A note on the Carleman condition for determinacy of moment problems

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0. Introduction

A distribution function is a real-valued, bounded and non-decreasing function ψ defined on some subinterval of the real line. The classical Stieltjes moment problem is defined as follows: Given a sequence $(\mu_n)_0^\infty$ of real numbers, find a distribution function ψ on $[0, \infty[$ such that

$$\mu_n = \int_0^\infty x^n \, d\psi(x), \quad n = 0, 1, \dots . \tag{0.1}$$

We get the Hamburger moment problem if ψ is a distribution function on $]-\infty, \infty[$ and (0.1) is replaced by

$$\mu_n = \int_{-\infty}^\infty x^n \, d\psi(x), \quad n = 0, 1, \dots. \tag{0.2}$$

These problems were studied by Stieltjes [8] and Hamburger [6], respectively. A general treatment of moment problems is found in [1] and [7]. See also [2].

A moment problem is said to be determinate if it has at most one solution ψ. Otherwise it is indeterminate. Carleman gave sufficient conditions for determinacy of the Stieltjes and Hamburger moment problems.

Theorem (Carleman [3,5]). *Let $(\mu_n)_0^\infty$ be a sequence of non-negative numbers satisfying*

$$\sum_1^\infty \frac{1}{\beta_n} = \infty, \quad \text{where} \quad \beta_n = \inf_{k \geqq n} \sqrt[2k]{\mu_k}. \tag{0.3}$$

Then the Stieltjes problem (0.1) *is determinate.*

We will consider weighted generalizations of the Carleman condition (0.3). Let $(C_n)_0^\infty$ be a sequence of non-negative numbers and denote by $PS(C_n)$ the following proposition:

If $(\mu_n)_0^\infty$ is any sequence of non-negative numbers satisfying

(0.4) $$\sum_1^\infty \frac{C_n}{\sqrt[2n]{\mu_n}} = \infty,$$

then the Stieltjes problem (0.1) is determinate.

It is the purpose of this paper to characterize the sequences $(C_n)_0^\infty$ for which $PS(C_n)$ is true. This characterization is given in Theorem 1.1 in Section 1 and proved in Section 2. In Section 3 we give the corresponding result for the Hamburger problem.

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1. Main result

Our main result is contained in the following theorem.

Theorem 1.1. *Let* $(C_n)_0^\infty$ *be a sequence of non-negative numbers.*

(i) *If*

(1.1) $$\sup_n \frac{1}{n} \cdot \sum_0^{n-1} C_k = \infty,$$

then $PS(C_n)$ *is false.*

(ii) *If the supremum in* (1.1) *is finite then* $PS(C_n)$ *is true with* (0.4) *replaced by*

(1.2) $$\sum_1^\infty \frac{C_n}{\beta_n} = \infty,$$

where β_n *is defined as in* (0.3).

Define $PS'(C_n)$ by replacing the condition (0.4) in $PS(C_n)$ by

$$\sum_1^\infty \frac{C_n}{\beta_n} = \infty$$

where β_n is defined as in (0.3).

Corollary. *The statements* (a)—(c) *are equivalent,*

(a) $PS(C_n)$ *is true,*

(b) $PS'(C_n)$ *is true,*

(c) *the supremum in* (1.1) *is finite.*

The corollary is an easy consequence of Theorem 1.1 and the definition of β_n.

The proof of the first part of the theorem uses results by Carleman on quasi analytic functions. In particular his construction of indeterminate Stieltjes problems in [4]. The condition (1.1) comes from Lemma 2.1 (Section 2) where it is shown to be equivalent to the existence of a certain non-increasing sequence of numbers.

Theorem 1.1 says that Carleman's theorem is sharp in a certain sense. Our next theorem expresses this fact in a different way.

Theorem 1.2. *Let* $(\mu_n)_0^\infty$ *be a sequence of non-negative numbers and define* β_n *as in* (0.3). *Then there exists* $(\lambda_n)_0^\infty$ *which generates an indeterminate Stieltjes problem and satisfies*

$$\lim_{n\to\infty} \sqrt[n]{\lambda_n/\mu_n} = 0$$

if and only if $\sum_1^\infty 1/\beta_n < \infty$.

Remark. A similar result is that if $(\mu_n)_0^\infty$ satisfies $\sum_1^\infty 1/\beta_n = \infty$ then there is a sequence $(\lambda_n)_0^\infty$, which also generates a determinate Stieltjes problem and satisfies $\lim_{n\to\infty} \sqrt[n]{\lambda_n/\mu_n} = \infty$.

2. Proof of Theorems 1.1 and 1.2

We will need the following lemma in the proof of Theorem 1.1.

Lemma 2.1. *Let* $(a_n)_1^\infty$ *be a sequence of non-negative numbers. Then there exists a non-negative and non-increasing sequence* $(b_n)_1^\infty$ *such that*

$$\sum_1^\infty a_n b_n = \infty \quad and \quad \sum_1^\infty b_n < \infty \tag{2.1}$$

if and only if

$$\sup_n \frac{1}{n} \cdot \sum_1^n a_k = \infty. \tag{2.2}$$

Proof. We first assume that (2.2) holds. Let $1 = n_1 < n_2 < \ldots < n_k < \ldots$ be integers to be specified below and define

$$A_k = \sum_{n_k \leqq j < n_{k+1}} a_j \quad \text{and} \quad m_k = (n_{k+1} - n_k)^{-1} \cdot A_k,$$

for $k = 1, 2, \ldots$. Put $b_j = (n_{k+1} - n_k)^{-1} \cdot m_k^{-1/2}$, $n_k \leqq j < n_{k+1}$, $k = 1, 2, \ldots$. Then

$$\sum_1^\infty a_j b_j = \sum_1^\infty \sqrt{m_k} \quad \text{and} \quad \sum_1^\infty b_j = \sum_1^\infty 1/\sqrt{m_k}. \tag{2.3}$$

By assumption (2.2) we can choose $(n_k)_1^\infty$ such that $(A_k)_1^\infty$ is nondecreasing and $(m_k)_1^\infty$ increases arbitrarily fast. Then by (2.3) the sequence $(b_j)_1^\infty$ has the required properties.

Conversely assume that the supremum in (2.2) is finite and equals M and that the sequence $(b_n)_1^\infty$ exists. Summation by parts then gives

$$\sum_1^n a_k b_k = A_n b_n + \sum_1^{n-1} A_k (b_k - b_{k+1}) \leqq M \sum_1^n b_k,$$

where $A_k = \sum_1^k a_j$. Letting n tend to infinity this contradicts (2.1) and completes the proof of the lemma.

Proof of Theorem 1.1. To prove (i) we assume that (1.1) holds. We must show the existence of a sequence $(\mu_n)_0^\infty$ satisfying (0.4) and which generates an indeterminate Stieltjes problem.

Lemma 2.1 implies that there exists a non-increasing sequence $(b_n)_0^\infty$ of non-negative numbers such that $\sum_0^\infty C_n b_n = \infty$ and $\sum_0^\infty b_n < \infty$. Put $M_n = (1/b_n)^n$, $n = 0, 1, \ldots$. Then by the Denjoy—Carleman Theorem [3, p. 422] the class $C\{M_n\}$ is not quasi analytic. See also [5]. Hence there exists an infinitely differentiable function f, not identically zero, such that $f^{(n)}(0) = f^{(n)}(1) = 0$, $n = 0, 1, \ldots$, and $|f^{(n)}(x)| \leqq k^n \cdot M_n$, $n = 0, 1, \ldots$, for all $0 \leqq x \leqq 1$ and some constant k. Put $\mu_n = \int_0^1 (f^{(n)}(x))^2\, dx$, for $n = 0, 1, \ldots$. Carleman [4, p. 188] proved that $(\mu_n)_0^\infty$ generates an indeterminate Stieltjes problem. The estimate

$$\sqrt[2n]{\mu_n} \leqq k \sqrt[n]{M_n} = \frac{k}{b_n}, \quad n = 1, 2, \ldots$$

implies that (0.4) holds and thereby part (i) is proved.

To prove part (ii) assume that (1.2) holds. Summation by parts gives

$$\sum_1^N \frac{C_n}{\beta_n} = S_N \frac{1}{\beta_N} + \sum_1^{N-1} S_n(\beta_n^{-1} - \beta_{n+1}^{-1}) \leqq M \sum_1^N \frac{1}{\beta_n},$$

where $S_n = \sum_1^n C_k$. It follows that the Carleman condition (0.3) is satisfied and hence the Stieltjes problem (0.1) is determinate. This completes the proof of Theorem 1.1.

The proof of Theorem 1.2 follows the same lines and we leave some of the details to the reader.

Proof of Theorem 1.2. To prove necessity let $(\mu_n)_0^\infty$ be given and assume that $(\lambda_n)_0^\infty$ exists with the stated properties. Then for some constant A

$$\alpha_n = \inf_{k \geqq n} \sqrt[2k]{\lambda_k} \leqq A \cdot \beta_n.$$

This gives $\sum 1/\beta_n \leqq A \cdot \sum 1/\alpha_n < \infty$ by Carleman's theorem, since $(\lambda_n)_0^\infty$ generates an indeterminate Stieltjes problem.

Next we assume that $\sum 1/\beta_n$ is convergent. There exists a non-negative and non-increasing sequence $(b_n)_0^\infty$ such that $\sum_0^\infty b_n < \infty$ and $\lim_{n \to \infty} b_n \cdot \beta_n = \infty$. Define $M_n = b_n^{-n}$. Then $C\{M_n\}$ is not quasi analytic and we can find an infinitely differentiable function f, not identically zero, with the same properties as in the proof of Theorem 1.1. Define $\lambda_n = \int_0^1 (f^{(n)}(x))^2\, dx$, $n = 0, 1, \ldots$. Then $(\lambda_n)_0^\infty$ generates an indeterminate Stieltjes problem and

$$\sqrt[n]{\lambda_n/\mu_n} \leqq k^2 \cdot (b_n \cdot \beta_n)^{-2}.$$

We conclude that $(\lambda_n)_0^\infty$ has the stated properties. Hence the condition $\sum 1/\beta_n < \infty$ is also sufficient and the proof is complete.

3. The Hamburger problem

The results in Section 1 carry over to the Hamburger problem. A sufficient condition for determinacy of the Hamburger problem is $\sum_1^\infty 1/\beta_n=\infty$, where $\beta_n=\inf_{k\geq n}\sqrt[2k]{\mu_{2k}}$. See [3] and [5]. For any sequence $(C_n)_0^\infty$ of non-negative numbers we denote by $PH(C_n)$ the following proposition:

If $(\mu_n)_0^\infty$ is any sequence of real numbers satisfying

$$\sum_1^\infty \frac{C_n}{\sqrt[2n]{\mu_{2n}}}=\infty \tag{3.1}$$

then the Hamburger problem (0.2) is determinate.

In complete analogy with Theorem 1.1 we have the following result.

Theorem 3.1. *Let* $(C_n)_0^\infty$ *be a sequence of non-negative numbers.*

(i) *If* (1.1) *holds then* $PH(C_n)$ *is false.*

(ii) *If the supremum in* (1.1) *is finite then* $PH(C_n)$ *is true with* (3.1) *replaced by*

$$\sum_1^\infty \frac{C_n}{\beta_n}=\infty,$$

where $\beta_n=\inf_{k\geq n}\sqrt[2k]{\mu_{2k}}$.

Proof. Every indeterminate Stieltjes problem can be transformed into an indeterminate Hamburger problem with the same moments, see [5, p. 81] or [7, p. 19]. Hence the first part of the theorem follows from Theorem 1.1. The second part is proved analogously to part (ii) of Theorem 1.1. The proof is complete.

Let $PH'(C_n)$ be the proposition $PH(C_n)$ with (3.1) replaced by

$$\sum_1^\infty \frac{C_n}{\beta_n}=\infty,$$

where β_n is defined as in the theorem.

Corollary. *The statements* (a)—(c) *are equivalent,*

(a) $PH(C_n)$ *is true,*

(b) $PH'(C_n)$ *is true,*

(c) *the supremum in* (1.1) *is finite.*

Also Theorem 1.2 has an obvious generalization to the Hamburger problem. We leave its statement and proof to the reader.

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