The two-sided complex moment problem

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Hamburger's theorem asserts that a function $\varphi\colon \mathbf{N}_0\to\mathbf{R}$ is the moment sequence of a measure on the real line if and only if φ is positive definite in the sense that the kernel $(n, m)\to\varphi(n+m)$ is positive semidefinite. (See [1], Theorem 6.2.2). The corresponding two-sided problem, consisting in characterizing those functions $\varphi\colon \mathbf{Z}\to\mathbf{R}$ which are the two-sided moment sequences of measures μ on $\mathbf{R}\setminus\{0\}$ in the sense of

$$\varphi(n) = \int x^n\, d\mu(x), \quad n\in\mathbf{Z},$$

has a similar solution: Such functions φ are precisely those which are positive definite in the sense that the kernel $(n, m)\to\varphi(n+m)$ is positive semidefinite. This was shown in [4]; see [1], Theorem 6.4.1, for a simple proof.

The complex moment problem, a natural analogue of the moment problem solved by Hamburger, requires the characterization of those functions $\varphi\colon \mathbf{N}_0^2\to\mathbf{C}$ which are complex moment sequences of measures μ on $\mathbf{C}$ in the sense that

$$\varphi(n, m) = \int z^n \bar{z}^m\, d\mu(z), \quad (n, m)\in\mathbf{N}_0^2.$$

In this case the pertinent concept of positive definiteness arises by considering the semigroup $(\mathbf{N}_0^2, +)$ with the involution (*) given by $(n, m)^*=(m, n)$ and agreeing to call a function $\varphi\colon \mathbf{N}_0^2\to\mathbf{C}$ positive definite if the kernel $(s, t)\to\varphi(s+t^*)$ on $\mathbf{N}_0^2\times\mathbf{N}_0^2$ is positive semidefinite. While every complex moment sequence is positive definite, there exist positive definite functions on $\mathbf{N}_0^2$ which are not complex moment sequences ([1], Theorem 6.3.5).

Observe that [1], Theorem 6.1.10 implies the following somewhat roundabout solution of the complex moment problem: A function $\varphi\colon \mathbf{N}_0^2\to\mathbf{C}$ is a complex moment sequence if and only if $\sum_{n,m} c_{n,m}\varphi(n, m)\geqq 0$ for each $(c_{n,m})\in\mathbf{C}^{(\mathbf{N}_0^2)}$ such that $\sum_{n,m} c_{n,m}z^n\bar{z}^m\geqq 0$ for all $z\in\mathbf{C}$.

Like Hamburger's moment problem, the complex moment problem has a two-sided companion, the problem of characterizing two-sided complex moment sequences, that is, functions $\varphi\colon \mathbf{Z}^2 \to \mathbf{C}$ representable in the form

$$\varphi(n, m) = \int z^n \bar{z}^m \, d\mu(z), \quad (n, m) \in \mathbf{Z}^2$$

with μ a measure on $\mathbf{C}\setminus\{0\}$.

Consider the group $\mathbf{Z}^2$ with the involution $(n, m)^* = (m, n)$ and define a function $\varphi\colon \mathbf{Z}^2 \to \mathbf{C}$ to be positive definite in case the kernel $(s, t) \to \varphi(s+t^*)$ on $\mathbf{Z}^2 \times \mathbf{Z}^2$ is positive semidefinite. We shall show a necessary and sufficient condition for a complex function on $\mathbf{Z}^2$ to be a two-sided complex moment sequence is that φ be positive definite. This is somewhat remarkable in view of the failure of the corresponding condition in the one-sided complex moment problem.

From [1] we recall a few definitions. A (abelian) **-semigroup* $(S, +, {}^*)$ consists of an abelian semigroup $(S, +)$ with zero 0 and an *involution*, that is, an involutory automorphism, in S, written as $s \to s^*$. A function $\varphi\colon S \to \mathbf{C}$ is *positive definite* if for any $N \in \mathbf{N}$ and any $s_1, \ldots, s_N \in S$ the $N \times N$ matrix $(\varphi(s_j + s_k^*))$, $j, k = 1, \ldots, N$, is positive semidefinite. A *character* (called 'semicharacter' in [1]) is a function $\varrho\colon S \to \mathbf{C}$ satisfying $\varrho(0) = 1$, $\varrho(s^*) = \overline{\varrho(s)}$, $\varrho(s+t) = \varrho(s)\varrho(t)$ for all $s, t \in S$. The set of characters on S, denoted by S^*, is considered with the topology of pointwise convergence. If μ is a Radon measure on S^* such that $\varrho \to \varrho(s)\colon S^* \to \mathbf{C}$ is μ-integrable for each $s \in \mathbf{C}$, the function $\hat{\mu}\colon S \to \mathbf{C}$ defined by

$$\hat{\mu}(s) = \int \varrho(s) \, d\mu(\varrho), \quad s \in S \tag{1}$$

is called a *moment function* and the measure μ is said to *represent* $\hat{\mu}$. The moment function $\hat{\mu}$ is *determinate* if no Radon measure on S^* other than μ represents $\hat{\mu}$. A *perfect semigroup* is a *-semigroup S such that every positive definite function on S is a determinate moment function.

Examples. (1) If S is an abelian group equipped with the involution $s^* = -s$, the characters on S are the usual group characters; by the discrete version of the Bochner—Weil theorem S is a perfect semigroup. In particular, the group $\mathbf{Z}$ with the involution $n^* = -n$ is perfect (Herglotz' Theorem, also known as Herglotz' Lemma).

(2) Let S be either of the semigroups $(\mathbf{N}_0, +)$ or $(\mathbf{Z}, +)$, equipped with the identical involution $n^* = n$. By the solution of Hamburger's moment problem or its two-sided version, every positive definite function on S is a moment function. Yet S is not perfect since there exist indeterminate moment functions, such as $n \to \exp(n^2)$ ([1], Example 6.4.6).

For a study of perfect semigroups we refer to [1], § 6.5.

If S and T are *-semigroups, we may consider the Cartesian product semigroup $S \times T$ with the involution given by $(s, t)^* = (s^*, t^*)$. By [1], Theorem 6.5.4, the product of two perfect semigroups is again perfect. It is natural to wonder whether the class of *-semigroups with the property that every positive definite function is a moment function is likewise stable under the formation of Cartesian products (of two factors).

A simple counterexample is provided by the semigroup $(\mathbf{N}_0, +)$: Although every positive definite function on $\mathbf{N}_0$ is a moment function (Hamburger's theorem), the semigroup $\mathbf{N}_0 \times \mathbf{N}_0$ admits a positive definite function which is not a moment function ([1], Theorem 6.3.4).

If, however, we require one factor to be a perfect semigroup and if we impose one additional constraint (dictated by proof technique rather than by examples showing its necessity), we obtain a positive result:

Proposition 1. *If S is a perfect semigroup and if T is a finitely generated *-semigroup such that every positive definite function on T is a moment function then every positive definite function on $S \times T$ is a moment function.*

Proof. The greater part of the proof consists in copying portions of the proof of [1], Theorem 6.5.4, to which we therefore find it convenient to refer.

Let $\varphi \colon S \times T \to \mathbf{C}$ be positive definite. The proof of [1], Theorem 6.5.4 up to and including line 7 from below on p. 205, except for the words 'uniquely determined' in the last line, holds verbatim in our situation. Hence, for each $t \in T$ there is a unique complex Radon measure μ_t on S^* such that

$$\varphi(s, t) = \int \varrho(s)\, d\mu_t(\varrho), \quad s \in S;$$

and for each Borel set A in S^* there is a Radon measure σ_A on T^* (denoted by τ_A in [1]) such that

$$\mu_t(A) = \int \zeta(t)\, d\sigma_A(\zeta), \quad t \in T.$$

In contrast to the situation of [1], Theorem 6.5.4 (where T is assumed to be perfect), our mapping $A \to \sigma_A$ cannot be expected to be even finitely additive. To remedy this, consider the set P of finite Borel partitions of S^*. The natural ordering on P is that of refinement, i.e. $\pi \leqq \pi'$ if and only if each element of π is the union of those elements of π' which it contains. With each Borel set A in S^* we associate an element π_A of P by $\pi_A = \{A, S^* \setminus A\}$ if $A \notin \{\emptyset, S^*\}$, $\pi_\emptyset = \pi_{S^*} = \{S^*\}$. For any $\pi \in P$ with $\pi \geqq \pi_A$ define a Radon measure $\sigma_{\pi, A}$ on T^* by

$$\sigma_{\pi, A} = \sum\nolimits_{D \in \pi \colon D \subset A} \sigma_D.$$

Since the elements of π contained in A form a Borel partition of A,

$$\hat{\sigma}_{\pi, A}(t) = \sum\nolimits_{D \in \pi \colon D \subset A} \hat{\sigma}_D(t) = \sum\nolimits_{D \in \pi \colon D \subset A} \mu_t(D) = \mu_t(A)$$

for each $t \in T$.

Select a universal subnet (π_i) of the identical net on P. For each Borel set A in S^* the net $(\sigma_{\pi_i, A})$, indexed by those i for which $\pi_i \geqq \pi_A$, is a universal net in the set of representing measures of the positive definite function $t \to \mu_t(A)$ on T. The latter set is weakly compact, T being finitely generated ([1], Proposition 6.1.7). Hence,

$$\sigma_{\pi_i, A} \to \tau_A$$

weakly for some Radon measure τ_A on T^* with $\hat{\tau}_A(t) = \mu_t(A)$, $t \in T$. We claim the mapping $A \to \tau_A$ has the following three properties:

(i) $\tau_\emptyset = 0$;

(ii) $\tau_A = \sum_{n \geqq 1} \tau_{A_n}$ whenever $A_1, A_2, \ldots$ are pairwise disjoint Borel sets in S^* with union A;

(iii) $\tau_A = \sup \{\tau_C | C$ compact, $C \subset A\}$ for each Borel set A in S^*.

First note that if A_1, A_2 are disjoint Borel sets in S^* then

$$\sigma_{\pi, A_1 \cup A_2} = \sigma_{\pi, A_1} + \sigma_{\pi, A_2}$$

whenever $\pi \in P$ is such that $\pi \geqq \pi_{A_1}$ and $\pi \geqq \pi_{A_2}$. It follows that $\tau_{A_1 \cup A_2} = \tau_{A_1} + \tau_{A_2}$. Thus the mapping $A \to \tau_A$ is finitely additive and, in particular, increasing.

Condition (i) needs no proof. To prove (ii), observe that the set function $\eta\colon \mathscr{B}(T^*) \to [0, \infty]$ ($\mathscr{B}(T^*)$ denoting the Borel σ-field in T^*) given by

$$\eta(B) = \sum_{n=1}^{\infty} \tau_{A_n}(B), \quad B \in \mathscr{B}(T^*)$$

is dominated by τ_A. By [1], 2.1.28, it follows that η is a Radon measure. Since

$$\eta(T^*) = \sum_{n=1}^{\infty} \tau_{A_n}(T^*) = \sum_{n=1}^{\infty} \mu_0(A_n) = \mu_0(A) = \tau_A(T^*),$$

we conclude that $\eta = \tau_A$.

To prove (iii), define $\vartheta\colon \mathscr{B}(T^*) \to [0, \infty]$ by

$$\vartheta(B) = \sup \{\tau_C(B) | C \text{ is a compact subset of } A\}$$

for $B \in \mathscr{B}(T^*)$. Then $\vartheta \leqq \tau_A$; by [1], 2.1.29, it follows that ϑ is a Radon measure. The equality

$$\vartheta(T^*) = \sup_C \tau_C(T^*) = \sup_C \mu_0(C) = \mu_0(A) = \tau_A(T^*)$$

shows $\vartheta = \tau_A$.

As shown in the proof of [1], Theorem 6.5.4, conditions (i), (ii), (iii) imply the existence of a Radon measure $\varkappa$ on $S^* \times T^*$ such that $\tau_A(B) = \varkappa(A \times B)$ for all $A \in \mathscr{B}(S^*)$, $B \in \mathscr{B}(T^*)$; and

$$\varphi(s, t) = \int \varrho(s)\zeta(t) d\varkappa(\varrho, \zeta), \quad (s, t) \in S \times T.$$

Thus if $(S \times T)^*$ is identified with $S^* \times T^*$ via the homeomorphism $(\varrho, \zeta) \to \varrho \otimes \zeta\colon S^* \times T^* \to (S \times T)^*$ given by $\varrho \otimes \zeta(s, t) = \varrho(s)\zeta(t)$ then the measure $\varkappa$ represents φ.

Remarks. (1) The hypothesis that T be finitely generated is used only to ensure that the set of representing measures of any positive definite function on T be weakly compact. That the latter condition holds if only T is countable is easy to verify, using the fact ([3], p. 53) that any countable projective system of Radon measures has a limit in the form of a Radon measure. Possibly, the assumption that T be finitely generated could be omitted altogether.

(2) Suppose S and T are *-semigroups such that every positive definite function on $S\times T$ is a moment function. It is fairly easy to verify that any positive definite function on S or T is a moment function. We conjecture that S or T must be perfect.

(3) Equation (1) makes sense for any measure μ defined on a σ-field in S^* rendering the integrands measurable — provided, of course, that the integrals exist. Suppose we redefine the terms 'moment function' and 'perfect semigroup' by admitting this wider class of representing measures (see [2] for a development of this idea). Then Proposition 1 holds without the assumption that T be finitely generated.

Theorem 1. *Consider the group* $(\mathbf{Z}^2, +)$ *with the involution* $(n, m)^*=(m, n)$. *A function* $\varphi\colon \mathbf{Z}^2\to\mathbf{C}$ *is positive definite if and only if there exists a Radon measure* μ *on* $\mathbf{C}\setminus\{0\}$ *such that for each* $(n, m)\in\mathbf{Z}^2$ *we have* $\int |z|^{n+m}\,d\mu(z)<\infty$ *and*

$$\varphi(n, m) = \int z^n\bar{z}^m\,d\mu(z). \tag{5}$$

Proof. The mapping $z\to\varrho_z\colon \mathbf{C}\setminus\{0\}\to(\mathbf{Z}^2)^*$ given by $\varrho_z(n, m)=z^n\bar{z}^m$, $z\in\mathbf{C}\setminus\{0\}$, $(n, m)\in\mathbf{Z}^2$, is a homeomorphism of $\mathbf{C}\setminus\{0\}$ onto $(\mathbf{Z}^2)^*$. Thus the functions φ on $\mathbf{Z}^2$ having a representation of the form (5) are just the moment functions on $\mathbf{Z}^2$; in particular, they are positive definite.

To prove the converse, let S denote the group $(\mathbf{Z}, +)$ equipped with the involution $n^*=-n$ and let T denote the group $(\mathbf{Z}, +)$ equipped with the involution $n^*=n$. We consider the group $G=S\times T$ with the product involution, i.e. $(p, q)^*=(-p, q)$.

The mapping $h\colon \mathbf{Z}^2\to G$ defined by $h(n, m)=(n-m, n+m)$ is a *-isomorphism of $\mathbf{Z}^2$ onto the *-stable subgroup H of G given by $H=\{(p, q)\in G\,|\,p+q\in 2\mathbf{Z}\}$. It therefore suffices to show that every positive definite function on H is a moment function.

By Herglotz' theorem, S is a perfect semigroup; by the above-mentioned solution of the two-sided Hamburger moment problem every positive definite function on T is a moment function. Now Proposition 1 implies that every positive definite function on G is a moment function.

Let $\varphi\colon H\to\mathbf{C}$ be positive definite and define $\Phi\colon G\to\mathbf{C}$ by $\Phi|H=\varphi$, $\Phi|(G\setminus H)=0$. Then Φ is positive definite. To see this, let $c_1, \ldots, c_n\in\mathbf{C}$ and

$s_1, \dots, s_n \in G$ be given. Note that H is of index 2 in G and that the element $a=(1,0)$ represents the coset $G \setminus H$. Defining $t_j = s_j + a$ and noting that $t_j + t_k^* = s_j + s_k^*$, we find

$$\sum_{j,k=1}^{n} c_j \overline{c_k} \Phi(s_j+s_k^*) = \sum_{s_j+s_k^* \in H} c_j \overline{c_k} \varphi(s_j+s_k^*)$$

$$= \sum_{s_j, s_k \in H} c_j \overline{c_k} \varphi(s_j+s_k^*) + \sum_{s_j, s_k \in G \setminus H} c_j \overline{c_k} \varphi(t_j+t_k^*) \geqq 0,$$

each term being nonnegative since in the first term the s_j, in the second term the t_j, are in H.

Since Φ is positive definite there is a Radon measure ν on G^* such that $\Phi = \hat{\nu}$. It follows that $\varphi = \hat{\mu}$ where μ is the image measure of ν under the 'projection' $\varrho \to \varrho | H \colon G^* \to H^*$.

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