Parabolic vector bundles on curves

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Seshadri introduced the notion of parabolic structures on vector bundles [4] and later constructed a moduli space for semistable parabolic vector bundles on curves [2]. In this small note we describe a different construction of the moduli space generalising the method of Gieseker [1]. This has some advantages. This construction is much simpler and shorter than in [2]. It avoids the use of unitary bundles and hence is applicable in positive characteristics. One does not need the introduction and comparison of different parabolic structures. Moreover, some computations which have to be repeated here (proposition 2) become simpler in this method. The generalisation to parabolic principal bundles will be considered in a subsequent paper.

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1. Preliminaries

Let X be an irreducible nonsingular projective curve.

Let S be the set of all parabolic semistable vector bundles of rank k, degree d, parabolic degree zero, a fixed quasi-parabolic structure at a given point x_0 in X with fixed weights $0<\alpha_1<\alpha_2<\ldots<\alpha_r<1$. One knows that S is bounded i.e. there exists m_0 such that for $m\geqq m_0$, we have $H^1(E(m))=0$ and $H^0(E(m))$ generates $E(m)$ for all E in S. Also for any real number q, the set of all subbundles F of elements of S with degree $\geqq q$ is bounded (p. 226, [2]).

Let g be the genus of X. Choose an integer $m\gg g$ so that $H^1(F(m))=0$ and $H^0(F(m))$ generates $F(m)$ for F in S or $F\subset E$, E in S and degree $F\geqq(-g-8)k$. Let $n=h^0(E)$, E in S. Let Q be the Hilbert scheme of coherent sheaves over X which are quotients of $\mathcal{O}_X^n$ and whose Hilbert polynomial is that of $E(m)$ with E in S. Let $\mathcal{F}$ be the universal family on $Q\times X$. Let R be the subscheme of Q consisting of points q in Q such that $H^1(\mathcal{F}_q)=0$, $h^0(\mathcal{F}_q)=n$, $\mathcal{F}_q$ is locally free and generically generated by global sections. Notice that R contains the set of points in Q corresponding to elements of S. Let $G(\mathscr{J})$ be the flag bundle over $R\times\{x_0\}$ of type determined by the quasiparabolic structure at x_0 and let $\tilde{R}$ be the total space of this flag bundle. $\tilde{R}$ has the local universal property for parabolic vector bundles. Let $\tilde{R}^{ss}$ (respectively $\tilde{R}^s$) be the set of points of $\tilde{R}$ corresponding to parabolic semistable (respectively stable) vector bundles. The group $G=SL(n)$ acts on R, $\tilde{R}$ keeping $\tilde{R}^{ss}$ and $\tilde{R}^s$ invariant.

Let $d_0=d+km$, $A=Pic^{d_0}(X)$, $g\colon X\times A\to A$ projection and M the Poincare bundle on $X\times A$. Let

$$Z = \mathbf{P}(\mathscr{H}om\,(\Lambda^k \mathcal{O}_A^n,\, g_*M)^*).$$

$SL(n)$ acts on Z preserving the fibres over A. Let Z^{ss} (resp. Z^s) denote the set of semistable points (resp. stable points) of Z for the linearisation of the action of $SL(n)$ with respect to the ample line bundle $\mathcal{O}_Z(m-g)$. Given a *'good pair'* (F, φ) where F is a flat family of vector bundles parametrised by S such that for all s in S, F_s is generated by global sections at the generic point of $X\times s$ and $\varphi\colon \mathcal{O}_S^n\to p_*F$ is an isomorphism, one gets a morphism $T(F,\varphi)\colon S\to Z$ ([1]; p. 57).

For s in S, $T(F,\varphi)(s)$ is the composite $\Lambda^k K^n \xrightarrow{\Lambda^k\varphi_s} \Lambda^k H^0(F_s) \xrightarrow{\Psi} H^0(\Lambda^k F_s)$, where Ψ is the natural map. Notice that if z is a closed point of Z with $g(z)=L$ in A, then z can be regarded as an element of $\operatorname{Hom}(\Lambda^k K^n, H^0(L))$.

Suppose, in addition, that F is a family of parabolic vector bundles with a fixed parabolic structure at x_0. The underlying quasiparabolic structure on F_s is given by a flag

$$(F_s)_{x_0} = F_1(F_s)_{x_0} \supset F_2(F_s)_{x_0} \supset \ldots \supset F_l(F_s)_{x_0} \supset 0.$$

This induces (via φ) a flag on $K^n=H^0(F_s)$

$$K^n = F_1(F_s) \supset F_2(F_s) \supset \ldots \supset F_l(F_s) \supset F_{l+1}(F_s),$$

where $F_{l+1}(F_s)$ is the kernel of the evaluation map $e_{x_0}\colon K^n\to(F_s)_{x_0}$ and $F_i(F_s)=e_{x_0}^{-1}(F_i(F_s)_{x_0})$ for all $i=1,2,\ldots,l$. Let $f_i=\dim F_i(F_s)$, $i=1,\ldots,l+1$. Let $G_r=\prod_i G_{n,f_i}$, where G_{n,f_i} denotes the Grassmannian of f_i-dimensional vector subspaces of K^n. Let L_i be the ample generator of $\operatorname{Pic} G_{n,f_i}$. On G_r, take the polarisation

$$L = \alpha_1 L_1 + (\alpha_2-\alpha_1)L_2 + \ldots + (\alpha_l-\alpha_{l-1})L_l + (1-\alpha_l)L_{l+1} = \textstyle\sum_i \varepsilon_i L_i.$$

One has a morphism $f\colon S\to G_r$ which associates to s in S the element $(F_i(F_s))$ in G_r. Thus we have a morphism $\tilde{T}(F,\varphi)\colon S\to Z\times G_r$, $\tilde{T}(F,\varphi)=T(F,\varphi)\times f$. Let T denote the morphism $\tilde{R}\to Z\times G_r$ thus obtained; it is $SL(n)$-invariant. On $Z\times G_r$, we take linearisation with respect to the ample line bundle $\mathcal{O}_Z(m-g)+L$.

Remarks and definitions 1

1(i) A point $(\tau, (F_i))$ in $Z\times G_r$ is semistable (or stable) if and only if for any subspace W of $V=K^n$, we have

$$\dim W\left(\textstyle\sum_i \varepsilon_i \dim F_i\right)-\dim V\left(\textstyle\sum_i \dim W\cap F_i\right)$$
$$+(m-g)(d\dim V-k\dim W)\geqq 0 \quad (\text{or } >0),$$

where d is the maximum of the cardinalities of τ-independent subsets of W.

Define $\operatorname{wt} V=\alpha_1(f_1-f_2)+\alpha_2(f_2-f_3)+\dots+\alpha_l(f_l-f_{l+1})$ $f_i=\dim F_i$; $\operatorname{wt} W=\sum \alpha_i(f_i'-f_{i+1}')$, $f_i'=\dim W\cap F_i$. Then $\operatorname{wt} V+f_{l+1}=\sum \varepsilon_i f_i$, $\operatorname{wt} W+f_{l+1}'=\sum \varepsilon_i f_i'$ and the above condition reduces to

$$\sigma_W\equiv \dim W(\operatorname{wt} V+f_{l+1})-\dim V(\operatorname{wt} W+f_{l+1}')$$
$$+(m-g)(d\dim V-k\dim W)\geqq 0 \quad (\text{or } >0).$$

1(ii) For E in $\tilde{R}$ and $F\subset E$, define

$$\chi_F=n[(m-g)\operatorname{rk} F+(\operatorname{rk} F-\operatorname{wt} F)]$$
$$-h^0(F(m))[(m-g)k+(\operatorname{rk} E-\operatorname{wt} E)]$$

where wt denotes the parabolic weight of the vector bundle.

1(iii) Let $W\subset V$, let $F(m)$ be the subbundle generated by W. Using the Riemann—Roch theorem and $\operatorname{wt} E=-\operatorname{degree} E$, one sees that

$$\chi_F=-n[\operatorname{paradeg} F+h^1(F(m))],$$

where paradeg denotes the parabolic degree i.e. degree+parabolic weight. In particular, if $H^1(F(m))=0$, $\chi_F=-n \operatorname{paradeg} F$.

1(iv) If $F(m)$ is generated by global sections, $W=H^0(F(m))$ and

$$H^1(F(m))=0,$$

then

$$\chi_F=\sigma_W.$$

2. The main results

Proposition 2.

(a) $q\in\tilde{R}^{ss}\Rightarrow T(q)\in(Z\times G_r)^{ss}$,

(b) $q\in\tilde{R}^{s}\Rightarrow T(q)\in(Z\times G_r)^{s}$,

(c) $q\in\tilde{R}$, $q\notin\tilde{R}^{ss}\Rightarrow T(q)\notin(Z\times G_r)^{ss}$

(d) $q\in\tilde{R}^{ss}-\tilde{R}^{s}\Rightarrow T(q)\notin(Z\times G_r)^{s}$.

Proof. *(c)* Let q correspond to a parabolic vector bundle E. Suppose E is not parabolic semistable. Then there exists a parabolic semistable subbundle F of E with $\text{paradeg}\, F > \text{paradeg}\, E = 0$. Let $W = H^0(F(m))$. As $m \gg g$, we have by 1(iii) and 1(iv),

$$\sigma_W = \chi_F = -n\, \text{paradeg}\, F < 0,$$

contradicting the semistability condition for $T(q)$.

(d) Suppose that E is parabolic semistable but not stable. Then there exists a parabolic stable subbundle F of E with $\text{paradeg}\, F = 0$.

As above, for $W = H^0(F(m))$, we have

$$\sigma_W = \chi_F = 0.$$

(a) and (b) Suppose E is a parabolic semistable (or stable) bundle. Let W be a subspace of V and let $F(m)$ be the subbundle of E generically generated by W.

Case 1. If W satisfies the conditions of 1(iv) we get $\sigma_W = \chi_F = -n\, \text{paradeg}\, F$

$$\geqq 0 \quad \text{or} \quad > 0$$

according to E being parabolic semistable or stable.

Case 2. If $\deg F \leqq (-g-8)k$, then we will show that $\sigma_W > 0$. Let $V'' = H^0(F(m))$. Using the fact that $d = \text{rk}\, F$ in 1(i), we get

$$(*) \qquad \begin{aligned} \sigma_W - \chi_F = {} & n(\text{wt}\, F - \text{rk}\, F + \dim W - f'_{l+1} - \text{wt}\, W) \\ & + (\dim V'' - \dim W)[k(m-g) + (k - \text{wt}\, E)]. \end{aligned}$$

Now, $\text{wt}\, E = \sum_{i=1}^{l} \alpha_i (f_i - f_{i+1}) < \sum (f_i - f_{i+1}) = k = \dim V - f_{l+1}$; similarly, $\text{wt}\, W < \dim W - f'_{l+1}$. Hence,

$$\sigma_W - \chi_F \geqq n(\text{wt}\, F - \text{rk}\, F) \geqq -n\, \text{rk}\, F \geqq -2k(m-g)\, \text{rk}\, F.$$

As in Lemma 4.3 [2], $\chi_F \geqq 3k(m-g)\, \text{rk}\, F$. So,

$$\sigma_W = (\sigma_W - \chi_F) + \chi_F \geqq k\, \text{rk}\, F(m-g) > 0.$$

Case 3. $\deg F > (-g-8)k$. In this case, by our choice of m, we have $H^1(F(m)) = 0$ and $V'' = H^0(F(m))$ generates $F(m)$. If $V'' = W$, we are through by case 1. Hence, may assume $V' \neq W$. By 1(iii), we have $\chi_F \geqq 0$ or > 0 according to E being parabolic semistable or stable. Soit suffices to show that $\sigma_W - \chi_F \geqq 0$. Using the commutative diagram

$$\begin{array}{ccccc} V'' & \longrightarrow & F(m)_{x_0} & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \\ W & \longrightarrow & \text{Im}\, W \approx W/W \cap F_{l+1}, & & \end{array}$$

we have $\dim V'' - \dim W \geqq \operatorname{rk} F - \dim W - f'_{i+1}$. From (*), we then have

$$\begin{aligned}\sigma_{W} - \chi_F &\geqq n(\operatorname{wt} F - \operatorname{wt} W) - n \dim (V'' - \dim W) \\ &\quad + n(\dim V'' - \dim W), \quad \text{as} \quad n = (k - \operatorname{wt} E) + k(m-g) \\ &= n(\operatorname{wt} F - \operatorname{wt} W) \\ &\geqq 0.\end{aligned}$$

This completes the proof of the proposition.

Proposition 3. *The morphism*

$$T\colon \tilde{R}^{ss} \to (Z \times G_r)^{ss}$$

is proper.

Proof. Let P be a closed point of a nonsingular curve C. Let $\Psi\colon C - P \to \tilde{R}^{ss}$ be a morphism such that $T \circ \Psi$ extends to a morphism $\overline{T \circ \Psi}\colon C \to (Z \times G_r)^{ss}$. By the valuative criterion for properness, to show $T/\tilde{R}^{ss}$ is proper, it suffices to show that the dotted arrow in the following commutative diagram can be realised.

$$\begin{array}{ccc} C - P & \xrightarrow{\Psi} & \tilde{R}^{ss} \\ \big\downarrow & \nearrow & \big\downarrow T \\ C & \xrightarrow{\overline{T \circ \Psi}} & (Z \times G_r)^{ss} \end{array} .$$

On $X \times (C - P)$ we have the surjective morphism $\varphi\colon \mathcal{O}^n \to \mathscr{J}' = (\operatorname{Id} x\, \Psi)^* \mathscr{J}$. By lemma 4.2 [1] or otherwise, we can extend $\mathscr{J}'$ to $\bar{\mathscr{J}}'$ over $X \times C$, flat over C, so that $\varphi_P\colon \mathcal{O}_X^n \to \bar{\mathscr{J}}'_{X \times P}$ is generically surjective. By the completeness of the flag variety, there exists a parabolic structure on $\bar{\mathscr{J}}'_{X \times P}$ (see the beginning of the proof of proposition 3.3, [2]). Thus $\tilde{T}_{(\mathscr{J}', \varphi)}$ extends to $\tilde{T}(\bar{\mathscr{J}}', \bar{\varphi})$ and $T(\bar{\mathscr{J}}', \bar{\varphi})P = \overline{T \circ \Psi}(P)$ is in $(Z \times G_r)^{ss}$. In particular, $\bar{\mathscr{J}}'_{X \times P}$ is generated at (x_0, P) by global sections of $\bar{\mathscr{J}}'_{X \times P}$. By proposition 2(c), it follows that $\bar{\mathscr{J}}'_{X \times P}$ is parabolic semistable and hence

$$(p_2)_*(\bar{\varphi})\colon \mathcal{O}_C^n \to (p_2)_* \bar{\mathscr{J}}'$$

is an isomorphism. Thus Ψ extends to a map $C \to \tilde{R}^{ss}$ realising the dotted arrow in the diagram. This proves that $T/\tilde{R}^{ss}$ is proper.

Theorem. *There exists a coarse moduli scheme M for the equivalence classes of semistable parabolic vector bundles (on an irreducible nonsingular complete curve X) of rank k, degree d, parabolic* degree 0 *and having a fixed parabolic structure at a given point x_0 in the curve X. The scheme M is a normal projective variety of dimension $k^2(g-1)+1+\dim F$ where F is the flag variety of type determined by the fixed* *quasiparabolic structure. The subset M^s of M corresponding to stable parabolic bundles is a smooth open subvariety.*

Proof. From the construction of $\tilde{R}^{ss}$ and lemma 4.3 [1] it follows that T is injective. Thus T is a proper injective and hence an affine morphism from $\tilde{R}^{ss}$ into $(Z\times G_r)^{ss}$ which is $SL(n)$-invariant. By proposition 3.12 [3], a good quotient of $\tilde{R}^{ss}$ exists if a good quotient of $(Z\times G_r)^{ss}$ by $SL(n)$ exists. The existence of a good quotient of $(Z\times G_r)^{ss}$ by $SL(n)$ is well-known. Hence the good quotient M of $\tilde{R}^{ss}$ by $SL(n)$ exists. Since $\tilde{R}^{ss}$ is a nonsingular quasiprojective variety of dimension $k^2(g-1)+1+n^2-1+\dim F$, it follows that M is a normal projective variety of dimension $k^2(g-1)+1+\dim F$. Also, $\tilde{R}^s\to M^s$ is a geometric quotient, M^s is nonsingular as $\tilde{R}^s$ is so.

3. Relation with Seshadri's method

In Theorem 4.1 [2] Seshadri gives an injective morphism $f\colon \tilde{R}^{ss}\to(H^N_{n,k}\times\prod_i H_{n,t_i})^{ss}$, where $H_{n,t}$ denotes the grassmannian of t-dimensional quotient spaces of K^n and N is a sufficiently large integer. Under duality isomorphism, one has $H_{n,t_i}\approx G_{n,f_i}$ in our notation i.e. $\prod_i H_{n,t_i}\approx G_r$. The induced map $\tilde{R}^{ss}\to H^N_{n,k}$ associates (roughly) to a vector bundle E the quotients $(H^0(E)\to E_{x_i})_i$, E_{x_i} being the fibre of E at a fixed point x_i of X, for $i=1, 2, \dots, N$. We shall show that f factors through T. The morphisms $\tilde{R}^{ss}\to G_r$ induced by T and f are the same under the above identification. Now, $Z=\mathbf{P}(\mathscr{H}om(\Lambda^k\mathcal{O}^n_A, g_*M)^*)$. The isomorphism $g^*(\mathrm{Hom}(\Lambda^k\mathcal{O}^n_A, g_*M))\xrightarrow{\sim}(\Lambda^k\mathcal{O}^n_{X\times A})^*\otimes g^*g_*M$ composed with the evaluation map $g^*g_*M\to M$ gives, for each x in X, a morphism $\Psi'_X\colon \mathrm{Hom}(\Lambda^k\mathcal{O}^n_A, g_*M)\to(\Lambda^k\mathcal{O}^n_A)^*\otimes(M/x\times A)$, and hence a rational map

$$\Psi_x\colon Z\to\mathbf{P}(\Lambda^k\mathcal{O}^n_A).$$

Roughly speaking $\Psi_x\circ T$ associates to a bundle E the one-dimensional quotient $\Lambda^k E_x$ of $\Lambda^k K^n$ i.e. an element of $H_{n,k}$ embedded in $\mathbf{P}(\Lambda^k K^n)$. Let $\Psi=\Psi_{x_1}\times\ldots\times\Psi_{x_N}$. The rational map Ψ is a morphism on $T(\tilde{R}^{ss})$ as bundles representing elements of $\tilde{R}^{ss}$ are generated by global sections. Clearly, $\Psi\circ T=f$. The main difficulty in [2] was that the morphism f could not be shown proper (remark 4.5, [2]). We have factored f though a proper morphism T (prop. 3) making a direct application of geometric invariant theory possible and thereby avoiding the complications in [2]. The following simple example for $k=2$ indicates that Ψ_{ss} is not proper in general, Ψ_{ss} being the restriction ot Ψ to the largest subset of Z^{ss} on which Ψ is a morphism.

Example 5. Let V and W be two vector spaces over $\mathbf{C}$, the dimension of V being even. Fix a basis (e_i) of W, let p_i denote the projection on the i^{th} coordinate i.e. $p_i\colon W\to \mathbf{C}$, $p_i(\sum_j w_j e_j)=w_i$. Then p_i induces a linear map $\operatorname{Hom}(\Lambda^2 V, W)\to (\Lambda^2 V)^*$ and hence a rational map

$$\Psi_i\colon P=\mathbf{P}(\operatorname{Hom}(\Lambda^2 V, W)^*)\to \mathbf{P}(\Lambda^2 V).$$

Let $\Psi=\prod_i \Psi_i$. $SL(V)$ acts linearly on both sides above, the action on W being trivial. It is easy to check that $\mathcal{O}(1)$ pulls back to $\mathcal{O}(1)$ under Ψ_i, $i=1, \dots, l$. So if we take the polarisation given by $\mathcal{O}(l)$ on P and by $\mathcal{O}(1)$ on each $\mathbf{P}(\Lambda^2 V)$, we get an induced map $\Psi\colon P^{ss}\to\prod_i \mathbf{P}(\Lambda^2 V)^{ss}$, ss denoting semistable points for the above linearisations of $SL(V)$ action. Let D denote the maximal subset of P on which Ψ is a morphism. Let bar '———' denote the image in projective space. Then for $T\in\operatorname{Hom}(\Lambda^2 V, W)$, $\overline{T}\in D$ iff $p_i\circ T\neq \circ\,\forall i$. We shall show that the morphism

$$\Psi_{ss}\colon D^{ss}\to (\prod_i \mathbf{P}(\Lambda^2 V))^{ss}$$

is not proper using the valuative criterion. The elements of $(\Lambda^2 V)^*$ can be regarded as alternating bilinear forms and the nondegenerate forms are semistable for $SL(V)$-action. Let $t\in(0, 1)$. Let $(\overline{T_i(t)})\in(\prod_i \mathbf{P}(\Lambda^2 V))^{ss}$ be given by $T_i(t)=$ the alternating form given by the matrix $\begin{bmatrix} 0 & t^{-a_i}I \\ t^{-a_i}I & 0\end{bmatrix}$, I being the identity matrix of rank$=\frac{1}{2}$ dim. V and a_is are integers *not* all equal. Then

$$\lim_{t\to 0}\overline{T_i(t)}=\lim_{t\to 0}(t^{a_i}T_i(t))^- = \begin{bmatrix} 0 & I \\ -I & 0\end{bmatrix}^-$$

belongs to $\mathbf{P}(\Lambda^2 V)^{ss}$ & hence $\lim_{t\to 0}(\overline{T_i(t)})_i\in(\prod \mathbf{P}(\Lambda^2 V))^{ss}$. Now, an element $T\in\operatorname{Hom}(\Lambda^2 V, W)$ is just an l-tuple of forms $(p_i\circ T)_{1\le i\le l}$. Define $T(t)$ by $p_i\circ T(t)=T_i(t)$, so $\Psi(\overline{T(t)})=(\overline{T_i(t)})$. Let $a_0=\max(a_i)$. Then

$$\overline{T_0}=\overline{\lim_{t\to 0}T(t)}=(\lim_{t\to 0}(t^{a_0}T(t)))^-=\overline{(b_i)}$$

where

$$b_i=\begin{cases}\begin{bmatrix} 0 & I \\ -I & 0\end{bmatrix} & \text{if } a_i=a_0,\\ 0, & \text{otherwise.}\end{cases}$$

Thus $T_0\notin D$ showing that Ψ_{ss} is not proper.

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