A note on Lipschitz functions and spectral synthesis

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1. Introduction and statement of results

Let $\mathbf{R}^k$ denote the Euclidean k-space and let $\varphi\in L^\infty(\mathbf{R}^k)$. The spectrum of φ, denoted by $\sigma(\varphi)$, is the set of points $t\in\mathbf{R}^k$ such that the function $e^{it\cdot x}$ is in the closed subspace spanned by the translates of φ in the weak-star topology of $L^\infty(\mathbf{R}^k)$ (being the dual of $L^1(\mathbf{R}^k)$). One says that φ admits spectral synthesis if φ belongs to the weak-star closed subspace spanned in $L^\infty(\mathbf{R}^k)$ by the functions $e^{it\cdot x}$, $t\in\sigma(\varphi)$. By duality, this happens if and only if the condition

$$f\in L^1(\mathbf{R}^k),\quad \hat{f}(t)=\int_{R^k} f(x)\,e^{-it\cdot x}\,dx = 0 \quad (t\in\sigma(\varphi)) \tag{1.1}$$

implies

$$f*\varphi(x)=\int_{R^k} f(x-y)\,\varphi(y)\,dy \equiv 0 \quad (x\in\mathbf{R}^k). \tag{1.2}$$

For other equivalent ways to formulate the spectral synthesis problem as well as for classical motivation of the problem, the reader is referred to Benedetto [1].

It is well-known that for $k\leqq 2$ (1.1) implies (1.2) if, in addition, $\hat{f}$ is sufficiently small in a neighborhood of $\sigma(\varphi)$, e.g. if, in the 1-dimensional space, $\hat{f}$ is in the Lipschitz class $\mathrm{Lip}_{1/2}$ in a neighborhood of $\sigma(\varphi)$ (see Herz [2]). On the other hand, for $k\geqq 3$ one can find a rapidly decreasing function f and a bounded function φ on $\mathbf{R}^k$ such that $\hat{f}(\sigma(\varphi))=\{0\}$, but $f*\varphi\neq 0$ (see Schwartz [8], Reiter [6]). This suggests that the set of integrable functions having Lipschitz continuous Fourier transforms might be large enough to test the synthesizability of a given function in $L^\infty(\mathbf{R}^k)$, at least for $k\geqq 3$. It turns out, however, that this conjecture is false. For the group of integers this observation is due to Lee [5]. In the present paper we prove the same result on $\mathbf{R}^k$, for every k, by using a different idea. More precisely, we show the existence of a function $\varphi\in L^\infty(\mathbf{R}^k)$ such that φ does not admit spectral synthesis although $f*\varphi\equiv 0$ whenever $f\in L^1(\mathbf{R}^k)$, $\hat{f}(\sigma(\varphi))=\{0\}$ and $\hat{f}\in\mathrm{Lip}_\alpha(\mathbf{R}^k)$ for some $\alpha>0$. This is a consequence of the following two theorems.

Theorem 1. *There exists a function* $\varphi\in\bigcap_{2<p\leqq\infty}L^p(\mathbf{R}^k)$ *with compact spectrum such that* φ *does not admit spectral synthesis.*

Theorem 2. *Let* $f\in L^1(\mathbf{R}^k)$ *and* $\varphi\in L^p\cap L^\infty(\mathbf{R}^k)$ *for some* p, $2\leqq p\leqq\infty$. *Suppose that* $\hat{f}(\sigma(\varphi))=\{0\}$ *and* $\hat{f}\in\mathrm{Lip}_{k/2-k/p}(\mathbf{R}^k)$. *Then* $f*\varphi\equiv 0$.

In Theorem 2 we define $\frac{1}{\infty}=0$. If $p=2$ then the assumption that $\hat{f}\in\mathrm{Lip}_{k/2-k/p}(\mathbf{R}^k)=C\cap L^\infty(\mathbf{R}^k)$ is trivially satisfied. Hence we have the sharp result that every bounded function in $L^2(\mathbf{R}^k)$ admits spectral synthesis.

As far as we know, Theorems 1 and 2 do not appear explicitly in the literature. However, for the group of integers similar results have been obtained by Kahane and Salem [3, pp. 121—123]. (Note that Lee's Theorem 1 follows immediately from their results.)

Before proving Theorems 1 and 2 we recall some simple and well-known properties of the spectrum. We consider fixed $\varphi\in L^\infty(\mathbf{R}^k)$ and $f\in L^1(\mathbf{R}^k)$.

(P1) $$\sigma(\varphi)=\{t\in\mathbf{R}^k|\hat{g}(t)=0 \quad \text{for every} \quad g\in L^1(\mathbf{R}^k) \text{ which satisfies } g*\varphi\equiv 0\}.$$

(P2) $$\sigma(f*\varphi)\subset \operatorname{supp}(\hat{f})\cap\sigma(\varphi).$$

(P3) $$\varphi=f*\varphi \text{ if } \hat{f}=1 \text{ in a neighborhood of } \sigma(\varphi).$$

(P4) $$\operatorname{supp}((f\varphi)\hat{\ })\subset\overline{\{t+\lambda|t\in\operatorname{supp}(\hat{f}),\ \lambda\in\sigma(\varphi)\}}.$$

2. Proof of Theorem 1

It is known (see Richards [7]) that there exists a real-valued function $g\in A(T)$ with the following properties: for some $\delta>0$,

$$C(x)\overset{\text{def}}{=}\sup_{j\in\mathbf{Z}}|(e^{ixg})\hat{\ }(j)|=O(e^{-\delta\sqrt{|x|}}),\quad x\in R,\ |x|\to\infty, \tag{2.1}$$

$$g\notin \operatorname{cl}\{g^2h|h\in A(T)\}. \tag{2.2}$$

One continuous linear functional on $A(T)$, which separates g from the closed ideal generated by g^2, is of the form

$$h\to\langle\delta',h\rangle\overset{\text{def}}{=}\int_{-\infty}^{\infty}ix\left(\frac{1}{2\pi}\int_0^{2\pi}h(t)\exp\big(ixg(t)\big)\,dt\right)dx.$$

Let χ be the characteristic function of the interval $(-\frac{1}{8},\frac{1}{8})$. Define

$$f(x)=\sum_{n=-\infty}^{\infty}\hat{g}(-n)\chi(x-n),$$

$$\psi(x)=\sum_{n=-\infty}^{\infty}\langle\delta',e^{int}\rangle\chi(x-n)\quad(x\in R).$$

Then, clearly, $f \in L^1(\mathbf{R})$. We will show that

$$\psi \in \bigcap_{2<p\leqq\infty} L^p(\mathbf{R}), \tag{2.3}$$

$$\hat{f}(\sigma(\psi)) = \{0\}, \tag{2.4}$$

$$f*\psi \neq 0. \tag{2.5}$$

To prove (2.3) pick any $p \in (2, \infty)$. It follows from Parseval's formula that

$$\sum_{n=-\infty}^{\infty} |(e^{ixg})^\wedge(n)|^p \leqq C(x)^{p-2}.$$

On the other hand, by Hölder's inequality

$$|\langle \delta', e^{int}\rangle| \leqq \int_{-\infty}^{\infty} |x|\, |(e^{ixg})^\wedge(-n)|\, dx$$

$$\leqq \left[\int_{-\infty}^{\infty} \left(\frac{|x|}{1+x^2}\right)^q dx\right]^{1/q} \left[\int_{-\infty}^{\infty} (1+x^2)^p |(e^{ixg})^\wedge(-n)|^p dx\right]^{1/p},$$

where $\dfrac{1}{p}+\dfrac{1}{q}=1$. Hence

$$\|\psi\|_p^p = \sum_{n=-\infty}^{\infty} \int_{n-1/2}^{n+1/2} |\psi(x)|^p dx = \tfrac{1}{4} \sum_{n=-\infty}^{\infty} |\langle \delta', e^{int}\rangle|^p$$

$$\leqq c\int_{-\infty}^{\infty} (1+x^2)^p \sum_{n=-\infty}^{\infty} |(e^{ixg})^\wedge(-n)|^p dx \leqq c\int_{-\infty}^{\infty} (1+x^2)^p C(x)^{p-2} dx$$

for some constant $c>0$. Therefore $\|\psi\|_p<\infty$ by (2.1). Moreover

$$\|\psi\|_\infty \leqq \int_{-\infty}^{\infty} |x|\, C(x)\, dx < \infty.$$

So (2.3) holds.

To prove (2.4) it suffices to show that $f*f*\psi \equiv 0$. Pick any $x \in \mathbf{R}$ and then an integer m such that $|x-m| \leqq \frac{1}{2}$. Then

$$f*f*\psi(x) = \sum_{n=-\infty}^{\infty} \langle \delta', e^{int}\rangle \int_{-1/8}^{1/8} f*f(x-y-n)\, dy$$

$$= \sum_{n=-\infty}^{\infty} \langle \delta', e^{int}\rangle \int_{-1/8}^{1/8} \left[\sum_{j=-\infty}^{\infty} \hat{g}(-j) \int_{-1/8}^{1/8} f(x-y-n-j-u)\, du\right] dy$$

$$= \int_{-1/8}^{1/8} \left[\int_{-1/8}^{1/8} \chi(x-y-u-m)\, du\right] dy \sum_{n=-\infty}^{\infty} \left[\sum_{j=-\infty}^{\infty} \hat{g}(-j)\, \hat{g}(n+j-m)\right] \langle \delta', e^{int}\rangle$$

$$= \int_{-1/8}^{1/8} \left[\int_{-1/8}^{1/8} \chi(x-y-u-m)\, du\right] dy\, \langle \delta', g^2 e^{imt}\rangle.$$

But $\langle \delta', g^2 e^{imt}\rangle = 0$ for every integer m. Hence $f*f*\psi \equiv 0$ and therefore $\hat{f}(\sigma(\psi)) = \{0\}$ by (P1).

Condition (2.5) follows from (2.2), since

$$f*\psi(0)=\tfrac{1}{4}\sum_{n=-\infty}^{\infty}\hat{g}(n)\langle\delta',e^{int}\rangle=\tfrac{1}{4}\langle\delta',g\rangle$$

and therefore $f*\psi(0)\neq 0$.

Let $(K_n)_{n\in\mathbf{N}}$ be the Fejér kernel on $\mathbf{R}$ (Katznelson [4, p. 124]). Then $f*K_n*\psi\neq 0$ for some n. For such an n define $\varphi=K_n*\psi$. Then, by (P2), we obtain the claimed result on the line. The general case now follows easily if we define

$$F(x_1,\ldots,x_k)=f(x_1)\ldots f(x_k),$$

$$\Phi(x_1,\ldots,x_k)=\varphi(x_1)\ldots\varphi(x_k).$$

This completes the proof of Theorem 1.

3. Proof of Theorem 2

We will use the standard Beurling—Pollard argument (cf. Herz [2, p. 710]).

Pick a rapidly decreasing function η on R^k such that $\eta(0)=1$ and $\operatorname{supp}(\hat{\eta})$ is contained in the open unit ball. Define $\eta_\varepsilon(x)=\eta(\varepsilon x)$, $\varphi_\varepsilon=(2\pi)^{-k}(\eta_\varepsilon\varphi)\hat{}$ $(\varepsilon>0)$. If $\sigma(\varphi)$ is compact then it follows from (P3) and (P4) that φ is continuous and φ_ε vanishes in the set $\{t\in\mathbf{R}^k|\inf_{\lambda\in\sigma(\varphi)}|t-\lambda|\geqq\varepsilon\}$. Hence

$$f*\varphi(x)=\lim_{\varepsilon\to 0+}\int_{\mathbf{R}^k}f(x-y)\,\eta_\varepsilon(y)\,\varphi(y)\,dy$$

$$=\lim_{\varepsilon\to 0+}\int_{H_\varepsilon}\hat{f}(t)\,\varphi_\varepsilon(t)\,e^{it\cdot x}\,dt,$$

where H_ε denotes the set $\{t\in\mathbf{R}^k|0<\inf_{\lambda\in\sigma(\varphi)}|t-\lambda|<\varepsilon\}$. An application of Schwarz' inequality and Plancherel's theorem shows that

$$|f*\varphi(x)|\leqq\liminf_{\varepsilon\to 0+}(2\pi)^{-k/2}\Big(\int_{H_\varepsilon}|\hat{f}(t)|^2\,dt\Big)^{1/2}\|\eta_\varepsilon\varphi\|_2.$$

Let ω be the modulus of continuity of $\hat{f}$, defined by $\omega(\varepsilon)=\sup_{|t-\lambda|\leqq\varepsilon}|\hat{f}(t)-\hat{f}(\lambda)|$. Let $|H_\varepsilon|$ denote the measure of H_ε. Then, by Hölder's inequality,

$$|f*\varphi(x)|\leqq\liminf_{\varepsilon\to 0+}(2\pi)^{-k/2}\omega(\varepsilon)\,|H_\varepsilon|^{1/2}\varepsilon^{k/p-k/2}\|\eta\|_{2p/(p-2)}\|\varphi\|_p. \tag{3.1}$$

But $\omega(\varepsilon)=O(\varepsilon^{k/2-k/p})$ and $|H_\varepsilon|=o(1)$ $(\varepsilon\to 0+)$. Hence it follows from (3.1) that $f*\varphi\equiv 0$.

If $\sigma(\varphi)$ is not compact we convolve φ with the Fejér kernel on $\mathbf{R}^k$, apply the above argument to these convolutions and then pass to the limit. This completes the proof of Theorem 2.

4. Remarks

Theorems 1 and 2 show that the set $\{f\in L^1(\mathbf{R}^k) \mid \hat{f}\in \bigcup_{\alpha>0} \mathrm{Lip}_\alpha (R^k)\}$ is not large enough to test the synthesizability of a given function in $L^\infty(\mathbf{R}^k)$. For the group of integers, Lee [5, Th. 2] obtained an even stronger result. Unfortunately, it seems that our method cannot be generalized to prove Lee's Theorem 2 on the line. However, we are able to improve our results somewhat. Let φ be as in our Theorem 2. Moreover, assume that $\sigma(\varphi)$ is compact and satisfies $|H_\varepsilon|=o(\varepsilon^a)$ for some $a\geqq 0$. Let $f\in L^1(\mathbf{R}^k)$ and assume that, for some constants α, $c>0$, $|\hat{f}(t)|\leqq c \inf_{\lambda\in\sigma(\varphi)} |t-\lambda|^\alpha$ in a neighborhood of $\sigma(\varphi)$. Then the proof of Theorem 2 above shows that $f*\varphi\equiv 0$ if

$$\alpha+\frac{a}{2}+\frac{k}{p}-\frac{k}{2}\geqq 0. \tag{4.1}$$

Now let φ be as in Theorem 1. It follows from (4.1) that $f*\varphi\equiv 0$ whenever $f\in L^1(\mathbf{R}^k)$ and $|\hat{f}(t)|\leqq c \inf_{\lambda\in\sigma(\varphi)} |t-\lambda|^\alpha$ in a neighborhood of $\sigma(\varphi)$, for any $\alpha>0$.

We mention finally that the inequality (4.1) is sharp for every $k\geqq 3$. This can be deduced from Schwartz' example [8]. We omit the details.

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