Weighted norm inequalities for a general maximal operator

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1. Introduction

Some studies about boundedness properties of maximal operators of Hardy—Littlewood type have been made recently (see [5], [6], [8]). In this note we study a very general operator which includes the known results as particular cases.

Let U, V be two arbitrary sets. Suppose we have some topological structure in the cartesian products $\mathbf{R}^n \times U$, $\mathbf{R}^n \times V$ and also suppose the existence of positive Borel measures $d\alpha(x,u)$ on $\mathbf{R}^n \times U$ and $d\beta(x,v)$ on $\mathbf{R}^n \times V$.

We shall denote by $L^p(\mathbf{R}^n \times U, d\alpha)$ the set of measurable functions in $\mathbf{R}^n \times U$ such that $\int_{\mathbf{R}^n \times U} |f(x,u)|^p \, d\alpha(x,u)$ is finite.

The α-measure of a set $E \subset \mathbf{R}^n \times U$ will be indicated by $\alpha(E)$ and the Lebesgue measure of $E \subset \mathbf{R}^n$ will be denoted by $|E|$.

Throughout this paper Φ, respectively Ψ, will be a set function from cubes in $\mathbf{R}^n$ into Borel sets in $\mathbf{R}^n \times U$, resp. $\mathbf{R}^n \times V$, satisfying:

(I) If Q_1, Q_2 are cubes with $Q_1 \cap Q_2 = \emptyset$ then $\Phi(Q_1) \cap \Phi(Q_2) = \emptyset$ and $\Psi(Q_1) \cap \Psi(Q_2) = \emptyset$.

(II) If $Q_1 \subset Q_2$ then $\Phi(Q_1) \subset \Phi(Q_2)$ and $\Psi(Q_1) \subset \Psi(Q_2)$.

(III) If $Q(x,r)$ denotes the cube with center x and side length r then, for any $x \in \mathbf{R}^n$

$$\bigcup_{r>0} \Phi(Q(x,r)) = \mathbf{R}^n \times U \quad \text{and} \quad \bigcup_{r>0} \Psi(Q(x,r)) = \mathbf{R}^n \times V.$$

We define the following maximal operator which applies functions in $\mathbf{R}^n \times U$ into functions in $\mathbf{R}^n \times V$:

$$Tf(x,v) = \sup \left\{ \frac{1}{|Q|} \int_{\Phi(Q)} |f(y,u)| \, d\alpha(y,u) \colon (x,v) \in \Psi(Q) \right\} \tag{1}$$

i.e. the supremum is taken over all cubes Q such that $(x,v) \in \Psi(Q)$.

Particular examples are the following:

A. If $\mathbf{R}^n\times U=\mathbf{R}^n\times V=\mathbf{R}^n$, $d\alpha(x,u)=dx$, where dx is the Lebesgue measure on $\mathbf{R}^n$ and $\Phi(Q)=\Psi(Q)=Q$, then T is the Hardy—Littlewood maximal operator.

B. If $\mathbf{R}^n\times U=\mathbf{R}^n$, $V=[0,\infty)$, $d\alpha(x,u)=dx$, $\Phi(Q)=Q$ and $\Psi(Q)=\tilde{Q}=\{(x,t)\colon x\in Q,\ 0\leqq t\leqq \text{side length of } Q\}$, then T is the operator

$$\mathcal{M}f(x,t)=\sup\left\{\frac{1}{|Q|}\int_Q |f(y)|\,dy\colon x\in Q,\quad 0\leqq t\leqq \text{side length of } Q\right\}$$

introduced by Fefferman—Stein [3] and studied in [5] and [6].

C. If $U=[0,\infty)$, $\mathbf{R}^n\times V=\mathbf{R}^n$, $\Phi(Q)=\tilde{Q}$, $\Psi(Q)=Q$, then T is the maximal operator

$$Cf(x)=\sup\left\{\frac{1}{|Q|}\int_{\tilde{Q}}|f(y,t)|\colon x\in Q\right\}$$

closely related with tent spaces (see [1]).

2. Main results

Theorem 1. *Let $\omega(x,u)$ be a positive function on $\mathbf{R}^n\times U$. The following conditions are equivalent:*

(i) *T is bounded from $L^p(\mathbf{R}^n\times U,\omega d\alpha)$ into weak-$L^p(\mathbf{R}^n\times V, d\beta)$ for some p, $1\leqq p<\infty$, i.e.*

$$\beta\big(\{(x,v)\in\mathbf{R}^n\times V\colon |Tf(x,v)|>\lambda\}\big)\leqq\frac{C}{\lambda^p}\int_{\mathbf{R}^n\times U}|f(x,u)|^p\omega(x,u)\,d\alpha(x,u).$$

(ii) *The weight ω satisfies that for any cube Q,*

$$\frac{\beta(\Psi(Q))}{|Q|}\left(\frac{1}{|Q|}\int_{\Phi(Q)}\omega(x,u)^{1-p'}\,d\alpha(x,u)\right)^{p-1}\leqq C$$

if $1<p<\infty$.

If $p=1$ the condition is that for any cube Q

$$\frac{\beta(\Psi(Q))}{|Q|}\leqq C\omega(x,u)\quad \alpha\text{-a.e. } (x,u)\in\Phi(Q).$$

We shall say that β is a ψ-Carleson measure if there exists a constant C such that

$$\beta(\Psi(Q))\leqq C|Q|\quad\text{for any cube } Q.$$

In this case, we can prove the following:

Theorem 2. *Let* $1<q<\infty$ *and* β *a* Ψ*-Carleson measure. Then, for* β*-almost every* $(x, v)\in\mathbf{R}^n\times V$

$$\sup\left\{\frac{1}{|Q|}\int_{\Psi(Q)}(T(f^q)(y, w))^{1/q}\,d\beta(y, w)\colon (x, v)\in\Psi(Q)\right\}\leqq C\big(T(f^q)(x, v)\big)^{1/q}.$$

Remark 1. If we define on $\mathbf{R}^n\times V$ the maximal operator

$$Mg(x, v)=\sup\left\{\frac{1}{|Q|}\int_{\Psi(\Omega)}|g(y, w)|\,d\beta(y, w)\colon (x, v)\in\Psi(Q)\right\},$$

then Theorem 2 ensures that the function $h(x, v)=\big(T(f^q)(x, v)\big)^{1/q}$ satisfies

$$Mh(x, v)\leqq Ch(x, v) \tag{2}$$

Observe that if $\mathbf{R}^n\times V=\mathbf{R}^n$, $\Psi(Q)=Q$ and $d\beta=dx$ then (2) says that $h(x)$ is a weight in the class A_1 of Muckenhoupt.

Theorem 3. *Suppose that* $\mathbf{R}^n\times U=\mathbf{R}^n$, $\Phi(Q)=Q$, $d\alpha(x, u)=dx$ *and* β *is a* Ψ*-Carleson measure on* $\mathbf{R}^n\times V$. *Then, the following vector-valued inequalities hold:*

(i) *For* $1<p, q<\infty$

$$\int_{\mathbf{R}^n\times V}\Big(\sum_{j=1}^{\infty}|Tf_j(x, v)|^q\Big)^{q/p}d\beta(x, v)\leqq C\int_{\mathbf{R}^n}\Big(\sum_{j=1}^{\infty}|f_j(x)|^q\Big)^{p/q}dx.$$

(ii) *For* $1<q<\infty$

$$\beta\Big(\Big\{(x, v)\in\mathbf{R}^n\times V\colon \sum_{j=1}^{\infty}|Tf_j(x, v)|^q>\lambda^q\Big\}\Big)\leqq\frac{C}{\lambda}\int_{\mathbf{R}^n}\Big(\sum_{j=1}^{\infty}|f_j(x)|^q\Big)^{1/q}dx.$$

Our last result is the strong version of Theorem 1.

Theorem 4. *Let* $1<p<\infty$ *and* $\omega(x, u)$ *a positive function in* $\mathbf{R}^n\times U$. *The following conditions are equivalent:*

(i) *T is bounded from* $L^p(R^n\times U, \omega\,d\alpha)$ *into* $L^p(\mathbf{R}^n\times V, d\beta)$.

(ii) *For any cube* Q,

$$\int_{\Psi(Q)}\big(T(\chi_{\Phi(Q)}\omega^{1-p'})(y, v)\big)^p\,d\beta(y, v)\leqq C\int_{\Phi(Q)}\omega(x, u)^{1-p'}\,d\alpha(x, u)<+\infty$$

where C is an absolute constant.

Remark 2. If $\mathbf{R}^n\times U=\mathbf{R}^n\times V=\mathbf{R}^n$, $\Phi(Q)=\Psi(Q)=Q$ and $d\alpha(x, u)=dx$, then Theorems 1 and 4 are very well known and due to Muckenhoupt [4] and Sawyer [8]. Theorem 2 is due to Coifman and Theorem 3 to Fefferman and Stein [3].

If $\mathbf{R}^n\times U=\mathbf{R}^n$, $V=[0, \infty)$, $\Phi(Q)=Q$, $\Psi(Q)=\tilde{Q}$ and $d\alpha(x, u)=dx$, Theorems 1, 3, 4 can be seen in [5], [6], [7].

If $\mathbf{R}^n\times V=\mathbf{R}^n$, $U=[0, \infty)$, $\Phi(Q)=\tilde{Q}$, $\Psi(Q)=Q$ then Theorem 2 is due to Deng [2].

Remark 3. If the measure $d\alpha$ is fixed we define the class $W_p^\alpha(T)$ (resp. $S_p^\alpha(T)$) as the set of pairs $(d\beta, \omega)$ such that the condition (ii) in Theorem 1 (resp. Theorem 4) is fulfilled.

In general, it is not true that for $p<q$, $W_p^\alpha(T)\subset W_q^\alpha(T)$ as the following example shows:

Take $\mathbf{R}^n\times V=\mathbf{R}^n$, $d\beta(x, v)=dx$, $\Psi(Q)=Q$ and let μ be a Φ-Carleson measure on $\mathbf{R}^n\times U$. Choose a function $\omega(x, u)$ such that for some Q and some $p<q$

$$\int_{\Phi(Q)} \omega(x, u)^{p'-q'}\, d\mu(x, u) = \infty.$$

If we put $d\alpha=\omega^{p'-1}d\mu$ then it is clear that $(dx, \omega\, d\alpha)\in W_q^\alpha(T)$ but

$$(dx, \omega\, d\alpha)\notin W_q^\alpha(T).$$

However, if α is a Φ-Carleson measure on $\mathbf{R}^n\times U$ then T is bounded from $L^\infty(\mathbf{R}^n\times U, \omega\, d\alpha)$ into $L^\infty(\mathbf{R}^n\times V, d\beta)$ and so, by using Marcinkiewicz's interpolation theorem and Theorems 1, 4 we obtain

$$W_1^\alpha(T)\subset \dots \subset S_p^\alpha(T)\subset W_p^\alpha(T)\subset \dots \subset S_q^\alpha(T)\subset W_q^\alpha(T)\subset \dots (1<p<q<\infty).$$

3. Proofs

Proof of Theorem 1. It is clear from the definition of T that for any cube Q,

$$\Psi(Q)\subset \left\{(x, v)\in \mathbf{R}^n\times V\colon\ Tf(x, v) \geqq \frac{1}{|Q|}\int_{\Phi(Q)} |f|\, d\alpha\right\}.$$

Then, if T satisfies part (i) of Theorem 1, in particular we shall have

$$(3)\qquad \beta(\Psi(Q)) \leqq C|Q|^p \left(\int_{\Phi(Q)} |f|\, d\alpha\right)^{-p} \int_{\mathbf{R}^n\times U} |f|^p \omega\, d\alpha.$$

Now, for $1<p<\infty$, we obtain (ii) of Theorem 1 putting $f=\chi_{\Phi(Q)}\omega^{1-p'}$ in (3).

In order to have part (ii) of Theorem 1 for $p=1$, observe that (3) says that

$$\int_{\Phi(Q)} |f| \frac{\beta(\Psi(Q))}{|Q|}\, d\alpha \leqq C\int_{\mathbf{R}^n\times U} |f|\omega\, d\alpha$$

for any f in $L^1(\mathbf{R}^n\times U, \omega\, d\alpha)$ and this implies that

$$\frac{\beta(\Psi(Q))}{|Q|} \leqq C\omega(x, u) \quad \alpha\text{-a.e.} \quad (x, u)\in\Phi(Q).$$

For the converse we shall need the dyadic cubes, i.e., the cubes of the form $\prod_{i=1}^n [x_i, x_i+2^k)$, where $x\in 2^k\mathbf{Z}^n$ for some k in $\mathbf{Z}$.

Let $1<p<\infty$ and $\lambda>0$. We want to prove the inequality

$$\beta(\{(x, v)\in \mathbf{R}^n\times V\colon Tf(x, v) > \lambda\}) \leqq \frac{C}{\lambda^p}\int_{\mathbf{R}^n\times U} |f|^p \omega\, d\alpha. \tag{4}$$

For $r>0$ we introduce the operators

$$T^r f(x, v) = \sup\left\{\frac{1}{|Q|}\int_{\Phi(Q)} |f|\, d\alpha\colon (x, v)\in \Psi(Q) \text{ and side length of } Q\leqq r\right\}.$$

Let A_λ^r be the set

$$A_\lambda^r = \{(x, v)\in \mathbf{R}^n\times V\colon T^n f(x, v) > \lambda\}.$$

If we can prove that

$$\beta(A_\lambda^r) \leqq \frac{C}{\lambda^p}\int_{\mathbf{R}^n\times U} |f(x, u)|^p \omega(x, u)\, d\alpha(x, u) \tag{5}$$

with constant C independent of r, then it is clear that the monotone convergence theorem will give us (4).

For each $(x, v)\in A_\lambda^r$ there exists a cube P such that $(x, v)\in \Psi(P)$ and

$$\frac{1}{|P|}\int_{\Phi(P)} |f|\, d\alpha > \lambda.$$

Let k be the only integer such that $2^{(k-1)n}<|P|\leqq 2^{kn}$. There exists at most 2^n dyadic cubes Q with $|Q|=2^{kn}$ and with nonvoid intersection with the interior of P. Then there exists at least a dyadic cube Q_0 with $|Q_0|=2^{kn}$ and such that

$$\frac{1}{|P|}\int_{\Phi(Q_0)} |f(y, u)|\, d\alpha(y, u) > \lambda 2^{-n}.$$

In particular, this cube Q_0 verifies

$$\frac{1}{|Q_0|}\int_{\Phi(Q_0)} |f(y, u)|\, dv(y, u) > \lambda 4^{-n}. \tag{6}$$

Now, there exists a dyadic cube Q_j such that $Q_0\subset Q_j$ and Q_j is a maximal dyadic cube for the condition (6), since, applying Hölder's inequality and condition $W_p^\alpha(T)$, the inequality

$$\lambda 4^{-n} < \frac{1}{|Q|}\int_{\Phi(Q)} |f| d\alpha = \frac{1}{|Q|}\left(\int_{\Phi(Q)} |f|^p \omega\, d\alpha\right)^{1/p}\left(\int_{\Phi(Q)} \omega^{-p'/p}\, d\alpha\right)^{1/p'}$$
$$\leqq C\beta(\Psi(Q))^{-1/p}\left(\int_{\mathbf{R}^n\times U} |f|^p \omega\, d\alpha\right)^{1/p}$$

for infinitely many dyadic cubes containing Q_0 would imply that

$$\beta(\mathbf{R}^n\times V) \leqq \frac{C}{\lambda^p}\int_{\mathbf{R}^n\times U}|f|^p\omega\,d\alpha$$

and in case (4) is obviously satisfied.

Moreover it is clear that $P\subset 3Q_j$ (where $3Q(x,r)=Q(x,3r)$).

In other words we have proved that $A_\lambda^r\subset\bigcup_j\Psi(3Q_j)$ where Q_j are disjoint cubes verifying (6).

Now, standard techniques in weight theory tell us that for $1<p<\infty$,

$$\beta(A_\lambda^r) \leqq C\sum_j\beta(\Psi(3Q_j))\frac{1}{\lambda^p}\left(\frac{1}{|Q_j|}\int_{\Phi(Q_j)}|f|\,d\alpha\right)^p$$

$$\leqq \frac{C}{\lambda^p}\sum_j\frac{\beta(\Psi(3Q_j))}{|Q_j|}\left(\int_{\Phi(Q_j)}|f|^p\omega\,d\alpha\right)\left(\int_{\Phi(3Q_j)}\omega^{-p'/p}\,d\alpha\right)^{p/p'} \leqq \frac{C}{\lambda^p}\int_{\mathbf{R}^n\times U}|f|^p\omega\,d\alpha.$$

So, (6) is proved and the proof of Theorem 1 is concluded. (Obvious modifications give the result for $p=1$.)

Proof of Theorem 4. This follows along the same lines as in [6].

The implication (i)$\Rightarrow$(ii) can be proved analogously to the corresponding one in Theorem 1.

For the converse, the first step is to prove the result for the "dyadic" operator

$$T_d f(x,v) = \sup\left\{\frac{1}{|Q|}\int_{\Phi(Q)}|f|\,d\alpha\colon (x,v)\in\Psi(Q),\ Q\ \text{dyadic}\right\}$$

and then the proof for T follows easily from the ensuing lemma (see [6], [8]):

Lemma. *We define for each* $z\in\mathbf{R}^n$ *the operator*

$${}^zT_d f(x,v) = \sup\frac{1}{|Q|}\int_{\Phi(Q)}|f(y,u)|\,d\alpha(y,u)$$

the supremum being taken in all cubes Q *with* $(x,v)\in\Psi(Q)$ *and such that the set* $Q-z=\{u-z\colon u\in Q\}$ *is a dyadic cube. Then,*

$$T^{2^k}f(x,v) \leqq C\int_{[-2^{k+2},2^{k+2}]^n}{}^zT_d f(x,v)\frac{dz}{2^{n(k+3)}}.$$

In order to prove the theorem for T_d, we introduce for $r>0$ the operators

$$T_d^r f(x,v) = \sup\frac{1}{|Q|}\int_{\Phi(Q)}|f(y,u)|\,d\alpha(y,u)$$

where the supremum is taken over all dyadic cubes Q such that $(x,v)\in\Psi(Q)$ and with side length less than r.

Let Ω_k be the set

$$\Omega_k = \{(x, v) \in \mathbf{R}^n \times V \colon T_d^r f(x, v) > 2^k\}, \quad k \in Z.$$

It is easy to show that the set Ω_k can be decomposed into $\Omega_k = \bigcup_{j \in J_k} \Psi(Q_j^k)$, where Q_j^k, $j \in J_k$, are disjoint dyadic cubes with side length less than r and satisfying:

$$\frac{1}{|Q_j^k|} \int_{\Phi(Q_j^k)} |f| \, d\alpha > 2^k.$$

Now, let us consider the disjoint sets

$$E_j^k = \Psi(Q_j^k) \setminus \Omega_{k+1}, \quad k \in Z, \; j \in J_k.$$

Then

$$\int_{\mathbf{R}^n \times V} |T_d^r f(x, v)|^p \, d\beta(x, v) \leqq \sum_{k,j} \int_{E_j^k} |T_d^r f|^p \, d\beta$$

$$\leqq \sum_{k,j} 2^{(k+1)p} \beta(E_j^k) \leqq 2^p \sum_{k,j} \beta(E_j^k) \left(\frac{1}{|Q_j^k|} \int_{\Phi(Q_j^k)} |f| \, d\alpha \right)^p.$$

We introduce the following notations (see [6] and references there):

$$\sigma(x, u) = \omega^{1-p'}(x, u), \quad \sigma(\Phi(Q)) = \int_{\Phi(Q)} \sigma \, d\alpha,$$

$$\gamma_{jk} = \beta(E_j^k) \left(\frac{\sigma(\Phi(Q_j^k))}{|Q_j^k|} \right)^p, \quad g_{jk} = \left(\frac{1}{\sigma(\Phi(Q_j^k))} \int_{\Phi(Q_j^k)} \frac{|f|}{\sigma} \sigma \, d\alpha \right)^p,$$

$X = \{(k, j) \colon k \in \mathbf{Z}, j \in J_k\}$ with atomic measure γ_{jk}, and $\Gamma(\lambda) = \{(k, j) \in X \colon g_{jk} > \lambda\}$.
Then we can write

$$\int_{\mathbf{R}^n \times V} |T_d^r f|^p \, d\beta \leqq 2^p \sum_{j,k} \gamma_{jk} g_{jk} = 2^p \int_0^\infty \left\{ \sum_{(k,j) \in \Gamma(\lambda)} \gamma_{jk} \right\} d\lambda$$

$$= 2^p \int_0^\infty \left\{ \sum_{k,j \in \Gamma(\lambda)} \int_{E_j} \left(\frac{\sigma(\Phi(Q_j^k))}{|Q_j^k|} \right)^p d\beta(x, v) \right\} d\lambda.$$

Calling Q_i the maximal cubes of the family $\{Q_j^k \colon (k, j) \in \Gamma(\lambda)\}$, this is less than

$$2^p \int_0^\infty \left(\sum_i \int_{\Psi(Q_i)} T_d^r (\sigma \chi_{\Phi(Q_i)})^p \, d\beta \right) d\lambda$$

and by hypothesis (ii) this is less than

$$2^p \int_0^\infty \left(\sum_i \int_{\Phi(Q_i)} \sigma \, d\alpha \right) d\lambda = 2^p \int_0^\infty \sigma\left(\bigcup_i \Phi(Q_i) \right) d\lambda.$$

The definition of $\Gamma(\lambda)$ states that

$$\bigcup_i \Phi(Q_i) \subset \left\{ (x, u) \in \mathbf{R}^n \times U \colon N\left(\frac{f}{\sigma} \right)(x, u)^p > \lambda \right\}$$

where

$$Ng(x, u) = \sup \frac{1}{\sigma(\Phi(Q))} \int_{\Phi(Q)} |g(x, u)| \, \sigma(x, u) \, d\alpha(x, u)$$

(the supremum being taken over all dyadic cubes in $\mathbf{R}^n$ such that $(x, u) \in \Phi(Q)$).

Then we have

$$\int_{\mathbf{R}^n \times V} |T_d^* f|^p \, d\beta \leqq 2^p \int_0^\infty \sigma\left(\left\{(x, u)\colon N\left(\frac{f}{\sigma}\right)(x, u)^p > \lambda\right\}\right) d\lambda$$

$$= 2^p \int_{\mathbf{R}^n \times U} N\left(\frac{f}{\sigma}\right)^p \sigma \, d\alpha \leqq 2^p \int_{\mathbf{R}^n \times U} \frac{|f|^p}{\sigma^p} \sigma \, d\alpha = 2^p \int_{\mathbf{R}^n \times U} |f|^p \omega \, d\alpha$$

where the last inequality is due to the fact that N is bounded from $L^p(\mathbf{R}^n \times U, \sigma d\alpha)$, $1 < p \leqq \infty$, into itself. This can be seen by interpolating the trivial result for $p = \infty$ with the (1, 1)-weak type inequality (which can be obtained with standard arguments involving dyadic cubes).

The monotone convergence theorem again gives (i) of Theorem 4 for T_d and the proof is finished.

Proof of Theorem 2. Let $1 < q < \infty$ and $g(x, u) = |f(x, u)|^q$. If a cube Q is fixed we decompose

$$g(x, u) = g_1(x, u) + g_2(x, u)$$

where $g_1(x, u) = g(x, u)\chi_{\Phi(3Q)}(x, u)$.

Since β is a Ψ-Carleson measure, Theorem 1 (with $\omega \equiv 1$) ensures that T is of weak type (1, 1) and then by the Kolmogorov inequality, we have for any δ with $0 < \delta < 1$,

$$\int_{\Psi(Q)} (Tg_1)^\delta \, d\beta \leqq C\beta(\Psi(Q))^{1-\delta} \left(\int_{\mathbf{R}^n \times U} |g_1(x, u)| \, d\alpha(x, u)\right)^\delta$$

$$\leqq C|Q|^{1-\delta} \left(\int_{\mathbf{R}^n \times U} |g_1(x, u)| \, d\alpha(x, u)\right)^\delta.$$

In particular,

$$\text{(7)} \qquad \frac{1}{|Q|} \int_{\Psi(Q)} (Tg_1)^\delta \, d\beta \leqq C \left(\frac{1}{|Q|} \int_{\Phi(Q)} |g| \, d\alpha\right)^\delta \leqq C(Tg(z, w))^\delta$$

for any $(z, w) \in \Psi(Q)$.

Now, let $(y, v) \in \Psi(Q)$. For any cube P such that $(y, v) \in \Psi(P)$ and $\frac{1}{|P|} \int_{\Phi(P)} |g_2| \, d\alpha \neq 0$, we have $\Psi(P) \cap \Psi(Q) \neq \emptyset$, $\Phi(P) \cap \Phi(3Q)^c \neq \emptyset$ and then properties I and II on Φ, Ψ imply that $P \cap Q \neq \emptyset$, $P \cap (3Q)^c \neq \emptyset$. This says that $Q \subset 3P$. So, for any $(z, w) \in \Psi(Q)$

$$Tg_2(y, v) \leqq 3^n Tg_2(z, w).$$

Thus,

(8)
$$\frac{1}{|Q|}\int_{\Psi(Q)}(Tg_2)^{\delta}\,d\beta \leqq C\frac{\beta(\Psi(Q))}{|Q|}\inf_{(z,w)\in\Psi(Q)}(Tg_2(z,w))^{\delta} \leqq C\inf_{(z,w)\in\Psi(Q)}(Tg_2(z,w))^{\delta}.$$

Finally, it is clear that (7) and (8) conclude the proof.

Proof of Theorem 3. We shall distinguish two cases:

1) *Part* (i) *with* $p>q$. In this case, for some nonnegative $h\in L^{r'}(\mathbf{R}^n\times V, d\beta)$ with $\|h\|_{L^{r'}(\mathbf{R}^n\times V), d\beta}=1$ (where $r=p/q$), we have

(9) $$\int_{\mathbf{R}^n\times V}\left(\sum_j|Tf_j(x,v)|^q\right)^{p/q}d\beta(x,v) = \left(\int_{\mathbf{R}^n\times V}\sum_j|Tf_j(x,v)|^q\,h(x,v)\,d\beta(x,v)\right)^r.$$

If we define

$$T^*h(x) = \sup_{x\in Q}\frac{1}{|Q|}\int_{\Psi(Q)}|h(y,v)|\,d\beta(y,v)$$

it is clear that the pair $(hd\beta, T^*h)$ belongs to the class $W_1^{\alpha}(T)$ (where $d\alpha=dx$) and then Remark 3 and Theorem 4 imply that the last member of (9) is less than

(10) $$\left(\int_{\mathbf{R}^n}\sum_j|f_j(x)|^q T^*h(x)\,dx\right)^r.$$

Moreover, the operator $h\mapsto T^*h$ is bounded from $L^p(\mathbf{R}^n\times V, dB)$ into $L^p(\mathbf{R}^n, dx)$. In fact, it is bounded from L^∞ into L^∞ (since β is Ψ-Carleson) and from L^1 into weak-L^1 (this can be deduced from Theorem 1).

Then, by using Hölder's inequality, (10) is less than

$$\int_{\mathbf{R}^n}\left(\sum_j|f_j(x)|^q\right)^r dx\cdot\|T^*h\|^{r/r'}_{L^{r'}(\mathbf{R}^n,dx)} \leqq C\int_{\mathbf{R}^n}\left(\sum_j|f_j(x)|^q\right)^{p/q}dx,$$

and, therefore, the theorem is proved in this case.

2) *Part* (i) *with* $p\leqq q$ *and part* (ii). We shall make use of a general theory of vector-valued singular integrals. We need the following proposition:

Proposition. *Let* $\mathbf{R}^n$, V, α *and* β *be as in Theorem* 3. *Let* E, F *be Banach spaces and* S *a linear operator bounded from* $L_E^{p_0}(\mathbf{R}^n, dx)$ *into* $L^{p_0}(\mathbf{R}\times V, d\beta)$ *for some* p_0, $1<p_0\leqq\infty$. *Suppose that there exists a function*

$$K\colon \mathbf{R}^n\times\mathbf{R}^n\times V\to\mathscr{L}(E,F)$$

($\mathscr{L}(E,F)$ *denotes the set of bounded linear operators from* A *into* B) *such that:*

(a) *For* $f\in L_E^\infty(\mathbf{R}^n, dx)$ *with support contained in a cube* Q, S *has the representation*

$$Sf(x,v) = \int_{\mathbf{R}^n}K(x,y,v)f(y)\,dy \quad \textit{for} \quad (x,v)\notin\Psi(Q).$$

(b) *There exist a natural number N and a constant C such that for any cube Q and* $k \in \mathbf{N}$

$$\|K(x, y, v)-K(x, y', v)\|_{\mathscr{L}(E,F)} \leqq \frac{C}{N^k|N^kQ|}$$
$$\textit{for} \quad (x, v) \notin \Psi(N^kQ), \quad y, y' \in Q.$$

Then,

(i) *For* $1<p\leqq q\leqq p_0$

$$\int_{\mathbf{R}^n\times V} \left(\sum_{j=1}^{\infty} \|Sf_j(x, v)\|_F^q\right)^{p/q} d\beta(x, v) \leqq C\int_{\mathbf{R}^n} \left(\sum_{j=1}^{\infty} \|f_j(x)\|_E^q\right)^{p/q} dx.$$

(ii) *For* $1<q\leqq p_0$

$$\beta\left(\left\{(x, v)\in \mathbf{R}^n\times V\colon \sum_{j=1}^{\infty} \|Sf_j(x, v)\|_F^q > \lambda^q\right\}\right) \leqq \frac{C}{\lambda}\int_{\mathbf{R}^n} \left(\sum_{j=1}^{\infty} \|f_j(x)\|_E^q\right)^{1/q} dx.$$

Before we sketch the proof of the proposition, we shall finish the proof of Theorem 3.

First of all, observe that in the definition of T we can restrict us to a numerable family, say I, of cubes (for instance, cubes with rational radius and center). If we consider a sequence $\{I_n\}_{n\in\mathbf{N}}$ with I_n(finite)$\nearrow I_2$ and we put

$$S^n f(x, v) = \left\{\int_{\mathbf{R}^n} \frac{1}{|Q|}\chi_{\Psi(Q)}(x, v)\,\chi_Q(y) f(y)\, dy\right\}_{Q\in I_n}$$

then it is clear that

$$\|S^n f(x, v)\|_{l^\infty(I_n)} \nearrow Tf(x, v) \quad (n\to\infty). \tag{11}$$

In particular, $\|S^n f(x, v)\|_{l^\infty(I_n)} \leqq Tf(x, v)$ and then, Theorem 1 says that S^n is bounded from $L^p(\mathbf{R}^n, dx)$ into $L^p_{l^\infty(I_n)}(\mathbf{R}^n\times V, d\beta)$ for $1<p<\infty$ (with bounds independent of n).

On the other hand, the kernel of the operator S^n (in the sense of the proposition) is the $\mathscr{L}(\mathbf{C}, l^\infty(I_n))\cong l^\infty(I_n)$-valued kernel given by

$$K^n(x, y, v) = \left\{\frac{1}{|P|}\chi_{\Psi(P)}(x, v)\,\chi_P(y)\right\}_{P\in I_n}.$$

Unfortunately, this kernel satisfies that for any cube Q and $k\in\mathbf{N}$

$$\|K^n(x, y, v)-K^n(x, y', v)\|_{l^\infty(I_n)} \leqq \frac{C}{|3^kQ|} \tag{12}$$
$$\text{for} \quad (x, v)\notin \Psi(3^kQ) \quad \text{and} \quad y, y'\in Q$$

and this condition is weaker than condition (b).

In order to have an operator in the conditions of the proposition we must smooth the kernel K^n: take a function φ: $\mathbf{R}\to\mathbf{R}$, $\varphi\in Q^1(\mathbf{R})$ such that

$$\chi_{[0,1]} \leqq \varphi \leqq \chi_{[-1,2]} \quad \text{and} \quad |\varphi'(t)| \leqq C/t.$$

We define the operator (y_p=center of P, $r(P)$=radius of P)

$$\tilde{S}^n f(x, v) = \left\{\int_{\mathbf{R}^n} \frac{1}{|P|} \chi_{\Psi(P)}(x, v)\, \varphi\left(\frac{|y-y_P|}{r(P)}\right) f(y)\, dy\right\}_{P\in I_n}.$$

It is easy to check that

(13) $$\|S^n f(x, v)\|_{l^\infty(I_n)} \leqq \|\tilde{S}^n f(x, v)\|_{l^\infty(I_n)} \leqq CTf(x, v)$$

and

(14) for any cube Q and $k\in\mathbf{N}$

$$\left\|\left\{\frac{1}{|P|} \chi_{\Psi(P)}(x, v) \left|\varphi\left(\frac{|y-y_P|}{r(P)}\right) - \varphi\left(\frac{|y'-y_P|}{r(P)}\right)\right|\right\}\right\|_{l^\infty(I_n)} \leqq \frac{C}{3^k|3^kQ|}$$

$$\text{for} \quad (x, v)\notin \Psi(3^kQ) \quad \text{and} \quad y, y'\in Q.$$

The second inequality in (13) implies that $\tilde{S}^n$ is bounded on L^{p_0} (with any p_0, $1<p_0<\infty$) and this fact together with (14) says that the proposition can be applied to $\tilde{S}^n$. Now, by the first inequality in (13) we obtain part (i) with $p\leqq q$ and part (ii) of Theorem 3 for $\|S^n f(.,.)\|_{l^\infty(I_n)}$.

Finally, note that all the constants are independent of n, and so the monotone convergence theorem and (11) conclude the proof of Theorem 3.

Proof of the Proposition. (Sketch.) For details in the case $\Psi(Q)=\tilde{Q}$ see [7]. Given a function $f\in L^1_E(\mathbf{R}^n, dx)$ and a positive number λ, we consider the set

$$\Omega_\lambda = \{(x, v)\in\mathbf{R}^n\times V\colon\ T_d(\|f\|_E)(x, v) > \lambda\}.$$

There exists a collection of dyadic cubes $\{Q_j\}$ such that $\Omega_\lambda=\bigcup_j \Psi(Q_j)$, $\lambda<1/|Q_j| \int_{Q_j} \|f(x)\|_E dx\leqq 2^n\lambda$ and $\|f(x)\|_E\leqq\lambda$ a.e. $x\notin\bigcup_j Q_j$.

We decompose the function

$$f = g+b = g+\textstyle\sum_j b_j$$

where

$$b_j(x) = \left(f(x)-\frac{1}{|Q_j|}\int_{Q_j} f\right)\chi_{Q_j}(x).$$

Now, we estimate the measures

$$\beta\big(\{(x, v)\colon\ \|Sg(x, v)\|_F > \lambda\}\big) \quad \text{and} \quad \beta\big(\{(x, v)\colon\ \|Sb(x, v)\|_F > \lambda\}\big).$$

Using that $\|g(x)\|_E\leqq C\lambda$ and the boundedness of S on L^{p_0}, it can be seen that the first measure is less than

$$\frac{C}{\lambda}\int_{\mathbf{R}^n}\|g(x)\|_E\,dx \leqq \frac{C}{\lambda}\int_{\mathbf{R}^n}\|f(x)\|_E\,dx. \tag{15}$$

About the second, we put $\Omega_\lambda^* = \bigcup_j \Psi(NQ_j)$ and

$$\beta(\{(x, v)\colon \|Sb(x, v)\|_F > \lambda\}) \leqq \beta(Q_\lambda^*) + \beta(\{(x, v)\notin \Omega_\lambda^*\colon \|Sb(x, v)\|_F > \lambda\}).$$

We already made in the proof of Theorem 1 the computation that shows

$$\beta(\Omega_\lambda^*) \leqq \frac{C}{\lambda}\int_{\mathbf{R}^n}\|f(x)\|_E\,dx. \tag{16}$$

Finally, the properties of K can be used in order to prove

$$\beta(\{(x, v)\notin \Omega_\lambda^*\colon \|Sb(x, v)\|_F > \lambda\}) \leqq \frac{C}{\lambda}\int_{\mathbf{R}^n}\|f(x)\|_E\,dx. \tag{17}$$

Pasting up together inequalities (15), (16), (17) we get that S is of weak type (1, 1) and then S maps $L_E^q(\mathbf{R}^n, dx)$ into $L_F^q(\mathbf{R}^n\times V, d\beta)$ for $1<q\leqq p_0$.

Now, we consider the sequence valued operator

$$\tilde{S}(\{f_j\}_{j=1}^\infty) = \{Sf_j\}_{j=1}^\infty.$$

It is obvious that S maps $L_{l^q(E)}^q(\mathbf{R}^n, dx)$ into $L_{l^q(F)}^q(\mathbf{R}^n\times V, d\beta)$ for $1<q\leqq p_0$. Moreover, $\tilde{S}$ has a kernel

$$\tilde{K}\colon \mathbf{R}^n\times\mathbf{R}^n\times V \to \mathscr{L}(l^q(E), l^q(F))$$

given by $\tilde{K}(x, y, v)[\{a_j\}_j] = \{K(x, y, v)\alpha_j\}_j$, $\{\alpha_j\}\subset E$. Then $\|\tilde{K}(x, y, v)\| = \|K(x, y, v)\|$ and so $\tilde{K}$ satisfies conditions (a) and (b). In particular, we can reproduce the proof made for S and we shall have that $\tilde{S}$ maps $L_{l^q(E)}^p(\mathbf{R}^n, dx)$ into $L_{l^q(F)}^p(\mathbf{R}^n\times V, d\beta)$, $1<p\leqq q\leqq p_0$ and $L_{l^q(E)}^1(\mathbf{R}^n, dx)$ into weak-$L_{l^q(F)}^1(\mathbf{R}^n\times V, d\beta)$, $1<q\leqq p_0$.

4. Examples

In addition to the examples A, B, C named in the introduction we shall mention the two following:

D. If $\mathbf{R}^n\times V = \mathbf{R}^n$, $U = \mathbf{R}^d$, $\Psi(Q) = Q$, $\Phi(Q) = Q\times Q'$ (where Q' is the cube in $\mathbf{R}^d$ with center 0 and radius Q' = radius Q) and $d\alpha(x, u) = dx\,du = dz$ (Lebesgue measure on $\mathbf{R}^{n+d} = \mathbf{R}^n\times\mathbf{R}^d$), then the operator

$$Tf(x) = \sup_{x\in Q\subset\mathbf{R}^n}\frac{1}{|Q|}\int_{Q\times Q'}|f(z)|\,dz$$

is the trace on the hyperplane $\mathbf{R}^n$ of the maximal fractional operator on $\mathbf{R}^{n+d}$

$$M_p f(y) = \sup_{y\in Q\subset \mathbf{R}^{n+d}} \frac{1}{|Q|^{1-(d/n+d)}} \int_Q |f(z)|dz,$$

that is, $Tf(x)=M_d f(x, 0)$. In fact, it is enough to observe that $|Q\times Q'|^{1-(d/n+d)}=|Q|$, $Q\subset \mathbf{R}^n$.

Then the general results can be applied and, for instance, Theorem 4 gives the following weighted norm inequalities:

Proposition. *Let* $1<p<\infty$.

$$\int_{\mathbf{R}^n} |M_d f(x, 0)|^p v(x)\, dx \leqq C \int_{\mathbf{R}^{n+d}} |f(z)|^p \omega(z)\, dz$$

if and only if for cube Q *in* $\mathbf{R}^n$

$$\int_Q (M_d(\chi_{Q\times Q'}\omega^{1-p'})(x, 0))^p v(x)\, dx \leqq C \int_{Q\times Q'} \omega(z)^{1-p'}\, dz < +\infty$$

E. If $\mathbf{R}^n\times U=\mathbf{R}^n$, $V=\mathbf{R}^n$, $\Phi(Q)=Q$, $\Psi(Q)=Q\times Q$ and $d\alpha(x, u)=dx$, then the operator T defined in (1) satisfies

$$Tf(x, y) \sim \mathcal{M}f(x, |x-y|), \quad x, y\in \mathbf{R}^n$$

where $\mathcal{M}$ is the operator introduced in the Example B, and so, by applying Theorem 4 we can get inequalities of the type

$$\int_{\mathbf{R}^n\times\mathbf{R}^n} \mathcal{M}f(x, |x-y|)^p\, d\beta(x, y) \leqq C \int_{\mathbf{R}^n} |f(x)|^p v(x)\, dx.$$

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