On the frequency of Titchmarsh's phenomenon for $\zeta(s)$-V

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1. Introduction

In a series of papers with the same title (as above) I, II, III, IV ([6], [7], [2] and [1]) we studied (sometimes jointly and sometimes individually) the quantity

$$\max_{T \leqq t \leqq T+H} |\zeta(\alpha+it)|$$

where $C \leqq \log\log T \leqq H \leqq T$, α is fixed in $\frac{1}{2} \leqq \alpha \leqq 1$ and C is a large positive constant. If $\alpha=1$ we could manage even with the condition $C \leqq \log\log\log T \leqq H \leqq T$. We also studied other quantities like

$$\min_{T \leqq t \leqq T+H} |\zeta(\alpha+it)| \quad \text{and} \quad \max_{\sigma \geqq \alpha,\, T \leqq t \leqq T+H} |\zeta(\sigma+it)|$$

and so on. In this paper we continue the investigations of the paper II. In II the second of us proved, amongst other things the following result. Let H be any preassigned quantity with $C \leqq \log\log T \leqq H \leqq \operatorname{Exp}\left(\frac{D \log T}{\log\log T}\right)$, where C is a large positive constant, α fixed $\left(\frac{1}{2}<\alpha<1\right)$, and D is a positive constant depending on α. Let I run over intervals (of length H) contained in $[T, 2T]$. Then

$$\log\log \left(\min_{I} \max_{t \in I} |\zeta(\alpha+it)|\right) \sim (1-\alpha) \log\log H.$$

Moreover in [8] the following result was proved amongst other things. Let $f_n = \max_{n \leqq t \leqq n+1} |\zeta(\alpha+it)|$ where n is an integer satisfying $T \leqq n \leqq 2T$. Then $\log f_n \leqq \frac{D_0 (\log T)^{1-\alpha}}{(\log\log T)^{\alpha}}$ with the exception of at most $0\left(\operatorname{Exp}\left(\log T - \frac{D_1 \log T}{\log\log T}\right)\right)$ values of n. Here D_1 is an arbitrary positive constant and D_0 is a positive constant depending on α and D_1. In [4] it is conjectured by H. L. Montgomery that for all

$t \geqq 1000$, $\log|\zeta(\alpha+it)|$ does not exceed a constant multiple of $\frac{(\log t)^{1-\alpha}}{(\log\log t)^{\alpha}}$. The object of this paper is to prove the following theorem. An idea in [2] is very useful in proving the theorem.

Theorem 1. *Let α be a fixed constant satisfying $\frac{1}{2}<\alpha<1$ and $E>1$ an arbitrary constant. Let $C \leqq H \leqq T/100$ and $K=\mathrm{Exp}\left(\frac{D\log H}{\log\log H}\right)$ where C is a large positive constant and D an arbitrary positive constant. Then there are $\geqq TK^{-E}$ disjoint intervals I of length K each contained in $[T, 2T]$ such that*

$$\frac{(\log K)^{1-\alpha}}{(\log\log K)^{\alpha}} \ll \max_{t\in I}|\log\zeta(\alpha+it)| \ll \frac{(\log K)^{1-\alpha}}{(\log\log K)^{\alpha}}.$$

Remark 1. Here $\log\zeta(s)$ is the analytic continuation along lines parallel to the σ-axis (we choose only those lines which do not contain a zero or pole of $\zeta(s)$) of $\log\zeta(s)$ in $\sigma \geqq 2$.

Remark 2. We can only prove that as I runs over all intervals of length K contained in $[T, 2T]$,

$$\frac{(\log K)^{1-\alpha}}{\log\log K} \ll \operatorname*{Min}_{I}\ \operatorname*{Max}_{t\in I}|\log\zeta(\alpha+it)| \ll \frac{(\log K)^{1-\alpha}}{(\log\log K)^{\alpha}}.$$

Hence for the logarithm of the middle quantity we have the asymptotic expression $(1-\alpha)\log\log K+0(\log\log\log K)$. We can also prove this formula for $\log\log(\mathrm{Min}_I\ \mathrm{Max}_{t\in I}|\zeta(\alpha+it)|)$.

Remark 3. Theorem 1 and the results mentioned in Remark 2 can be easily extended to ordinary L-series and also to abelian L-series of quadratic fields. They can also be extended to zeta-function of abelian extension of rationals and also zeta-function of abelian extension of quadratic fields. The last result mentioned in Remark 2 can be extended to $\sum_{n=1}^{\infty}(an+b)^{-s}$ where a and b are positive integers and also to zeta-function of a ray class in a quadratic field.

2. Outline of the proof

Let β_0, β_1 and β be constants satisfying $\frac{1}{2}<\beta_0<\beta_1<\beta<\alpha<1$. It is well-known that

$$\frac{1}{T}\int_T^{2T}\left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt = 0(\log T).$$

From this it follows that there are $\gg\frac{T}{H}$ disjoint intervals I_0 for t (ignoring a bit at one end) each of length $H+20(\log H)^2$ contained in $[T, 2T]$ for which

$$\int_{2\geqq\sigma\geqq\beta_0}\int_{t\in I_0}|\zeta(s)|^2\,dt\,d\sigma \ll H \tag{1}$$

and

$$\int_{t\in I_0}|\zeta(\beta_1+it)|^2\,dt \ll H. \tag{2}$$

From (1) and (2) it follows by standard methods that $N(\beta, I_0)\ll H^{1-\delta}$ where δ is a fixed positive constant depending only on α and β. Here $N(\beta, I_0)$ is the number of zeros ϱ of $\zeta(s)$ with $\operatorname{Re}\varrho\geqq\beta$ and $\operatorname{Im}\varrho$ lying in I_0. Hence if we divide I_0 into abutting intervals (ignoring a bit at one end) I_1 each of length $H^{\theta}+20(\log H)^2$ where $\theta=\delta/2$, the number of intervals I_1 is $\sim H^{1-\theta}$. Out of these we omit those I_1 for which $(\sigma\geqq\beta,\ t \text{ in } I_1)$ contains a zero of $\zeta(s)$. (They are not more than a constant times $H^{1-\delta}$). We now consider a typical interval I_1 which is such that $(\sigma\geqq\beta,$ t in $I_1)$ is zero-free. Let us designate this t interval by $[T_0-10(\log H)^2, T_0+H^{\theta}+10(\log H)^2]$. Put

$$H_1 = H^{\theta} \quad\text{and}\quad k=\left[\frac{C_1\log H}{\log\log H}\right], \tag{3}$$

where C_1 is a small positive constant. Then we prove the following theorem.

Theorem 2. *We have*

$$\int_{T_0}^{T_0+H_1}|\log\zeta(\alpha+it)|^{2k}\,dt > C_2^k A_k^{2k} H_1, \tag{4}$$

and

$$\int_{T_0}^{T_0+H_1}|\log\zeta(\alpha+it)|^{4k}\,dt < C_3^{2k}A_{2k}^{4k} < 2^{4k}C_3^{2k}A_k^{4k}H_1, \tag{5}$$

where $A_k=\dfrac{k^{1-\alpha}}{(\log k)^{\alpha}}$, *and C_2 and C_3 are positive constants independent of* C_1.

Corollary. *Divide $[T_0, T_0+H_1]$ into equal (abutting) intervals I each of length K (neglecting a bit at one end). Then the number N of intervals I for which*

$$\int^{I}|\log\zeta(\alpha+it)|^{2k}\,dt > \tfrac{1}{4}C_2^kA_k^{2k}K \tag{6}$$

satisfies $N\geqq -1+\dfrac{1}{16}\left(\dfrac{C_2}{4C_3}\right)^{2k}\dfrac{H_1}{K}$ *and so in these intervals*

$$\max_{t\in I}|\log\zeta(\alpha+it)| \gg A_k.$$

Proof. Put $J=\int_{t\in I}|\log\zeta(\alpha+it)|^{2k}\,dt$. Then

$$\textstyle\sum_I J > \frac{1}{2}C_2^kA_k^{2k}H_1$$

since the contribution from the neglected bit is not more than $K^{1/2}(2^{4k}C_3^{2k}A_k^{4k}H_1)^{1/2}$ on using (5). Let $\sum_2 J=\sum_{I,J>\frac{1}{4}C_2^kA_k^{2k}K} J$ and $\sum_1$ the sum over the remaining intervals I. Then

$$\sum_2 J > \tfrac{1}{4} C_2^k A_k^{2k} H_1.$$

Put $\sum_2 1=N$. Then by Hölder's inequality we have

$$\begin{aligned}\tfrac{1}{4} C_2^k A_k^{2k} H_1 &< N^{1/2}(\textstyle\sum_2 J^2)^{1/2}\\ &\leqq N^{1/2}(\textstyle\sum_2 \int_I |\log\zeta(\alpha+it)|^{4k}\,dt\,K)^{1/2}\\ &\leqq N^{1/2}K^{1/2}(2^{4k}C_3^{2k}A_k^{4k}H_1)^{1/2}.\end{aligned}$$

Hence $N>\dfrac{1}{16}\left(\dfrac{C_2}{4C_3}\right)^{2k}\dfrac{H_1}{K}$. This proves the corollary.

Theorem 3. *Let J_1 be the maximum over* $(\operatorname{Re} s\geqq\alpha,\ \operatorname{Im} s$ *in* $I)$ *of* $|\log\zeta(s)|^{2k}$. *Then with the notation introduced above and with $I=[a,b]$, we have,*

(7) $$\sum_2 J_1 \leqq (\log H)^2 \sum_2 \int_{a-1\leqq t\leqq b+1,}\int_{\sigma\geqq\alpha-1/(\log H)} |\log\zeta(s)|^{2k}\,d\sigma\,dt$$

$$\leqq 2(\log H)^2 \iint_{\substack{T_0-1\leqq t\leqq T_0+H_1+1\\ \sigma\geqq\alpha-1/(\log H)}} |\log\zeta(s)|^{2k}\,d\sigma\,dt \leqq (\log H)^2 C_4^k A_k^{2k} H_1,$$

where C_4 is independent of C_1.

Corollary. *Of any $\dfrac{N}{2}$ of the summands J_1 appearing in $\sum_2$, the minimum J_1 does not exceed*

(8) $$\frac{2(\log H)^2 C_4^k A_k^{2k} H_1}{-1+\dfrac{1}{16}\left(\dfrac{C_2}{4C_3}\right)^{2k}\dfrac{H_1}{K}}.$$

Hence the $\max_{t\in I}|\log\zeta(\alpha+it)|$ *over those intervals I is* $\ll A_k$.

Combining the corollaries to Theorems 2 and 3 we have $\geqq -1+\dfrac{1}{32}\left(\dfrac{C_2}{4C_3}\right)^{2k}\dfrac{H_1}{K}$ $(=M$ say) intervals I contained in I_1 for which there holds

(9) $$A_k \ll \max_{t\in I}|\log\zeta(\alpha+it)| \ll A_k.$$

Now by choosing C_1 small we have $M\geqq H_1K^{-E}$ where $H_1=H^\theta$ and the number of intervals I_1 is $\sim H^{1-\theta}$. Since I_1 is contained in I_0 and the number of intervals I_0 is $\gg TH^{-1}$, we have, in all

(10) $$\gg H^\theta K^{-E} H^{1-\theta} T H^{-1} = TK^{-E}$$

disjoint intervals of length K each where (9) holds.

This completes the proof of Theorem 1 provided we prove Theorems 2 and 3.

3. Preliminaries to the proofs of Theorems 2 and 3

From (1) using the fact that the absolute value of an analytic function at a point does not exceed its mean-value over a disc $\left(\text{say of radius } \frac{1}{\log H}\right)$ round that point as centre, we obtain

$$|\zeta(s)| \leqq H^2 \quad \text{in} \quad \left(\sigma \geqq \beta + \frac{1}{\log H},\ T_0 - 9(\log H)^2 \leqq t \leqq T_0 + H_1 + 9(\log H)^2\right).$$

Hence in this region $\operatorname{Re}\log\zeta(s) \leqq 2\log H$. Now $\log\zeta(2+it) = 0(1)$ and hence by Borel—Carathéodory theorem we have

$$\log\zeta(s) = 0(\log H) \quad \text{in} \quad (\sigma \geqq (\alpha+\beta)/2,\ T_0 - 8(\log H)^2 \leqq t \leqq T_0 + H_1 + 8(\log H)^2).$$

Now put

$$X = (\log H)^B \tag{11}$$

where B is a large positive constant.

We have

$$\sum_p p^{-s} \exp\left(-\frac{p}{X}\right) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \log\zeta(s+w) X^w \Gamma(w)\, dw + 0(1)$$

where $T_0 - 7(\log H)^2 \leqq t \leqq T_0 + H_1 + 7(\log H)^2$ and $\sigma = \alpha$. Here first break off the portion $|\operatorname{Im} w| \geqq (\log H)^2$ and move the rest of the line of integration to $\operatorname{Re} w$ given by $\operatorname{Re}(s+w) = (\alpha+\beta)/2$. Also observe that

$$\sum_{p \geqq X^2} p^{-s} \exp\left(-\frac{p}{X}\right) = 0(1).$$

Collecting our results we have (since $|\Gamma(w)| \ll \exp(-|\operatorname{Im} w|)$)

$$\log\zeta(s) = \sum_{p \leqq X^2} p^{-s} \exp\left(-\frac{p}{X}\right) + 0(1) \tag{12}$$

where $\sigma = \alpha$ and $T_0 - 7(\log H)^2 \leqq t \leqq T_0 + H_1 + 7(\log H)^2$. Let

$$X^{2k} \leqq H_1^{1/2} \quad \text{and} \quad \left(\sum_{p \leqq X^2} p^{-s} \exp\left(-\frac{p}{X}\right)\right)^k = \sum_{n \leqq X^{2k}} a_k(n) n^{-s} = F(s), \tag{13}$$

say.

Then we have

$$|F(s)|^2 \leqq (|\log\zeta(s)| + C_5)^{2k} \leqq 2^{2k}|\log\zeta(s)|^{2k} + (2C_5)^{2k}, \tag{14}$$

and also

$$|\log\zeta(s)|^{2k} \leqq 2^{2k}|F(s)|^2 + (2C_5)^{2k}. \tag{15}$$

We now integrate these inequalities from $t=T_0$ to $t=T_0+H_1$. Also we note that these inequalities are valid even when $\frac{11}{10}\geqq \mathrm{Re}\, s \geqq \alpha-\frac{1}{\log H}$, $T_0-6(\log H)^2\leqq t\leqq T_0+H_1+6(\log H)^2$. Now in $\sigma\geqq\frac{11}{10}$, we have $|\log\zeta(s)|\ll 2^{-\sigma}$ and so

(16) $$\iint_{\sigma\geqq(11/10),\, T_0-1\leqq t\leqq T_0+H_1+1} |\log\zeta(s)|^{2k}\, d\sigma\, dt \ll \iint 2^{-2k\sigma} C_6^k\, d\sigma\, dt \ll H_1 C_6^k.$$

Therefore in order to prove Theorem 3, it suffices to consider

(17) $$2^{2k}\iint |F(s)|^2\, d\sigma\, dt + H_1 C_7^k$$

where the area integral extends over $\frac{11}{10}\geqq \mathrm{Re}\, s\geqq\alpha-\frac{1}{\log H}$, $T_0-1\leqq t\leqq T_0+H_1+1$. By a simple computation, we have since $X^{2k}\leqq H_1^{1/2}$,

(18) $$G(\sigma)\ll \frac{1}{H_1}\int_{T_0-1}^{T_0+H_1-1} |F(s)|^2\, dt \ll G(\sigma)$$

where

(19) $$G(\sigma) = \sum_{n\leqq X^{2k}} \frac{(a_k(n))^2}{n^{2\sigma}}.$$

4. Upper and lower bounds for $G(\sigma)$

Things similar to $G(\sigma)$ were first studied by H. L. Montgomery (see [5]). We consider upper and lower bounds for $(G(\sigma))^{1/2k}$. Let $p_1=2, p_2=3, \ldots, p_k$ be the first k primes. By prime number theorem

(20) $$p_1\cdots p_k = \exp(p_k+0(k)) = \exp(k\log k+k\log\log k+0(k)).$$

Taking only the contribution to $G(\sigma)$ from $n=p_1\ldots p_k$, we have since $\exp\left(-\frac{p_i}{X}\right)\geqq\frac{1}{2}$ $(i=1$ to $k)$,

(21) $$(G(\sigma))^{1/2k} \geqq \left(\frac{(k!)^2 2^{-2k}}{(p_1\cdots p_k)^{2\sigma}}\right)^{1/2k} \gg \frac{k^{1-\sigma}}{(\log k)^\sigma} = A_k(\sigma) \quad \text{say.}$$

This proves the lower bound

(22) $$G(\sigma)\geqq (A_k(\sigma))^{2k} C_8^{2k}.$$

As regards the upper bound we write

(23) $$\sum_{p\leqq X^2} p^{-s}\exp\left(-\frac{p}{X}\right) = \sum_1 + \sum_2$$

where $\sum_1$ extends over $p\leqq k\log k$ and $\sum_2$ the rest.

Note that

(24) $$|F(s)|^2 \leqq 2^{2k}\left|\sum\nolimits_1\right|^{2k}+2^{2k}\left|\sum\nolimits_2\right|^{2k}.$$

Put

(25) $$\sum\nolimits_1^k = \sum \frac{b_k(n)}{n^s} = F_1(s) \quad \text{say,}$$

and

(26) $$\sum\nolimits_2^k = \sum \frac{c_k(n)}{n^s} = F_2(s) \quad \text{say.}$$

By a simple computation we have

(27) $$\frac{1}{H_1}\int_{T_0-1}^{T_0+H_1+1} |F_1(s)|^2\,dt \ll G_1(\sigma) \quad \text{and} \quad \frac{1}{H_1}\int_{T_0-1}^{T_0+H_1+1} |F_2(s)|^2\,dt \ll G_2(\sigma)$$

where

(28) $$G_1(\sigma) = \sum \frac{(b_k(n))^2}{n^{2\sigma}} \leqq \left(\sum \frac{b_k(n)}{n^\sigma}\right)^2 = \left(\sum\nolimits_{p\leqq k\log k} p^{-\sigma}\exp\left(-\frac{p}{X}\right)\right)^{2k}$$

and

(29) $$G_2(\sigma) = \sum \frac{(C_k(n))^2}{n^{2\sigma}} \leqq k!\sum \frac{C_k(n)}{n^{2\sigma}} \leqq k!\left(\sum\nolimits_{p\geqq k\log k} p^{-2\sigma}\exp\left(-\frac{p}{X}\right)\right)^{2k}.$$

If $\sigma<1$ we have easily

(30) $$(G_1(\sigma))^{1/2k} \ll \frac{(k\log k)^{1-\sigma}}{\log k} = \frac{k^{1-\sigma}}{(\log k)^\sigma}$$

and by Sterling's approximation for $k!$ we also have

(31) $$(G_2(\sigma))^{1/2k} \ll k^{1/2}\left(\frac{(k\log k)^{1-2\sigma}}{\log k}\right)^{1/2} = \frac{k^{1-\sigma}}{(\log k)^\sigma}.$$

This proves the upper bound

(32) $$(G(\sigma))^{1/2k} \ll \left(\frac{1}{H_1}\int_{T_0-1}^{T_0+H_1+1} |F(s)|^2\,dt\right)^{1/2k} \ll A_k(\sigma)$$

which in turn gives an upper bound for $(G(\sigma))^{1/2k}$ if $\beta_0\leqq\sigma\leqq 1-\delta_1$ uniformly in σ for every $\delta_1>0$. If $\frac{11}{10}\geqq\sigma\geqq 1-\delta_1$ the bounds for the area integral are negligible if δ_1 is small since it is

$$\leqq 2\left(\sum \frac{a_k(n)}{n^{\sigma_1}}\right)^2 \leqq 2\left(\sum\nolimits_{p\leqq X^2} p^{-\sigma_1}\exp\left(-\frac{p}{X}\right)\right)^{2k}$$

where $\sigma_1=1-\delta$.

This completes the proofs of Theorems 2 and 3. Thus the proof of Theorem 1 is complete.

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