(z_j, r_j)$, $1 \leq j \leq m$, satisfying $r_j \leq r \leq r_0$.

(6.15) \[ |z_j - z_k| \geq 2r \ (j \neq k), \quad h_p(A_j) \geq r_j^{(1/2 + \delta)}. \]

We have to prove that

(6.16) \[ m \geq r^{-e}. \]

Without loss of generality we can assume that

(6.17) \[ z_j \in \partial \Omega, \quad j = 1, \ldots, m; \]

and $A$ is large enough to satisfy the following geometrical condition.

Let $a, b, z$ be three points in a sector of the annulus $R(0; 1, e^A)$ with central angle $\frac{\pi}{4}$ and $|a| = 1$, $|b| = e^A$. Then the angle $<(azb)$ is greater than $\frac{\pi}{4}$ (See Figure 3.)

**Fig. 3**

**Lemma 1.** Let

(6.18) \[ 2 \leq \frac{|z_j - z_k|}{re^{\nu A}} \leq e^A - 1 \]

and

(6.19) \[ re^{(\nu + 1)A} < \frac{1}{2} \text{diam } \partial \Omega. \]

Then at least one of the two pairs $(z_j, v)$ and $(z_k, v)$ is exceptional.

**Proof.** Suppose that the pair $(z_j, v)$ is not exceptional. By hypothesis, $\partial \Omega$ meets both components of $C \setminus R$, $R = R(z_j; e^{\nu A}, e^{(\nu + 1)A})$ so the curve $\partial \Omega$ should pass inside a sector of $R$ of aperture $\frac{\pi}{4}$. Let $a$ and $b$ be two points of $\partial \Omega$ lying on different circumferences of $\partial R$. By (6.17) and (6.18), $z_k$ lies in the sector so we have

\[ <(az_k b) > \frac{\pi}{4}. \]

Hence the pair $(z_k, v)$ is exceptional. \qed
For \( v = 0, 1, \ldots, N \),

\[
N = 1 + \left( A^{-1} \log \frac{1}{r} \right),
\]

let \( \mathcal{G}_v \) denote the lattice of all squares \( Q_v \) of side \( s_v = \frac{1}{2} re^{vA} \) such that the coordinates of the vertices are multiples of \( s_v \). Since \( e^A \) is an integer, \( \mathcal{G}_{v+1} \) is a sublattice of \( \mathcal{G}_v \).

We will assume that no point \( z_j \) lies on the boundary of a square \( Q_0 \in \mathcal{G}_0 \). By (6.15) each \( Q_0 \) contains at most one point \( z_j \). By (6.10) we can also assume that all \( z_j \) lie in a single square \( Q_N \in \mathcal{G}_N \).

Given \( Q_{v+1} \in \mathcal{G}_{v+1} \) we define a set \( S(Q_{v+1}) \) consisting of some squares \( Q_v \subset Q_{v+1} \) as follows

**Definition.** If, for all \( z_j \in Q_{v+1} \), the pair \((z_j, v)\) is exceptional, we put \( S(Q_{v+1}) = \emptyset \). Otherwise,

\[
S(Q_{v+1}) = \{ Q_v \in \mathcal{G}_v : Q_v \subset Q_{v+1} \text{ and } Q_v \cap A(z_j, 2re^{vA}) \neq \emptyset \text{ for any } z_j \in Q_{v+1} \text{ such that } (z_j, v) \text{ is non-exceptional} \}.
\]

Since a disc of radius \( 2re^{vA} = 4s_v \) meets at most 100 squares of \( \mathcal{G}_v \), we have

\[
(6.20) \quad \text{card } S(Q_{v+1}) \leq 100.
\]

**Lemma 2.** Let \( v < N - 1 \) and \( z_k \in Q_v \subset Q_{v+1} \). If the pair \((z_k, v)\) is non-exceptional, then

\[
Q_v \subset S(Q_{v+1}).
\]

**Proof.** Let \( z_j \in Q_{v+1} \) and let the pair \((z_j, v)\) be non-exceptional. We should verify that

\[
(6.21) \quad Q_v \cap A(z_j, 2re^{vA}) \neq \emptyset.
\]

Since (6.10) and the inequality \( v < N - 1 \) imply (6.19), we can make use of Lemma 1. We thus have either

\[
|z_j - z_k| < 2re^{vA},
\]

or

\[
|z_j - z_k| > re^{(v+1)A} - re^{vA}.
\]

The first inequality implies (6.21) while the second is ruled out because \( z_j \) and \( z_k \) lie in the same square \( Q_{v+1} \). \( \Box \)

Each point \( z_j \) defines (and is determined by) a sequence of squares

\[
(6.22) \quad Q_0 \subset Q_1 \subset \ldots \subset Q_N
\]

with \( z_j \in Q_v \). According to Lemma 2

\[
Q_v \subset S(Q_{v+1})
\]
for all $v<N-1$ for which the pair $(z_j, v)$ is non-exceptional. By Lemma 6.7 the number of exceptional pairs does not exceed

$$A^{-2} \log \frac{1}{r} \leq \frac{N}{A}.$$ 

Thus $m$ does not exceed the number of all possible sequences (6.22) with $Q_v \subset S(Q_{v+1})$ except for at most $\left(\frac{N}{A} + 1\right)$ indices $v$. Therefore by (6.20),

$$m \leq \sum_{k=0}^{[\frac{N}{A}-1]+1} \binom{N}{k} e^{2Ak} 100^{N-k} \leq \exp \left\{ 2A \left( \frac{N}{A} + 1 \right) \right\} \sum_{k=0}^{N} \binom{N}{k} 100^{N-k}$$

$$= \exp \left\{ 2N + N \log 101 + 2A \right\} \leq \left( \frac{1}{r} \right)^{20A-1}$$

provided that $r \leq r_0$ and $r_0$ is sufficiently small. By the choice of $A$, we have (6.16). □

7. Concluding remarks

7.1. Radial growth of $f'$ and $1/f'$. Most of the results on boundary distortion obtained in this paper are consequences of the corresponding results on the radial growth of the reciprocal of the derivative of a univalent function. Now we wish to list the latter results explicitly and briefly discuss their counterparts for the derivative itself. The problem is to estimate the maximal dimension of the set on which the order of growth of the derivative (or its reciprocal) is greater than the given one. This problem admits different versions. As variable we prefer to choose the dimension of the exceptional set. For $p \in (0, 1]$ we define the following quantities. ($\mathcal{S}$ denotes the usual class of univalent functions.)

$$\alpha(p) = \sup \left\{ \alpha \geq 0 : \exists f \in \mathcal{S} \quad \forall E \subset \mathcal{T} \quad \text{such that} \quad H_p(E) > 0 \quad \text{and} \quad \liminf_{r \to 1-} \frac{|f'(r\zeta)|}{(1-r)^\alpha} = 0 \quad \text{for} \quad \zeta \in E \right\};$$

$$\alpha_1(p) = \sup \left\{ \alpha \geq 0 : \exists f \in \mathcal{S} \quad \forall E \subset \mathcal{T} \quad \text{such that} \quad H_p(E) > 0 \quad \text{and} \quad \lim_{r \to 1-} \frac{|f'(r\zeta)|}{(1-r)^\alpha} = 0 \quad \text{for} \quad \zeta \in E \right\};$$

$$\beta(p) = \sup \left\{ \beta \geq 0 : \exists f \in \mathcal{S} \quad \forall E \subset \mathcal{T} \quad \text{such that} \quad H_p(E) > 0 \quad \text{and} \quad \limsup_{r \to 1-} |f'(r\zeta)| (1-r)^\beta = \infty \quad \text{for} \quad \zeta \in E \right\};$$

$$\beta_1(p) = \sup \left\{ \beta \geq 0 : \exists f \in \mathcal{S} \quad \forall E \subset \mathcal{T} \quad \text{such that} \quad H_p(E) > 0 \quad \text{and} \quad \lim_{r \to 1-} |f'(r\zeta)| (1-r)^\beta = \infty \quad \text{for} \quad \zeta \in E \right\}.$$
Obviously, all four functions are decreasing and $\alpha(p) \equiv \alpha_1(p)$, $\beta(p) \equiv \beta_1(p)$. It is probable that they are continuous, strictly decreasing and that $\alpha(p) = \alpha_1(p)$, $\beta(p) = \beta_1(p)$, but I do not know the proof. Some estimates of these functions can be traced in the literature.

The distortion theorem trivially implies that
\[
\alpha(p) \equiv 1, \quad \beta(p) \equiv 3.
\]
A result of A. Beurling [4] shows that
\[
|f'(r\zeta)| = o\left(\frac{1}{1-r}\right)
\]
outside possible exceptional sets of logarithmic capacity zero, and hence
\[
\beta(p) \equiv 1 \quad \text{for all} \quad p > 0.
\]
W. Seidel and J. Walsh [33] proved that
\[
\beta(1) \equiv \frac{1}{2},
\]
and this result has recently been improved up to
\[
\beta(1) = 0, \quad \alpha(1) = 0
\]
by J. Clunie and T. MacGregor [10] (see also [17] and [22]). The only lower bound I know goes back to A. Lohwater and G. Piranian [19]. They constructed an example of $f \in S$ with
\[
\lim_{r \to 1^-} |f'(r\zeta)|(1-r)^{1/2} = \infty
\]
for all $\zeta$ in a set of positive capacity, but the proof shows, in fact, that
\[
\alpha_1(p) > 0 \quad \text{and} \quad \beta_1(p) > 0 \quad \text{for all} \quad p < 1.
\]

The results of the present paper provide estimates of $\alpha(p)$ and $\alpha_1(p)$. Although these results have only been established for conformal mappings onto Jordan domains, they can easily be extended to arbitrary univalent functions. We do not include the details. Remark that $\alpha(p)$ and $\alpha_1(p)$ are connected with the function $d(p)$ through the inequalities
\[
\frac{p}{1 + \alpha(p)} \equiv d(p) \equiv \frac{p}{1 + \alpha_1(p)},
\]
cf. Theorems 0.5 and 0.6. Our estimates of $\alpha(p)$ and $\alpha_1(p)$ are the following.

(7.1) $\alpha(p) \equiv \alpha_1(p) \equiv 1 - p$ (see Section 3.1).

(7.2) $\alpha(p) \asymp \alpha_1(p) \asymp (1-p)^{1/2}$ as $p \to 1$ (see Section 5).
In particular

(7.3) \[ \alpha(0+) = \alpha_1(0+) = 1, \]
(7.4) \[ \alpha(1-) = \alpha_1(1-) = 0. \]

Finally

(7.5) \[ \alpha_1(p) \equiv \alpha(p) < 1 \quad \text{for} \quad p > 0 \quad \text{(Theorem 6.2)}. \]

An alternative way to prove (7.5) is to verify that

(7.6) \[ \inf_{\gamma \geq 0} x(\gamma) = 1, \]

where

\[ x(\gamma) = \inf \{ x \geq 0 : \int |f'(r)^{-\gamma}|d\zeta| = O((1-)^{n^{x-\gamma}}) \quad \text{for any} \ f \in (S) \} \]

(cf. Lemma 5.1). A stronger conjecture is that \( x(2) = 1 \), cf. [7].

Arguments similar to that used in the paper allow to obtain some estimates of \( \beta \) and \( \beta_1 \):

(7.1') \[ \beta(p) \equiv \beta_1(p) \equiv 1 - p, \]
(7.2') \[ \beta(p) \asymp \beta_1(p) \asymp (1-p)^{1/2} \quad \text{as} \quad p \to 1, \]
(7.3') \[ \beta(0+) = \beta_1(0+) = 1, \]
(7.4') \[ \beta(1-) = \beta_1(1-) = 0. \]

As to the counterpart of (7.5),

(7.5') \[ \beta_1(p) \equiv \beta(p) < 1 \quad \text{for} \quad p > 0, \]

it admits a considerable amplification

(7.7) \[ \beta(p) \equiv 1 - \frac{p}{2} \]

with a very elementary proof. Together with (7.1) this gives a suitable approximation to \( 1 - \beta(p) \) as \( p \to 0^+ : \)

\[ 1 - \beta(p) \asymp 1 - \beta_1(p) \asymp p. \]

I do not know whether a similar result is also true for \( \alpha(p) \).

**Proof of (7.7).** With the help of an elementary transformation, any simply-connected domain (other than \( C \)) can be mapped onto a bounded domain. The composition with an elementary function does not have an influence on the dimen-
Conformal mapping and Hausdorff measures

sion of the set on which the derivative of a univalent function has a given order of growth. Thus we can assume that
\[ \int \int |f'|^2 \, dm_2 < \infty. \]

Then from the distortion theorem or from the maximal theorem, it follows that
\[ \int |f'(r_\zeta)|^2 |d\zeta| = O\left(\frac{1}{1-r}\right) \quad \text{as} \quad r \to 1-. \tag{7.8} \]

Repeating the reasoning in the proof of Lemma 5.1, we obtain from (7.8)
\[ \dim \{\zeta \in \mathbb{T} : \limsup_{r \to 1} |f'(r_\zeta)| (1-r)^\beta > 0\} \leq 1 + 1 - 2\beta = 2(1 - \beta). \]

By definition of \( \beta(p) \) (7.7) follows. \( \square \)

7.2. On dominating subsets. In [22], [23] the following question on dominating subsets was studied. Let \( \Omega \) be a simply-connected domain and \( \varphi \) be a measure function. Does there exist a dominating subset \( A \) of \( \Omega \) satisfying
\[ \sum_{\lambda \in A} \varphi(\delta_\lambda) < \infty \tag{7.9} \]

where \( \delta_\lambda = \text{dist} (\lambda, \partial \Omega) \)? See [23] for the background of the problem and [32] for the definition and properties of dominating subsets. Recall that the property of a set to be dominating is a conformal invariant and that \( A \subset \mathbb{D} \) is a dominating subset of the unit disc iff there exists a subset \( E \subset \mathbb{T} \) of full Lebesgue measure such that the intervals \( I(\lambda), \lambda \in \Lambda \), cover \( E \) with infinite multiplicity. (Here, for \( \lambda \in \mathbb{D} \), \( I(\lambda) \) stands for the interval \( I \) such that \( \lambda = a_I \).) This characterization of dominating subsets of \( \mathbb{D} \) makes obvious the following observation.

Let \( f : \mathbb{D} \to \Omega \) be a conformal mapping onto \( \Omega \) and \( \varphi \) be a measure function. There exists a dominating subset \( A \) of \( \Omega \) satisfying (7.9) if and only if there exists a subset \( E \subset \mathbb{T} \) and \( D_\varphi(E) = 0 \).

It was proved in [22], Lemma 3.1 that the existence of a dominating subset satisfying (7.9) implies the singularity of the harmonic measure \( \omega = h_t^\varphi \) with respect to the Hausdorff measure \( \Lambda_\varphi \) and it was asked whether the converse is true. A (positive) answer has only been known for \( \varphi(t) = t \) (see [20], Theorem 4).

There exists another version of the problem under consideration. The argument in Section 2.4 implies in fact that
\[ \omega \preceq H_\varphi \Rightarrow H_1 \preceq D_\varphi \tag{7.10} \]

(see also [23], Lemma 2.3). The question is whether the converse of (7.10) is true. By Theorem 2.3 (or by [20], Theorem 4) the answer is "yes" for \( \varphi(t) = t \).

The technique applied in the proof of Theorem 2.3 enables us to answer both questions in the affirmative for sufficiently regular measure functions \( \varphi \). The restriction
we impose on $\varphi$ is certainly unnecessarily severe and could easily be weakened considerably. At the same time, without any regularity condition, I am still unable to settle the question.

**Theorem.** Let $f$ be a conformal mapping of $D$ onto a Jordan domain $\Omega$ and $\varphi$ be a logarithmico-exponential measure function. The following are equivalent.

1) $\omega \ll H_\varphi$.
2) $H_1 \ll D^f_\varphi$.
3) For almost all $\zeta \in \Gamma$,

$$\liminf_{r \to 1^-} \frac{|f'(r\zeta)|}{\psi(1-r)} > 0$$

where $\psi = t^{-1} \varphi^{-1}(t)$. Similarly, the following three statements are equivalent.

1) $\omega$ is singular with respect to $H_\varphi$.
2) There exists a dominating subset $\Lambda$ of $\Omega$ with

$$\sum_{\lambda \in \Lambda} \varphi(\delta \lambda) < \infty.$$ 

3) For almost all $\zeta \in \Gamma$,

$$\liminf_{r \to 1^-} \frac{|f'(r\zeta)|}{\psi(1-r)} = 0.$$

We shall confine ourselves to the proof of only the first part of the theorem. First we make some preliminary remarks.

a) The class of logarithmico-exponential functions (or L-functions) was introduced by G. H. Hardy [15], see also the excellent exposition in [6]. Roughly speaking, a measure function $\varphi$ is an L-function if it is defined on an interval $(0, t_0)$ by an expression consisting of a finite number of log's and exp's together with some composition and arithmetic operations. By the main property of L-functions, any two of them are comparable, i.e. either $\varphi_1 = o(\varphi_2)$ or $\varphi_2 = o(\varphi_1)$, or $\varphi_1 \sim C \varphi_2$ as $t \to 0^+$.

b) The equivalence 2\(\Rightarrow\)3 follows from [23], Lemma 2.3. Hence the only assertion still to be proved is 2\(\Rightarrow\)1.

c) The proof of 2\(\Rightarrow\)1 will rely on certain results from [22], which we now recall. If $\varphi(t) = o(t)$ as $t \to 0$, any $\Omega$ admits a dominating subset satisfying (6.8), see [23], Lemma 3.1. In this case, $H_1 \ll D^f_\varphi$ and 2\(\Rightarrow\)1 is trivial. On the other hand, if

$$t \exp \{ |\log t|^{3/4} \} = o(\varphi(t)) \quad \text{as} \quad t \to 0,$$
then by [22], Theorem 1, \( \omega \prec H_\varphi \), and \( 2 \Rightarrow 1 \) is trivial again. Thus we can assume in the sequel that

\[
7.11 \quad c t \leq \varphi(t) \leq C t \exp \{ [\log t]^{3/4} \}
\]
as \( t \to 0 \).

**Proof of Theorem.** For the reason pointed above, we shall only prove \( 2 \Rightarrow 1 \) assuming (7.11) being valid. Define the L-function \( \chi \) by

\[
\varphi(t) = t \chi(t).
\]

We derive from (7.11) that

\[
c \leq \chi(t) \leq C \exp \{ [\log t]^{3/4} \}.
\]

It is easy to see that for L-functions the latter implies

\[
7.12 \quad \chi \left( \frac{t}{[\log t]} \right) \leq C \chi(t) \quad \text{as} \quad t \to 0.
\]

Assume that \( \omega \prec H_\varphi \), i.e. there is a subset \( e_0 \subset \partial \Omega \) with \( \omega(e_0) > 0 \) and \( H_\varphi(e_0) = 0 \). For any \( \varepsilon > 0 \) there exists a covering of \( e_0 \) by discs \( A_\varepsilon \) of radii \( r_\varepsilon \leq r_0 \) such that

\[
\sum \varphi(r_\varepsilon) < \varepsilon.
\]

We proceed further as in the proof of Theorem 2.3. For any \( v \) there are subarcs \( \sigma_j^{(v)} \) of \( \partial A_\varepsilon \), \( 1 \leq j \leq N(v) \),

\[
N(v) \leq k_0 \log \frac{1}{r_v},
\]

which are crosscuts of \( \Omega \) and separate the subarcs \( \beta_j^{(v)} = f(I_j^{(v)}) \) of \( \partial \Omega \) satisfying

\[
7.13 \quad \omega(A_\varepsilon \setminus \bigcup_{j=1}^{N(v)} \beta_j^{(v)}) \leq \varphi(r_\varepsilon).
\]

By Proposition 1.5

\[
\sum_{j=1}^{N(v)} |I_j^{(v)}| |f'(a_j^{(v)})| \leq C \sum_{j=1}^{N(v)} \text{diam } \sigma_j^{(v)} \equiv C r_v.
\]

Since \( \varphi \) is an L-function, (7.12) implies

\[
7.14 \quad \sum_{j=1}^{N(v)} \varphi(I_j^{(v)} |f'(a_j^{(v)})|) \leq C \varphi(r_\varepsilon).
\]

Let \( e \) denote the set

\[
e_0 \cap \left[ \bigcup_v \bigcup_{j=1}^{N(v)} \beta_j^{(v)} \right]
\]
and \( E = f^{-1}e \). Then by (7.13)

\[
7.15 \quad \omega(e \setminus e_0) \leq \varepsilon.
\]
and

\[ |E| = \frac{1}{2} \omega(e_0) > 0 \]

provided \( \epsilon \) is small. On the other hand, by (7.14),

\[ D_{\phi}(E) \leq \sum \sum_{j=1}^{N(V)} \varphi(|I_j|^\epsilon |f'(a_j)|) \leq C \sum r_v \leq C \epsilon. \]

Since \( \epsilon \) is arbitrary, (7.15) and (7.16) imply that

\[ H_1 \subseteq D_{\phi}' \quad \square \]

References

20. MAKAROV, N. G., Defining subsets, the support of harmonic measure, and perturbations of the spectra of operators in Hilbert space, *Dokl. Akad. Nauk SSSR* 274 (1984), 1033-1037 [Russian].

Received August 18, 1985

N. G. Makarov
Department of Analysis
ELTE
Múzeum krt. 6—8
H—1088 Budapest
Hungary

Current address:
LOMI
Fontanka 27
191011 Leningrad
USSR