Spaces of Carleson measures: duality and interpolation

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1. Introduction

In [12], [13] Coifman, Meyer and Stein have developed a theory of "tent spaces" with interesting applications. Their theory has led to a unification and simplification of some basic techniques in harmonic analysis.

The theory of "tent spaces" is closely related to the one of Hardy spaces. In this paper we consider the relationship of these tent spaces with spaces of Carleson measures. In particular we identify the spaces of Carleson measures as the duals of certain tent spaces.

We also compute the real interpolation spaces of some extreme tent spaces by means of computing the corresponding K functionals of Peetre. As pointed out in [13] the interpolation theory of tent spaces can be used to derive the corresponing theory for H^p spaces and Lipschitz spaces.

Let us briefly explain the motivation of this paper (we refer to §2 for detailed definitions). A basic inequality valid for Carleson measures on $\mathbf{R}^{n+1}_+$ is

$$(1.1)\qquad |\mu|\{(x,t)\big||f(x,t)|>\lambda\}\leqq c\big|\{x/A_\infty(f)(x)>\lambda\}\big|$$

where $A_\infty(f)(x)=\sup_{\Gamma(x)}|f(y,t)|$, $\Gamma(x)=$cone with vertex x. Then

$$(1.2)\qquad \left|\int f(x,t)\,d\mu(x,t)\right|\leqq c\left(\sup_B\frac{1}{|B|}\int_{T(B)}d|\mu|\right)\|A_\infty(f)\|_1$$

where $T(B)=$"tent with base B".

Let $\|\mu\|_{V^1}=\|C_1(\mu)\|_\infty$, where $C_1(\mu)(x)=\sup_{B\ni x}\dfrac{1}{|B|}\int_{T(B)}d|\mu|$. Then (1.2) can

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be interpreted as a duality result between $T^1_\infty=\{f|A_\infty(f)\in L^1\}$ and V^1=space of Carleson measures on $\bar{\mathbf{R}}^{n+1}_+$, i.e., $(T^1_\infty)^*=V^1$ (cf. [2]).

It is of interest to study inequalities of the form (1.2) by means of considering spaces of Carleson measures defined by the boundedness of the functionals $\sup_{B\ni x}\frac{1}{|B|^\alpha}\int_{T(B)}d|\mu|$, $\alpha>0$. These spaces, V^α, have been considered by several authors: Duren [15], Barker [3], Amar and Bonami [2], Johnson [18].

In this paper we develop this remark systematically and use it to compute the duals of certain "tent spaces" as spaces of Carleson measures. These extreme "tent spaces" therefore play an important role in the study of maximal operators. We also develop the interpolation theory of these extreme spaces completing the results of Coifman, Meyer and Stein [13] as well as those of Amar and Bonami [2].

We should mention that while this paper was being prepared for publication we received a preprint by Bonami and Johnson [5] that contains some of our results. Our approach is, however, different.

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2. Preliminaries

We shall work on $\mathbf{R}^{n+1}_+$ but most of our results are valid, and useful, in the more general context of homogeneous spaces.

Let Ω be an open set in $\mathbf{R}^n$, we let $T(\Omega)$ be the subset of $\mathbf{R}^{n+1}_+$ defined by $T(\Omega)=\{(x,t)|B(x,t)\subset\Omega\}$, where $B(x,t)$ denotes the ball with centre x and radius t. As usual, $|\Omega|$ will stand for the Lebesgue measure of the set Ω.

Given $\alpha\in\mathbf{R}$, we say that a measure w in $\bar{\mathbf{R}}^{n+1}_+$ is a Carleson measure of order α, if $\forall\,\Omega\subset X$, Ω open bounded,

$$|w|\big(T(\Omega)\big)\leqq c|\Omega|^\alpha \tag{2.1}$$

where c is an absolute constant independent of Ω. In fact, we let

$$\|w\|_{V^\alpha}=\inf\{c\colon (2.1)\ \text{holds}\} \tag{2.2}$$

and $V^\alpha=\{w|w$ measure on $\bar{\mathbf{R}}^{n+1}_+$, $\|w\|_{V^\alpha}<\infty\}$. Notice, in particular, that $V^0=\{$finite measures on $\bar{\mathbf{R}}^{n+1}_+\}$.

The case $\alpha=1$ corresponds to the measures studied by Carleson [7].

We consider kernels $P_t(x,y)$ on $\mathbf{R}^n\times\mathbf{R}^n$ and the corresponding integral operators associated with them $P_tf(x)=\int P_t(x,y)f(y)\,dy$. The maximal operators associated with these operators are given by

$$A_\infty P(f)(x) = \sup_{(y,t)\in\Gamma(x)} |P_t f(y)|.$$

The kernels we consider always satisfy:

$$\|A_\infty P(f)\|_p \leqq c_p \|f\|_p, \quad 1<p<\infty. \tag{2.3}$$

Important examples are provided by the Poisson kernels in different settings and also by $P_t^0(x,y)=|B(x,t)|^{-1}\chi_{B(x,t)}(y)$. As is well known conditions of the form (2.1) are useful to estimate the distribution function of $P_t f$ in terms of $A_\infty P(f)$. In fact, given $\lambda>0$, let $\Omega_\lambda=\{x|Mf(x)>\lambda\}$, where M denotes the maximal operator of Hardy—Littlewood. Then

$$\{(x,t)|P_t f(x)>\lambda\}\subset T(\Omega_\lambda). \tag{2.4}$$

Using (2.4) and the usual notation of Lorentz spaces (cf. [17]) we get, for $\alpha>0$,

$$\|P_t f\|_{L^{p\cdot\alpha}(dw)} \leqq c\|w\|_{V^\alpha}\|f\|_{L(p,p\cdot\alpha)}. \tag{2.5}$$

In particular the choice $\alpha=\frac{1}{p}$, gives (cf. [2])

$$\int |P_t f(x)|\, d|w|(x,t) \leqq c\|w\|_{V^{1/p}}\|f\|_{L(p,1)} \tag{2.6}$$

which can be interpreted as a duality formula.

The tent spaces T_q^p, $0<p\leqq\infty$, $0<q\leqq\infty$ are defined using two families of functionals. The A_q functionals defined by:

$$A_q(f)(x) = \left\{\int_{\Gamma(x)} |f(y,t)|^q \frac{dy\,dt}{t^{n+1}}\right\}^{1/q}, \tag{2.7}$$

$$A_\infty(f)(x) = \sup_{\Gamma(x)} |f(y,t)|, \tag{2.7$'$}$$

where, as usual, $\Gamma(x)$ is the cone with vertex x, $\{(y,t)\big||x-y|<t\}$.

The C_q functionals are given by

$$C_q(f)(x) = \sup\left\{\frac{1}{|B|}\int_{T(B)} |f(y,t)|^q \frac{dy\,dt}{t}\right\}^{1/q} \tag{2.8}$$

where the sup is taken over all balls containing x.

Let us observe that the A_1 and C_1 functionals can be defined for measures. This observation will be particularly useful for us in what follows. (See 2.9).

The tent spaces T_q^p are defined by the condition $A_q(f)\in L^p$, $0<q\leqq\infty$, $0<p<\infty$, (if $q=\infty$ we impose some additional restrictions such as the continuity of f (cf. [13] and (7.1) below). One defines similarly the T_q^∞ spaces using C_q functionals (cf. [13]).

In view of (2.5) it is of interest to define tent spaces based on Lorentz spaces; this can be done simply by replacing the L^p norm condition by an $L(p,r)$ norm condition in the appropriate definitions. The spaces thus obtained will be denoted $T_q^{p,r}$ and will be shown to appear naturally when we interpolate T_q^p spaces.

Let us note in passing that this description of the $T_q^{p,r}$ spaces as interpolation spaces, proves their completeness and provides the duality theory for them, at least when $1<p<\infty$, $1\leqq q<\infty$, from the duality theory developed for the T_q^p spaces in [13]. Alternatively one could deduce this part of the theory using similar arguments to those given in [13] (cf. also [15]).

Our attention in this paper will be devoted to the study of the extremal spaces T_1^∞, T_∞^p, $T_\infty^{p,1}$ and their duals. Let us start by making explicit the remark that followed (2.8).

(2.9) Definition. The A_1, C_1 functionals for measures are given by

$$A_1(w)(x) = \int_{\Gamma(x)} t^{-n}\, d|w|(y,t) \tag{2.10}$$

$$C_1(w)(x) = \sup_{B\ni x} \frac{1}{|B|}\int_{T(B)} d|w|(y,t). \tag{2.11}$$

For $0<p<\infty$, $0<q\leqq\infty$, let $\tau_1^{p,q}=\{w|A_1(w)\in L^{p,q}\}$, $\tau_1^\infty=\{w|C_1(w)\in L^\infty\}$ where w is a measure on $\bar{\mathbf{R}}_+^{n+1}$. One easily cheks that $\tau_1^\infty=V^1$, on the other hand, if $1<p<\infty$, $\tau_1^{p,p}=\tau_1^p$ coincides with the space $W^{1/p'}$ introduced by Amar and Bonami in [2].

The A_1 and C_1 functionals are related by

$$C_1(w)(x) \leqq cM(A_1(w))(x).$$

In particular,

$$\left|\{x|C_1(w)(x)>\lambda\}\right| \leqq \left|\{x|cM(A_1(w))(x)>\lambda\}\right| \tag{2.12}$$

where M is the Hardy—Littlewood maximal operator. This follows readily from the definitions (cf. [3]).

Finally, in our estimates of K functionals we need the well-known concept of nonincreasing rearrangment (cf. [17]): for a given function f the nonincreasing rearrangement f^* is the generalized inverse of the distribution function $\left|\{x||f(x)|>t\}\right|$ therefore $\left|\{x||f(x)|>f^*(t)\}\right|\leqq t$. The double star function f^{**} is simply the average of f^* at time t; $f^{**}(t)=t^{-1}\int_0^t f^*(s)\,ds$, $t>0$, and is also decreasing.

3. Interpolation of T_∞^p spaces

In this section we study the interpolation properties of the T_∞^p spaces. Our results complement earlier work by Coifman, Meyer and Stein [13] and Amar and Bonami [2].

We shall consider the real method of interpolation using Peetre's K functionals.

We review briefly some pertinent facts about real interpolation. For more information on interpolation theory we refer the reader to [4].

Let B_0, B_1 be quasi normed spaces embedded in a suitable topological Hausdorff vector space V, and define for $f\in B_0+B_1$, $t>0$,

$$K(t, f, B_0, B_1) = \inf_{\substack{f=f_0+f_1\\ f_i\in B_i}} \{\|f_0\|_{B_0}+t\|f_1\|_{B_1}\}.$$

For $0<\theta<1$, $0<q\leqq\infty$, let

$$(B_0, B_1)_{\theta,q} = \left\{f\in B_0+B_1/\|f\|_{(B_0,B_1)_{\theta,q}} = \left\{\int_0^\infty [t^{-\theta}K(t, f, B_0, B_1)]^q \frac{dt}{t}\right\}^{1/q} <\infty\right\}.$$

We shall also use, in §4, an important complement to the reiteration theorem obtained recently by T. Wolff [22]: Let A_i, $i=0, \dots,$ be quasi Banach spaces continuously embedded in a common topological Hausdorff vector space; let $0<\theta<\eta<1$, $\theta=\lambda\eta$, $\eta=(1-\mu)\theta+\mu$, $0<\lambda$, $\mu<1$, then if $A_1=(A_0, A_2)_{\lambda,p}$, $A_2=(A_1, A_3)_{\mu,q}$ we have $A_1=(A_0, A_3)_{\theta,p}$, $A_2=(A_0, A_3)_{\eta,q}$. In other words Wolff's theorem allows to treat each end point space separately.

When considering the $\{T^p_\infty\}_{p>0}$ scale it is natural to let $T^\infty_\infty=L^\infty$ since $A_\infty(f)\in L^\infty\Leftrightarrow f\in L^\infty$. Our main result concerning this scale is

(3.1) Theorem. *Let* $0<p<\infty$, $0<\theta<1$, $0<r\leqq\infty$, $\dfrac{1}{p_\theta}=\dfrac{1-\theta}{p}$, *then*

(i) $\exists c_1, c_2>0$ *such that*

$$c_1\left\{\int_0^{t^p} A_\infty(f)^{*p}(s)\,ds\right\}^{1/p} \leqq K(t,f,T^p_\infty, L^\infty) \leqq c_2\left\{\int_0^{t^p} A_\infty(f)^{*p}(s)\,ds\right\}^{1/p}.$$

In other words,

$$K(t,f,T^p_\infty, L^\infty) \approx K(t, A_\infty(f), L^p, L^\infty).$$

(ii) $$(T^p_\infty, L^\infty)_{\theta,r} = T^{p_\theta,r}_\infty.$$

Proof. The second half of the theorem is an easy consequence of the first half.

Consider the proof of (i). Let $f=f_1+f_2$ be a decomposition of f with $f_0\in T^p_\infty$, $f_1\in L^\infty$, then

$$\left\{\int_0^{t^p} A_\infty(f)^{*p}(s)\,ds\right\}^{1/p} \leqq c\left[\left\{\int_0^{t^p} A_\infty(f_0)^{*p}(s)\,ds\right\}^{1/p}+\left\{\int_0^{t^p} A_\infty(f_1)^{*p}(s)\,ds\right\}^{1/p}\right]$$
$$\leqq c[\|f_0\|_{T^p_\infty}+t\|f_1\|_\infty].$$

Thus, taking infimum over all decompositions gives the first inequality of (i).

Let us construct a nearly optimal decomposition to prove the main estimate of (i). Let $t>0$, $f\in T^p_\infty+L^\infty$, define $\Omega_t=\{x|A_\infty(f)(x)>A_\infty(f)^*(t^p)\}$, and consider the decomposition of f given by $f=f_0+f_1$, where $f_0=f\chi_{T(\Omega_t)}$. Then,

$$\|f_0\|_{T^p_\infty} = \|A_\infty(f_0)\|_p \leqq c\|\chi_{\Omega_t}A_\infty(f)\|_p \leqq c\left\{\int_0^{t^p} A_\infty(f)^{*p}(s)\,ds\right\}^{1/p}. \tag{3.2}$$

Moreover,

$$t\|f_1\|_\infty \leqq \left\{\int_0^{t^p} A_\infty(f)^{*p}(s)\,ds\right\}^{1/p}.$$

Combining these estimates we conclude that

$$\begin{aligned} K(t, f, T^p_\infty, L^\infty) &\leqq \|f_0\|_{T^p_\infty} + t\|f_1\|_\infty \\ &\leqq c\left\{\int_0^{t^p} A_\infty(f)^{*p}(s)\,ds\right\}^{1/p}. \end{aligned}$$

4. Interpolation of τ^p_1 spaces

In this section we extend the results of [13] (cf. also [2]) concerning the interpolation of the $\{\tau^p_1\}$ scale.

(4.1) Theorem. *Let* $0<p<\infty$, $0<\theta<1$, $0<r\leqq\infty$, $\dfrac{1}{p_\theta}=\dfrac{1-\theta}{p}$, *then*

$$(\tau^p_1, \tau^\infty_1)_{\theta,r} = \tau^{p_\theta,r}_1. \tag{4.2}$$

The proof of (4.2) is based on the following estimates for the K functionals for couples of τ^p_1 spaces

(4.3) Theorem. (i) *Let* $1\leqq p<\infty$, *then*

$$K(t, w, \tau^p_1, \tau^\infty_1) \leqq c\left\{\int_0^{t^p} M\big(A_1(w)\big)^{*p}(s)\,ds\right\}^{1/p} \tag{4.4}$$

where c is a positive constant independent of w.

(ii) *Let* $0<p_0<p_1<\infty$, *then*

$$K(t, w, \tau^{p_0}_1, \tau^{p_1}_1) \approx K\big(t, A_1(w), L^{p_0}, L^{p_1}\big). \tag{4.5}$$

Using (i) and (ii) we can now provide the

Proof of (4.1). Observe that for $1<p<\infty$, we have (cf. [13])

$$\|A_1(w)\|_p \sim \|C_1(w)\|_p. \tag{4.6}$$

Consider the operator C_1, then by (4.6) we have

$$C_1\colon \tau^p_1 \to L^p, \quad C_1\colon \tau^\infty_1 \to L^\infty, \quad p>1. \tag{4.7}$$

Interpolating the estimates in (4.7) and taking into account (4.6) once again gives

$$(4.8)\qquad (\tau_1^p, \tau_1^\infty)_{\theta,r} \subseteq \tau_1^{p_\theta,r}, \frac{1}{p_\theta} = \frac{1-\theta}{p}, \quad \text{if} \quad p > 1.$$

The reverse inclusion can now be obtained from (4.4) and Hardy's inequality. Therefore we have obtained

$$(4.9)\qquad (\tau_1^p, \tau_1^\infty)_{\theta,r} = \tau_1^{p_\theta,r}, \frac{1}{p_\theta} = \frac{1-\theta}{p}, \quad 1 < p < \infty.$$

Using (4.5) we readily derive

$$(4.10)\qquad (\tau_1^{p_0}, \tau_1^{p_1})_{\theta,r} = \tau_1^{p_\theta,r}, \quad \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad 0 < p_0 < p_1 < \infty.$$

We can now invoke Wolff's theorem to combine (4.9) and (4.10) to obtain (4.2).

We are now ready to provide the

Proof of (4.3). We prove only (4.4) (the assertion (4.5) being similar). Moreover, we only need to consider the case $p=1$.

The proof of (4.4) consists in exhibiting a nearly optimal decomposition. Let $t>0$, $w\in\tau_1^1+\tau_1^\infty$, define $\Omega_t=\{x|C_1(w)(x)>cM(A_1(w))^*(t)\}$, and consider the decomposition $w=\mu_0+\mu_1$, where $\mu_0=w\chi_{T(\Omega_t)}$. We have,

$$\|\mu_0\|_{\tau_1^1} = \|A_1(\mu_0)\|_1 \leqq \|A_1(w)\chi_{\Omega_t}\|_1 \leqq \int_0^{|\Omega_t|} M(A_1(w))^*(s)\,ds.$$

Now, recall that by (2.12)

$$|\Omega_t| \leqq c\left|\{x|M(A_1(w))(x) > M(A_1(w))^*(t)\}\right| \leqq ct, \; c > 1.$$

Therefore, using the fact that $f^{**}(t)\downarrow$, we obtain

$$(4.11)\qquad \|\mu_0\|_{\tau_1^1} \leqq c\int_0^t M(A_1(w))^*(s)\,ds.$$

Consider now $C_1(\mu_1)(x)$; clearly if $x\in\Omega_t^c$ we get

$$tC_1(\mu_1)(x) \leqq tC_1(w)(x) \leqq tM(A_1(w))^*(t)$$
$$\leqq \int_0^t M(A_1(w))^*(s)\,ds.$$

Suppose now that $x\in\Omega_t$. In order to estimate $C_1(\mu_1)(x)$ we use a Whitney decomposition of the open set Ω_t. (See [21], p. 167.) Therefore, $\Omega_t=\bigcup_{k=1}^\infty Q_k$, where the Q_k's are disjoint cubes such that dist $(Q_k, \Omega_t^c)\approx$diam (Q_k). More precisely, for each cube Q_k let x_k^0 be its center and $2r_k$ its diameter, we also let $\bar{Q}_k$ be the cube concentric with Q_k and such that diam $\bar{Q}_k=10r_k$; then, $\frac{3}{4}(2r_k)<$dist $(\bar{Q}_j, \Omega_t^c)$, and dist $(Q_k, \Omega_t^c)\leqq$ $\leqq 4(2r_k)$.

Let $B=B(x, r)$ be a ball centered at x with radius r. If $T(B)\subset T(\Omega_t)$, then we clearly have $1/|B| \int_{T(B)} d|\mu_1|=0$. Thus, from now on we suppose that $T(B)\cap (T(\Omega_t))^c\neq\emptyset$. Therefore, there exists $(y, s)\in T(B)$ such that $(y, s)\notin T(\Omega_t)$. If $(y, s)\notin T(\Omega_t)$ then $B(y, s)\not\subset T(\Omega_t)$. Two possibilities arise: a) $y\notin \Omega_t$ or b) $y\in\Omega_t$ but $\operatorname{dist}(y, \Omega_t^c)<s$.

We consider the case b) first. By our assumption $y\in Q_k$ for some $k\in\mathbf{N}$. Therefore,

$$\left(\tfrac{3}{4}\right)(2r_k) < \operatorname{dist}(y, \Omega_t^c) < s < r$$

where the last inequality follows from the fact that $(y, s)\in T(B)$.

Let $x_k\in\Omega_t^c$ be such that $\operatorname{dist}(x_k, Q_k)\leqq 8r_k$. It follows that $x_k\in\bar{B}=B(x, 10r)$. In fact,

$$\begin{aligned}|x-x_k| &\leqq |x-y|+|y-x_k^0|+|x_k^0-x_k|\\ &\leqq r+r_k+10r_k < 9r.\end{aligned}$$

Consequently,

$$\begin{aligned}\frac{1}{|B|}\int_{T(B)} d|\mu_1| &\leqq \frac{c}{|\bar{B}|}\int_{T(B)} d|\mu_1|\\ &\leqq cC_1(\mu_1)(x_k) \leqq cM\big(A_1(w)\big)^*(t).\end{aligned}$$

Finally if $y\notin\Omega_t$, then

$$\frac{1}{|B|}\int_{T(B)} d|\mu_1| \leqq C_1(\mu_1)(y) \leqq M\big(A_1(w)\big)^*(t).$$

Our analysis shows that if $x\in\Omega_t$,

$$tC_1(\mu_1)(x) \leqq c\int_0^t M\big(A_1(w)\big)^*(s)\,ds.$$

Therefore

$$t\|\mu_1\|_{\tau_1^\infty} \leqq c\int_0^t M\big(A_1(w)\big)^*(s)\,ds. \tag{4.12}$$

Combining (4.11) and (4.12) we obtain

$$K(t, w, \tau_1^1, \tau_1^\infty) \leqq c\int_0^t M\big(A_1(w)\big)^*(s)\,ds$$

as desired.

In particular, we have the results of [2] for the (V^0, V^1) couple. In fact since $\tau_1^1=V^0$, $\tau_1^\infty=V^1$, we get from (4.2)

$$(V^0, V^1)_{\theta,\infty} = \tau_1^{p_\theta,\infty} = V^{1/p_\theta}, \quad \frac{1}{p_\theta} = 1-\theta. \tag{4.13}$$

Moreover, we also have (cf. §2)

$$(V^0, V^1)_{\theta,p_\theta} = W^{1/p_\theta}, \quad \frac{1}{p_\theta} = 1-\theta\,. \tag{4.14}$$

5. Spaces of Carleson measures and duality of $T^{p,q}_\infty$ spaces

We take up the study of the duals of T^p_∞ spaces. Recall the result of Coifman, Meyer and Stein [13]: $(T^1_\infty)^*=\tau_1^\infty=V^1$. In this section we consider more generally, the duals of $T^{p,q}_\infty$ spaces.

We begin our analysis with the case $p\geqq 1$.

(5.1) Theorem.

(i) $(T^{p,1}_\infty)^*=V^{1/p},\quad 1\leqq p<\infty$

(ii) $(T^p_\infty)^*=\tau_1^{p'},\quad 1<p<\infty.$

Proof. (i) The basic inequality is a reformulation of (2.5) for $\alpha=1/p$, namely

$$\left|\int f(x,t)\,dw(x,t)\right|\leqq c\|w\|_{V^{1/p}}\|A_\infty(f)\|_{p,1}.$$

It follows that $V^{1/p}\subset(T^{p,1}_\infty)^*$.

To prove the reverse inclusion we argue as in [13]. Let $l\in(T^{p,1}_\infty)^*$, let K be a compact set of $\overline{\mathbf{R}}^{n+1}_+$ and consider the restriction l_K of l to $C(K)$; then there exists a measure on K representing l_K, i.e. $l_K(f)=\int f(x,t)\,d\mu_K(x,t)\forall f\in C(K)$; by letting $\{K_n\}$ be an increasing sequence of compact sets covering $\overline{\mathbf{R}}^{n+1}_+$ we obtain a measure μ on $\overline{\mathbf{R}}^{n+1}_+$.

We claim that $\mu\in V^{1/p}$. We may suppose that $\mu\geqq 0$. Then given any open bounded set Ω, let $\{f_n\}$ be a sequence of continuous bounded functions with compact support such that $f_n\uparrow\chi_{T(\Omega)}$; then $\left|\int f_n(x,t)\,d\mu\right|\leqq\|l\|\|f_n\|_{T^{p,1}}$, and therefore

$$\left|\int_{T(\Omega)}d\mu\right|\leqq\|l\|\|A_\infty(\chi_{T(\Omega)})\|_{p,1}=p\|l\||\Omega|^{1/p}.$$

(ii) The basic inequality here is the elementary inequality

$$\left|\int_{\mathbf{R}^{n+1}_+}f(x,t)\,d\mu(x,t)\right|\leqq c\int_{\mathbf{R}^n}A_\infty(f)(x)A_1(\mu)(x)\,dx. \tag{5.2}$$

The proof of (5.2) runs as follows:

$$\int A_\infty(f)(x)A_1(\mu)(x)\,dx=\int_{\mathbf{R}^n}\sup_{\Gamma(x)}|f(y,t)|\int_{\Gamma(x)}\frac{d|\mu|(y,t)}{t^n}\,dx$$

$$\geqq\int_{\mathbf{R}^n}\int_{\Gamma(x)}\frac{|f(y,t)|}{t^n}\,d|\mu|(y,t)\,dx=c\int_{\mathbf{R}^n}\int_0^\infty\int_{|x-y|<t}\frac{|f(y,t)|}{t^n}\,d|\mu|(y,t)\,dx$$

$$=c\int_{\mathbf{R}^{n+1}_+}|f(y,t)|\,d|\mu|(y,t).$$

It follows that $\tau_1^{p'}\subset(T^p_\infty)^*$. Let $l\in(T^p_\infty)^*$ and argue as in the proof of (i) to obtain a measure μ representing l. We shall show that $A_1(\mu)\in L^{p'}$. Assume that μ is positive.

Let us observe that A_1 is the adjoint of the operator P_t^0 defined in §2: $P_t^0 f(x) = \frac{1}{t^n} \int_{|y| \leqq t} f(x-y)\,dy$. Therefore,

$$\|A_1(\mu)\|_{p'} = \sup_{\|g\|_p \leqq 1} \int A_1(\mu)(x) g(x)\,dx = \sup_{\|g\|_p \leqq 1} \int P_t^0 g(x)\,d\mu(x, t).$$

At this point we use the fact that μ represents the functional l, and obtain

$$\|A_1(\mu)\|_{p'} \leqq \sup_{\|g\|_p \leqq 1} \|P_t^0 g\|_{T_\infty^p} \|l\| \leqq \|l\| \sup_{\|g\|_p \leqq 1} \|Mg\|_p \leqq c\|l\|$$

where in the last inequality we use the Hardy—Littlewood maximal theorem.

To deal with the case $0<p<1$ we use the atomic theory of T_∞^p spaces as developed in [13]. A p-atom is a function $a(x, t)$ such that there exists a ball B with $\operatorname{supp} a \subset T(B)$ and $\|a\|_\infty \leqq |B|^{-1/p}$. Observe that if a is a p-atom then $\|a\|_{T_\infty^p} \leqq 1$. Moreover, notice that these atoms do not have zero mean.

(5.3) Theorem (cf. [13]). *Suppose that* $f \in T_\infty^p$, $0<p\leqq 1$, *then* $f = \sum_{j=1}^\infty \lambda_j a_j$ *in the sense of* $\mathcal{S}'$, *where the* a_j*'s are p-atoms,* $\lambda_j \in \mathbf{C}$, *and* $\left(\sum_{j=1}^\infty |\lambda_j|^p\right)^{1/p} \leqq c\|f\|_{T_\infty^p}$. *In fact,* $\|f\|_{T_\infty^p} \approx \inf\left\{\left(\sum_{j=1}^\infty |\lambda_j|^p\right)^{1/p} : f = \sum_{j=1}^\infty \lambda_j a_j,\ a_j \text{ } p\text{-atom}\right\}$.

Finally one more observation is needed before stating the duality theorem. It is easy to check, using for example Whitney's decompositions, that $w \in V^r$ with $r \geqq 1$ if and only if for some $c>0$ and all balls $B \subset \mathbf{R}^m$,

$$|w|(T(B)) \leqq c|B|^r. \tag{5.4}$$

(5.5) Theorem. *Suppose* $0<p<1$, *then*

$$(T_\infty^p)^* = V^{1/p}. \tag{5.6}$$

Proof. Let a be a p-atom, $\operatorname{supp} a \subset T(B)$, for some ball $B \subset R^n$, then if $\mu \in V^{1/p}$

$$\left|\int_{T(B)} a(x, t)\,d\mu(x, t)\right| \leqq |B|^{-1/p} \int_{T(B)} d|\mu|(x, t) \leqq \|\mu\|_{V^{1/p}}.$$

Therefore, by (5.3), we see that $V^{1/p} \subset (T_\infty^p)^*$. The reverse inclusion is obtained as in the proof of the case $p \geqq 1$ above using the remark (5.4).

6. Applications

We consider applications of our results to some problems in harmonic analysis.

A. Multiplier operators and maximal operators on V^α, $\tau_1^{p,q}$ and T_∞^p spaces

Let X denote any of the spaces V^α, $\tau_1^{p,q}$ or T_∞^p. A natural problem that arises is the study of multipliers of X: i.e., let $\mu(x, t)$ be a function on $\mathbf{R}_+^{n+1}$, then under what conditions is the map $w \to \mu w$ well defined and continuous from X into itself?

Observe that when $X=T^p_\infty$ a necessary condition is the continuity of μ. In other cases the necessity of this condition is not so explicit but is natural in view of our next.

(6.1) Lemma (cf. [18]). *Let* $P=(x_0, t_0)\in \mathbf{R}^{n+1}_+$ *be fixed and let* δ_P *be the Dirac measure concentrated in P. Then,* $\delta_P\in V^\alpha$, $\delta_P\in \tau_1^{p,q}$. *Moreover,*

(i) $\|\delta_P\|_{V^\alpha} = \dfrac{c}{t_0^{n\alpha}}, \quad c=c(n,\alpha)$

(ii) $\|\delta_P\|_{\tau_1^{p,q}} = c t_0^{n((1/p)-1)}, \quad c=c(n,p,q).$

Proof. Let $\Omega\subset\mathbf{R}^n$ be a bounded open set, then $\delta_P(T(\Omega))$ vanishes unless $(x_0, t_0)\in T(\Omega)$. Thus, if $\delta_P(T(\Omega))\neq 0$ we see that $ct_0^n\leqq|\Omega|$, and consequently, for $\alpha>0$

$$\delta_P(T(\Omega)) \leqq \frac{1}{(ct_0^n)^\alpha}|\Omega|^\alpha.$$

On the other hand $\delta_P(T(B(x_0, t_0)))=1=|B(x_0, t_0)|^\alpha/(ct_0^n)^\alpha$. These two observations prove (i).

The second half of the lemma follows readily from the observation that $A_1(\delta_P)(x)=t_0^{-n}\chi_{B(x_0,t_0)}(x)$.

Let us denote by M_X the space of multipliers of X as a subset of $C(\mathbf{R}^{n+1}_+)$. Let $B(\mathbf{R}^{n+1}_+)$ be the space of bounded continuous functions on $\mathbf{R}^{n+1}_+$. We then have

(6.2) Theorem. $M_X=B(\mathbf{R}^{n+1}_+)$.

Proof. It is clear that $B(\mathbf{R}^{n+1}_+)\subseteqq M_X$. Now, if μ is a multiplier of V^α, then according to (6.1),

$$|\mu(x_0, t_0)|\,\|\delta_P\|_{V^\alpha} = \|\mu\delta_P\|_{V^\alpha} \leqq C_\mu\|\delta_P\|_{V^\alpha}.$$

Thus, $\mu\in B(\mathbf{R}^{n+1}_+)$.

The same argument can be used to show that any continuous multiplier of $\tau_1^{p,q}$ must be bounded. We also readily see that $M_{T^p_\infty}=B(\mathbf{R}^{n+1}_+)$.

We introduce now two weighted maximal functions, which are variants of the one considered in [13].

Let μ be a function defined on $\mathbf{R}^{n+1}_+$. Given $f\in T^p_\infty$, $0<p\leqq 1$, let

$$N_\mu(f)(x) = \sup_{t>0}|\mu(x,t)f(x,t)|.$$

If φ is a function in the Schwartz class $S(\mathbf{R}^n)$ such that $\int\varphi(x)\,dx=1$, let $\varphi_t(x)=t^{-n}\varphi(x/t)$. Given $f\in H^p(\mathbf{R}^n)$, $0<p\leqq 1$, define

$$N_\mu^\varphi(f)(x) = \sup_{t\geqq 0}|\mu(x,t)(\varphi_t*f)(x)|.$$

(6.3) Theorem. *Let* $0<p\leqq 1$. *The following conditions are equivalent*

(i) $N_\mu\colon T^p_\infty \to L^p$, *boundedly.*

(ii) $\sup 1/|B| \int_B (\sup_{t<r_B} |\mu(x,t)|)^p dx < \infty$, *where the* sup *is taken over all balls and* r_B=*radius of* B.

Proof. Suppose that (i) holds. Given a ball B, let f be a continuous function such that $\chi_{T(B)} \leqq f \leqq \chi_{T(2B)}$ where $2B$ denotes the ball with the same center as B and radius $=2r_B$. We have,

$$\chi_B(x)\sup_{t<r_B} |\mu(x,t)| = \sup_{t>0} |\mu(x,t)\chi_{T(B)}(x,t)| \leqq N_\mu(f)(x).$$

Thus,

$$\int_B (\sup_{t<r_B} |\mu(x,t)|)^p\,dx \leqq c_\mu \|\chi_{T(2B)}\|^p_{T^p_\infty} = c_\mu |B|$$

which is (ii).

Conversely, suppose that μ satisfies (ii). Let f be a p-atom supported in $T(B)$, for some ball B. Then,

$$N_\mu(f)(x) \leqq \frac{1}{|B|^{1/p}} \sup_{t>0} |\mu(x,t)\chi_{T(B)}(x,t)|$$

$$\leqq \frac{\chi_B(x)}{|B|^{1/p}} \sup_{t<r_B} |\mu(x,t)|.$$

Consequently,

$$\|N_\mu(f)\|^p_p \leqq \sup_B \frac{1}{|B|} \int_B (\sup_{t<r_B} |\mu(x,t)|)^p\,dx$$

and (i) follows.

(6.4) Corollary. *Suppose that* μ *satisfies condition* (ii) *of* (6.9). *Then,* N^φ_μ *defines a bounded operator,* $N^\varphi_\mu\colon H^p \to L^p$, $0<p\leqq 1$.

Proof. Indeed, if $f\in H^p$, then $\varphi_t * f \in T^p_\infty$ (see [16], p. 183). Consequently, $N^\varphi_\mu(f) = N_\mu(\varphi_t * f)$ and (6.4) follows from (6.3).

B. Interpolation of H^p spaces and Lipschitz spaces

As pointed out in [13] the theory of tent spaces can be used to derive the interpolation theory (real and complex) of H^p spaces. In fact, using a "trick" of A. P. Calderón (cf. [6]) we can "invert" the π_φ operators (cf. [13]). Let $\varphi\in C^\infty_0(\mathbf{R}^n)$ be radial satisfying all the conditions of [13, p. 328]. Moreover, suppose φ satisfies $\int \varphi(\xi t)|\xi| e^{-2\pi|\xi|t}\,dt = -\frac{1}{2\pi}$, $\forall \xi\neq 0$. Then for sufficiently regular f, $\pi_\varphi\left(t\frac{d}{dt}P_t f\right)=f$, where $P_t f$= Poisson integral of f. (This can be easily checked using Fourier transforms). Now, one observes that $f \in H^p \Leftrightarrow t\dfrac{d}{dt}(P_t f)\ P_t f \in T_2^p$ to obtain the interpolation theorems for H^p spaces from the ones for T_2^p spaces.

The complex interpolation theory for the T_∞^p and τ_∞^p spaces can be readily derived from the results of §3, §4 and the methods of [20].

(6.5) Theorem. *Let* $1<p<p_0<\infty$, $0<\theta<1$, *then*

(i) $[T_\infty^p, T_\infty^{p_0}]_\theta = T_\infty^{p_\theta}$, $\dfrac{1}{p_\theta} = \dfrac{1-\theta}{p} + \dfrac{\theta}{p_0}$.

(ii) $[\tau_1^p, \tau_1^{p_0}]_\theta = \tau_1^{p_\theta}$, $\dfrac{1}{p_\theta} = \dfrac{1-\theta}{p} + \dfrac{\theta}{p_0}$.

Proof. We shall only provide the details of the proof of (ii). (The proof of (i) is similar.)

Using (i), and our duality theorem (5.1) (ii) we have

$$\tau_1^{p_\theta} = [T_\infty^{p'}, T_\infty^{p_0'}]_\theta^* = [\tau_1^p, \tau_1^{p_0}]_\theta.$$

C. Balayages

Let P_t be a kernel on $\mathbf{R}^n \times \mathbf{R}^n$ (see §2). The balayage $P*w$ is defined formally by (cf. [2])

$$(P*w)(y) = \int_{\mathbf{R}_+^{n+1}} P_t(x, y)\, dw(x, t) \tag{6.6}$$

where w is a measure on $\overline{\mathbf{R}}_+^{n+1}$.

Therefore, we have the duality

$$\int_{\mathbf{R}_+^{n+1}} P_t f(x)\, dw(x, t) = \int_{\mathbf{R}^n} f(y)(P*w)(y)\, dy. \tag{6.7}$$

In particular, if $P_t f$ is the Poisson integral of f, we have

$$P_t\colon H^p \to T_\infty^p, \quad 0<p\leqq 1 \tag{6.8}$$

and by duality,

$$P*\colon V^{1/p} \to \operatorname{Lip} n\left(\frac{1}{p}-1\right), \quad 0<p<1, \tag{6.9}$$

where the Lip space is modulo polynomials of degree $\left[n\left(\frac{1}{p}-1\right)\right]$. More precisely, in order to give a sense to (6.6) it is necessary in general to suppose that the measure w is finite.

Let us observe that (6.9) also holds for any kernel P_t satisfying (6.8). In particular, we will give a direct proof of the following result.

(6.10) Theorem. *Let P_t be a kernel on $\mathbf{R}^n \times \mathbf{R}^n$ satisfying for some $N=1, 2, 3, \dots$, the following conditions:*

(i) $|D_y^\beta P_t(x, y)| \leqq ct(t+|x-y|)^{-n-1-|\beta|}, \quad |\beta| \leqq N-1$

(ii) $\left|D_y^\beta P_t(x, y) - \sum_{|\gamma+\beta|<N} D_y^{\gamma+\beta} P_t(x, y_0) \frac{(y-y_0)^\gamma}{\gamma!}\right| \leqq c \frac{|y-y_0|^N}{(t+|x-y_0|)^{n+N+|\beta|}}$

if $t+|x-y_0| > 2|y-y_0|, \ |\beta| \leqq N-1.$

Let $w \in V^\alpha$, $\alpha \geqq 1$ be a finite measure. Then, if $1+\frac{N-1}{n}<\alpha<1+\frac{N}{n}$, the balayage $P*w$ of w belongs to Lip $n(\alpha-1)$.

Proof. Let us first show that $P*w$ is well defined a.e. In fact, condition (6.10) (i) with $\beta=0$ implies that $\sup_{x,t} \int |P_t(x, y)|\, dy < \infty$. Since w is a finite measure, we deduce that

$$\int_{\mathbf{R}^n} |P*w(y)|\, dy \leqq |w|(\mathbf{R}_+^{n+1})$$

which implies that $P*w$ is finite a.e. Now, given $|\beta|=N-1$, we will show that $D^\beta P*w$ is a bounded function satisfying a Lipschitz condition

$$|D^\beta(P*w)(y) - D^\beta(P*w)(y_0)| \leqq c|y-y_0|^{n(\alpha-1)-N+1}.$$

If B denotes $B(y, 1)$, we see that (see [2], p. 33),

$$\left|\int_{T(B)} D_y^\beta P_t(x, y)\, dw(x, t)\right| + \left|\int_{T(B)} D_y^\beta P_t(x, y)\, dw(x, t)\right| < \infty.$$

According to (6.10) (i), since $\alpha n > n+N-1$, we can appeal to Lemma 2(1) in [2] to deduce that the first term is bounded. In $T(B)^c$ we have $t+|x-y|>1$; thus, the second integral can be majorized by $c|w|(\mathbf{R}_+^{n+1}) < \infty$.

Finally, let us verify the Lipschitz condition. Fix y, y_0 and let $B=B(y_0, 2|y-y_0|)$. Then, we estimate

$$\int_{T(B)} [D_y^\beta P_t(x, y) - D_y^\beta P_t(x, y_0)]\, dw(x, t) + \int_{T(B)^c} [D_y^\beta P_t(x, y) -$$

$$- D_y^\beta P_t(x, y_0)]\, dw(x, t) = (I)+(J).$$

The term (I) can be majorized by the sum of two integrals over tents

$$(I) \leqq \int_{T(B(y,3|y-y_0|))} |D_y^\beta P_t(x,y)|\,d|w|(x,t)$$
$$+\int_{T(B)} |D_y^\beta P_t(x,y_0)|\,d|w|(x,t)$$

and each of these integrals can be estimated in the same way using (6.10) (i) and Lemma 2 in [2]. In fact, let us consider the first one, say (I_1).

$$(I_1) \leqq c\int_{T(B(y,3|y-y_0|))} (t+|x-y|)^{-n-N+1}\,d|w|(x,t).$$

Since $\alpha n > n+N-1$, we are again in the conditions of Lemma 2(2) in [2] and consequently the last integral is bounded by a constant times $|y-y_0|^{n(\alpha-1)-N+1}$.

To estimate (J), we observe that over $T(B)^c$ we have $t+|x-y_0|>2|y-y_0|$ and therefore, we can use (6.10) (ii), to get

$$(J) \leqq c|y-y_0|^N \int_{T(B)^c} (t+|x-y_0|)^{-n-2N+1}\,d|w|(x,t).$$

Since this time $\alpha n < n+N \leqq n+2N-1$, we can apply Lemma 2(1) in [2], to obtain $(J) \leqq c|y-y_0|^{(\alpha-1)n-N+1}$. This completes the proof of the theorem.

(6.11) Remark. When $\frac{1}{p}$ in (6.9) has the critical values $\frac{N+n}{n}$, $N=1,2,\ldots$, one gets that $D^\beta(P*w)$, $|\beta|=N-1$, belongs to the Zygmund space Lip. 1 However, when $N\geqq 2$, the same technique of Theorem (6.10) gives the improvement

$$|D^\beta(P*w)(y)-D^\beta(P*w)(y_0)| \leqq c|y-y_0|.$$

Let us observe that the hypothesis that w is finite can be dropped in (6.10) by considering a regularization of the balayage (see [18]). More precisely, if $1+\frac{N-1}{n} < \alpha < 1+\frac{N}{n}$, we consider for $|\beta| \leqq N-1$,

$$\int \Big[D_y^\beta P_t(x,y) - \sum_{|\beta+\gamma|<N} D_y^{\beta+\gamma} P_t(x,y_0)\frac{(y-y_0)^\gamma}{\gamma!}\chi_{T(\bar{B})^c}(x,t)\Big]\,dw(x,t),$$

where $B=B(y_0,r_0)$, $\bar{B}=B(y_0,2r_0)$. This integral can be seen to be finite for a.e. $y\in B$. In fact,

$$\int_{T(\bar{B})} |D_y^\beta P_t(x,y)|\,d|w|(x,t) \leqq c\int_{T(B(y,3r_0))} (t+|x-y|)^{-n-|\beta|}\,d|w|(x,t).$$

Since $\alpha n > n+N-1 \geqq n+|\beta|$, we can apply Lemma 2(2) in [2] to estimate it by $cr_0^{(\alpha-1)n-|\beta|}$. On the other hand,

$$\left|\int_{T(B)^c}\left[D_y^\beta P_t(x, y)-\sum_{|\beta+\gamma|<N} D_y^{\beta+\gamma} P_t(x, y_0)\frac{(y-y_0)^\gamma}{\gamma!}\right] dw(x, t)\right|$$

$$\leqq c\int_{T(B)^c} |y-y_0|^N (t+|x-y_0|)^{-n-N-|\beta|}\, d|w|(x, t).$$

This time we have $\alpha n < n+N \leqq n+N+|\beta|$. Thus, applying Lemma 2(1) in [2], the last term is bounded by

$$c|y-y_0|^N r_0^{(\alpha-1)n-N-|\beta|}.$$

Taking $r_0=|y-y_0|$, these calculations also show that the increment can be majorized by $c|y-y_0|^{(\alpha-1)n-N+1}$, when $|\beta|=N-1$.

7. Carleson measures on product spaces

The theory developed in this paper can be easily adapted to the study of certain tent spaces and spaces of Carleson measures on product spaces. A theory of Carleson measures on product spaces has been developed by L. Carleson, S. Y. A. Chang, R. Fefferman (cf. [8], [9], [10], [11]).

In this section we shall outline an extension of some of our results to the two parameter setting. A more detailed theory of tent spaces on product spaces will be developed elsewhere,

We will work on $\mathbf{R}_+^m \times \mathbf{R}_+^n = \{(x, t) | x=(x_1, x_2)\in \mathbf{R}^m\times\mathbf{R}^n,\ t=(t_1, t_2)\in\mathbf{R}^2,\ t_1, t_2>0\}$. Given $(x, t)\in \mathbf{R}_+^m\times\mathbf{R}_+^n$, $B(x, t)$ stands for the product of balls $B(x_1, t_1)\times B(x_2, t_2)\subset \mathbf{R}^m\times\mathbf{R}^n$. Moreover, given $x\in\mathbf{R}^m\times\mathbf{R}^n$, $\Gamma(x)$ will denote the product of cones $\Gamma(x_1)\times \Gamma(x_2)\subset\mathbf{R}_+^m\times\mathbf{R}_+^n$.

Given a subset $\Omega\subset\mathbf{R}^m\times\mathbf{R}^n$, the tent over Ω, $T(\Omega)$, is defined as

$$\{(x, t)\in\mathbf{R}_+^m\times\mathbf{R}_+^n | B(x, t)\subset\Omega\}.$$

(7.1) Definition. In accordance with the one parameter case, we define

$$A_\infty(f)(x) = \sup_{\Gamma(x)} |f(y, t)|.$$

For $0<p<\infty$, $0<r\leqq\infty$, $T_\infty^{p,r}$ denotes the class of functions f which are continuous in $\mathbf{R}_+^m\times\mathbf{R}_+^n$, for which $A_\infty(f)(x)\in L(p, r)$ and $\|A_\infty(f_\varepsilon-f)\|_{p,r}\to 0$, as $\varepsilon\to 0$, where $f_\varepsilon(x, t)=f(x, t+\varepsilon)$, $\varepsilon=(\varepsilon_1, \varepsilon_2)$, $\varepsilon_1, \varepsilon_2>0$. These spaces are normed spaces for $1<p<\infty$, $1\leqq r\leqq\infty$.

As in the one parameter case, we have the inequality

$$|f(x,t)| \leqq \inf_{y\in B(x,t)} A_\infty(f)(y).$$

By integration on $B(x,t)$, we get

$$(7.2) \qquad |f(x,t)| \leqq C t_1^{-m/p} t_2^{-n/p} \|A_\infty(f)\|_p.$$

This shows that T^p_∞ is a complete space.

The spaces T^p_∞, $0<p\leqq 1$, can be characterized through an atomic decomposition. First, a suitable notion of atom is needed.

A function $a(y,t)$ is a p-atom if there exists an open subset $\Omega\subset\mathbf{R}^m\times\mathbf{R}^n$ of finite measure such that

(i) $\operatorname{supp}(a)\subset T(\Omega)$,

(ii) $\|a\|_\infty\leqq|\Omega|^{-1/p}$.

It is readily seen that a p-atom a belongs to T^p_∞ and $\|A_\infty(a)\|_p\leqq 1$.

(7.3) Theorem. *Given a function f defined in $\mathbf{R}^m_+\times\mathbf{R}^n_+$, the following statements are equivalent:*

a) $f\in T^p_\infty$,

b) $f=\sum\lambda_k a_k$ *in the sense of* $\mathscr{D}'(\mathbf{R}^m_+\times\mathbf{R}^n_+)$, *where a_k are p-atoms,* $\sum|\lambda_k|^p\leqq c\|A_\infty(f)\|^p$.

Moreover, $\|f\|^p_{T^p_\infty}\approx\inf\{\sum|\lambda_k|^p : f=\sum\lambda_k a_k\}$, *as in the one parameter case.*

Proof. We only need to show a)$\Rightarrow$b). First observe that given a function f defined in $\mathbf{R}^m_+\times\mathbf{R}^n_+$, we have for each $\lambda>0$,

$$(7.4) \qquad \{(x,t)\,|\,|f(x,t)|>\lambda\}\subset T\{x\,|\,A_\infty(f)(x)>\lambda\}.$$

Now, given $f\in T^p_\infty$, let us consider for each $k\in\mathbf{Z}$, $O_k=\{x|A_\infty(f)>2^k\}$. O_k is an open subset of $\mathbf{R}^m\times\mathbf{R}^n$ of finite measure and $O_{k+1}\subset O_k$. Let $\mathcal{O}=\bigcup_k T(O_k)$. From (7.4) it follows that $\operatorname{supp}(f)\subset\mathcal{O}$. Now, let $\mathcal{O}_k=T(O_k)\setminus T(O_{k+1})$. The sets $\mathcal{O}_k$ are a disjoint covering of $\operatorname{supp}(f)$. Moreover, on $\mathcal{O}_k$ we have $|f(x,t)|\leqq 2^{k+1}$. Thus, we can write $f=\sum\lambda_k a_k$, where $\lambda_k=2^{k+1}|O_k|^{1/p}$, $a_k=2^{-k-1}|O_k|^{1/p}\chi_{\mathcal{O}_k}f$. It is easy to see that a_k is a p-atom supported on $\mathcal{O}_k$. Moreover, let $\Delta_j=O_j\setminus O_{j+1}$. Then, $O_k=\bigcup_{j=k}^\infty\Delta_j$ and

$$\begin{aligned}
\sum_{k=-\infty}^\infty \lambda_k^p &= 2^p\sum_{k=-\infty}^\infty 2^{pk}|O_k| \\
&= 2^p\sum_{k=-\infty}^\infty 2^{pk}\sum_{j=k}^\infty|\Delta_j| \\
&\leqq 2^p\sum_{k=-\infty}^\infty 2^{pk}\sum_{j=k}^\infty 2^{-pj}\int_{\Delta_j} A_\infty(f)(x)^p\,dx \\
&= 2^p\sum_{j=-\infty}^\infty 2^{-pj}\sum_{k=-\infty}^{j} 2^{pk}\int_{\Delta_j} A_\infty(f)(x)^p\,dx \\
&= c\sum_{j=-\infty}^\infty\int_{\Delta} A_\infty(f)(x)^p\,dx \\
&= c\|A_\infty(f)\|^p_{L^p}.
\end{aligned}$$

This completes the proof of the theorem.

We want now to identify the duals of the spaces T^p_∞, $T^{p,1}_\infty$. In analogy with the one parameter case, a suitable notion of Carleson measure in the product space $\bar{\mathbf{R}}^m_+ \times \bar{\mathbf{R}}^n_+$ plays an important role.

(7.5) Definition. Given a measure w in $\bar{\mathbf{R}}^m_+ \times \bar{\mathbf{R}}^n_+$, the A_1 functional is defined as

$$A_1(w)(x) = \int_{\Gamma(x)} \frac{d|w|(x, t)}{t_1^m t_2^n}.$$

For $\alpha \geqq 0$, we also define

$$C_{1,\alpha}(w)(x) = \sup \frac{1}{|\Omega|^\alpha} \int_{T(\Omega)} d|w|(y, t)$$

where the supremum is taken over all the open subsets of $\mathbf{R}^m \times \mathbf{R}^n$ of finite measure containing x. For $0<p,\ \alpha<\infty$, let $\tau^p_1 = \{w|A_1(w)\in L^p\}$, $V^\alpha = \{w|C_{1,\alpha}(w)\in L^\infty\}$.

According to [8], V^1 is the space of Carleson measures. If P_t denotes the tensor product of the usual Poisson kernels in $\mathbf{R}^m_+$ and $\mathbf{R}^n_+$, we have the following characterization of V^1. A measure $w\in V^1$ if and only if (see [19])

$$(7.6) \qquad \int_{\mathbf{R}^m_+\times\mathbf{R}^n_+} |P_t(f)((y)^p\, d|w|(y, t) \leqq c_p \int_{\mathbf{R}^m\times\mathbf{R}^n} |f(y)|^p\, dy \quad 1 \leqq p < \infty.$$

The functionals A_∞ and $C_{1,\alpha}$ are related by the following inequality

$$(7.7) \qquad \int |f(y, t)|^{1/\alpha}\, d|w|(y, t) \leqq \int A_\infty(f)(x)\, C_{1,\alpha}(w)(x)^{1/\alpha}\, dx.$$

In fact, the proof follows readily from (7.4).

In the spirit of (7.6) we also have a characterization of V^α, $0<\alpha<1$. In fact, a measure $w\in V^\alpha$ if and only if

$$(7.7) \qquad \int |P_t(f)(y)|\, d|w|(y, t) \leqq c\|f\|_{1/\alpha,1}.$$

Finally, a measure $w\in\tau^p_1$, $1<p<\infty$, if and only if

$$(7.8) \qquad \int_{\mathbf{R}^m_+\times\mathbf{R}^n_+} |P_t(f)(y)|\, d|w|(y, t) \leqq c\left(\int_{\mathbf{R}^m\times\mathbf{R}^n} |f(x)|^{p'}\, dx\right)^{1/p'}.$$

All the above assertions follow readily as in the one parameter case.

(7.9) Theorem.

a) *Let* $0<p\leqq 1$, *then* $(T^p_\infty)^* = V^{1/p}$.
Let $1<p<\infty$, *then*
b) $(T^p_\infty)^* = \tau^{p'}_1$.
c) $(T^{p,1}_\infty)^* = V^{1/p}$.

Proof. The only change needed in the theory of §5 is to replace in the proof of (5.1) (ii) the maximal operator of Hardy Littlewood by the strong maximal operator.

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