The propagation of singularities for pseudo-differential operators with self-tangential characteristics

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0. Introduction

In this paper, we study the propagation of singularities for a class of pseudo-differential operators having characteristics of variable multiplicity. We do not assume the characteristics to be in involution, in the sense that their Hamilton fields satisfy the Frobenius integrability condition. Instead, we assume that the characteristic set is a union of hypersurfaces tangent of exactly order $k_0 \geqq 1$ along an involutive submanifold of codimension $d_0 \geqq 2$. This means that the Hamilton fields are parallel at the intersection, and their Lie brackets vanish of at least order k_0 there. We also assume a version of the generalized Levi condition. One example, with $k_0 = 1$, is the wave operator for uniaxial crystals, i.e. trigonal, tetragonal and hexagonal crystals. The main result is stated in Theorem 1.3, and it shows that the wave front set of the solution is propagated along the union of the Hamilton fields of the characteristic surfaces.

The method of proof is to reduce the operator to a first order diagonalizable system — see Proposition 2.3. By the geometry of the problem and the Levi condition, this can be done using the general symbol classes of the Weyl calculus. For this system the Cauchy problem is well posed, and the parametrix is constructed using Lax' method of oscillatory solutions — see Proposition 3.4. The oscillatory solutions are conormal distributions with non-standard symbols, so we need some calculus lemmas in the appendix. The special symbol classes make it possible to "blow up" the singularity of the characteristics as in [10]. The contributions outside the singularity may then be taken care of, and we are left with solving a micro-local system of pseudo-differential operators along the leaves of the singularity, which is done in Section 4. Finally, the singularities of the parametrix are analyzed in Section 5.

There have been many studies of singularities of solutions of symmetrizable hyperbolic systems, see [15] and references there. Nosmas [12] has studied the involutive case. Kumano-go and Taniguchi [8] have constructed parametrices for diagonalizable systems, but since they consider classical symbols, their results are not directly applicable here. The results on the propagation of singularities for the system in Proposition 2.3 may be obtained by the method of energy estimates of Ivrii [6] (see also [16]). For scalar operators, the case when the characteristics have transversal involutive self-intersection has been analyzed in [1], [9], [13] and [14]. Melrose and Uhlmann [10] considered the case of conical involutive singularity of the characteristic set. Morimoto [11] studied operators on the form (2.12) below, but with involutive characteristics. Ivrii [7] considered operators with L^∞ bounds on the Poisson brackets at double characteristic points.

In this paper, we shall consider classical, or polyhomogeneous, pseudo-differential operators. These have symbols which are asymptotic sums of homogeneous terms. But we shall also use the more general symbol classes of the Weyl calculus. Since all our metrics are split, we can use the standard calculus of pseudo-differential operators with these symbol classes. For notation and calculus results, see [5, Chapter 18].

1. Statement of result

We are going to study the pseudo-differential operator $P\in\Psi^m_{phg}(X)$ on a C^∞ manifold X. Let $p=\sigma(P)$ be the principal symbol and $\Sigma=p^{-1}(0)$ the characteristic set. Assume, microlocally near $(x_0,\xi_0)\in\Sigma$,

(1.1) $\Sigma=\bigcup_{j=1}^{r_0}S_j$, $r_0\geqq 2$, where S_j are non-radial hypersurfaces

tangent at $\Sigma_2=\bigcap_{j=1}^{r_0}S_j$ of exactly order $k_0\geqq 1$.

This means that the Hamilton fields of S_j do not have the radial direction $\langle\xi,\partial_\xi\rangle$. Also, the k_0:th jets of S_j coincide on Σ_2, but no k_0+1:th jet does, and the surfaces only intersect at Σ_2 in a neighborhood of (x_0,ξ_0). Observe that the surfaces need not be in involution, in the sense that their Hamilton fields satisfy the Frobenius integrability condition. Since p is homogeneous in ξ, Σ_i and S_j are conical. Next we assume, microlocally near (x_0,ξ_0),

(1.2) Σ_2 is an involutive manifold of codimension $d_0\geqq 2$,

and $\Pi(\Sigma_2)=X$, where Π is the projection: $T^*(X)\to X$.

Clearly the codimension cannot be equal to 1, and by non-degeneracy Σ_2 is a manifold near (x_0, ξ_0). In order to obtain conditions on lower order terms of P on the multiple characteristic set we assume the following version of the Levi condition. For $j=1, \ldots, r_0$ there exist $m_j \in \mathbf{N}$, with the property that, if $\varphi_j \in C^\infty$, $(x, d_x\varphi_j) \in S_j$ near x_0, $d_x\varphi_j(x_0)=\xi_0$ then

$$(1.3) \qquad |e^{-i\varrho\varphi_j}P(e^{i\varrho\varphi_j}a)| \leqq C\big(1+\varrho\delta^{k_0+1}(x, d_x\varphi_j)\big)^{m_0-m_j}(1+\varrho)^{m-m_0}, \quad \varrho \to \infty,$$

$\forall a \in C^\infty$ supported near x_0. Here $m_0=\sum_{j=1}^{r_0} m_j$, and $\delta(dx, d\xi)$ is the homogeneous distance to Σ_2, i.e. the distance with respect to the metric $|dx|^2+|d\xi|^2/(1+|\xi|^2)$. This means that p vanishes of order m_j at $S_j \setminus \Sigma_2$, of order m_0 at Σ_2, and P satisfies the Levi conditions on S_j and Σ_2 (see [2]). We also have uniform conditions on lower order terms on $\Sigma_1=\Sigma\setminus\Sigma_2$ when approaching Σ_2. In order to avoid extra zeroes of the principal symbol at Σ_2, we assume

$$(1.4) \qquad d^{m_0}p \neq 0 \quad \text{at} \quad \Sigma_2, \quad m_0 = \sum_{j=0}^{r_0} m_j,$$

microlocally near (x_0, ξ_0), where $d^k p$ is the k:th differential of p.

Clearly, (1.1), (1.2) and (1.4) are invariant under multiplication with elliptic pseudo-differential operators and conjugation by elliptic Fourier integral operators corresponding to canonical transformations preserving the projection condition: $\Pi(\Sigma_2)=X$. In order to obtain the invariance of (1.3) we need the following

Lemma 1.1. *Condition* (1.3) *is invariant under multiplication of P with elliptic pseudo-differential operators and conjugation of P by elliptic Fourier integral operators corresponding to canonical transformations preserving the projection condition.*

Proof. It suffices to check how $F \in I^k(Y\times X, \Gamma')$ transforms $u(x, \varrho)e^{i\varrho\psi(x)}$, when Γ is the graph of a homogeneous canonical transformation χ preserving the projection condition. Here $u(x, \varrho)=\sum u_j(x)\varrho^j \in C^\infty$ satisfies

$$(1.5) \qquad |u(x, \varrho)| \leqq C\big(1+\varrho\delta^{k_0+1}(x, d_x\psi(x))\big)^{\mu}(1+\varrho)^{\nu}, \quad \varrho \to \infty,$$

where $\mu, \nu \geqq 0$. By [5, Prop. 25.3.3] we may assume $|\partial y/\partial x| \neq 0$ so that χ is given by a homogeneous generating function φ. We obtain

$$(1.6) \qquad F(e^{i\varrho\psi}u) = (2\pi)^{-n}\iint e^{i(\varphi(y,\xi)-\langle x,\xi\rangle+\varrho\psi(x))}a(y, \xi)u(x, \varrho)\,dx\,d\xi$$
$$= (2\pi)^{-n}\varrho^n\iint e^{i\varrho(\varphi(y,\xi)-\langle x,\xi\rangle+\psi(x))}a(y, \varrho\xi)u(x, \varrho)\,dx\,d\xi,$$

with $a \in S^k$. The critical points for the exponent are given by

$$(1.7) \qquad \mathscr{C} = \begin{cases} x = \partial_\xi\varphi(y, \xi) \\ \xi = \psi'(x), \end{cases}$$

which are non-degenerate. In fact, by differentiating (1.7) with $y(x)$ as function of x, we obtain that the determinant of the Hessian of the exponent is equal to

$$(1.8) \qquad |\mathrm{Id}-\partial_\xi^2\varphi(y,\xi)\psi''(x)| = |\partial_y\partial_\xi\varphi(y,\xi)|\,|\partial y/\partial x| \neq 0 \quad \text{at} \quad \mathscr{C}.$$

The section $(x, d\psi)$ is mapped on $(y, d\Phi)$ by χ, where $\Phi(y(x))=\psi(x)$, since $\sum \xi_j\,dx_j$ is preserved by homogeneity. The method of stationary phase gives $F(e^{i\varrho\psi}u)=e^{i\varrho\Phi}v$, with

$$(1.9) \qquad v(y,\varrho)\sim w(y)\sum_{j=0}^{\infty}\varrho^{-j}L_j\big(a(y,\varrho\xi)u(x,\varrho)\big)\Big|_{\substack{\xi=\psi'(x)\\ x=\partial_\xi\varphi(y,\xi)}}, \quad \varrho\to\infty,$$

where L_j are differential operators of order $2j$ in (x,ξ) and $w(y)\neq 0$ (see [5, Th. 7.7.1 and 7.7.6]). Since $k_0\geqq 1$ in (1.1), $v(y,\varrho)=\sum v_j(y)\varrho^j$ satisfies (1.5) with $\nu+k$ instead of ν. In fact, (1.5) is equivalent to

$$|u_j(x)| \leqq C_j\delta\big(x, d\psi(x)\big)^{(j-\nu)_+(k_0+1)},$$

where $t_+=\max(t,0)$. Then

$$|D_x^\beta u_j(x)| \leqq C_{j\beta}\delta\big(x, d\psi(x)\big)^{(j-\nu-i)_+(k_0+1)}, \quad |\beta|\leqq 2i.$$

Since χ is a diffeomorphism, this completes the proof of the lemma.

We shall now state the result for propagation of singularities for P. Since the surfaces are tangent at Σ_2, their Hamilton fields are parallel. Since Σ_2 is involutive and $\Sigma_2=\cap S_j$, the Hamilton fields of S_j are tangent to Σ_2, and they define the same flow there.

Definition 1.2. *The Hamilton flow on Σ is the union of the Hamilton flows on S_j, $j=1,\dots,r_0$.*

The following is the main result of the paper.

Theorem 1.3. *Assume $P\in\Psi_{phg}^m(X)$ satisfies* (1.1)—(1.4) *microlocally near $w\in\Sigma$. If $u\in\mathscr{D}'(X)$, then $WF\,u\setminus WF\,Pu$ is invariant under the Hamilton flow on $\Sigma=p^{-1}(0)$ near w.*

On Σ_1 this follows from the fact that the characteristics have constant multiplicity, see [2, Th. 1.1]. In the next section, we shall reduce P to a first order system, microlocally near Σ_2. First we shall give an example.

Example 1.4. We consider Maxwell's equations in uniaxial crystals

$$(1.10) \qquad \begin{cases} \varepsilon\,\partial_t e-\operatorname{curl} h = 0\\ \mu\,\partial_t h+\operatorname{curl} e = 0\\ \operatorname{div}(\varepsilon e) = \operatorname{div}(\mu h) = 0.\end{cases}$$

Here e, h are distributions with values in $\mathbf{C}^3$ and ε, μ are positive definite, constant

3×3 matrices, such that $\tilde{\varepsilon}=\mu^{-1/2}\varepsilon\mu^{-1/2}$ has two different eigenvalues $\alpha, \beta>0$. By choosing new fiber and x variables, we may assume $\mu=\mathrm{Id}$ and

$$\varepsilon = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}.$$

The system (1.10) has characteristic set included in $\{\tau\neq 0\}$. If we skip the divergence equations, which are redundant when $\tau\neq 0$, the resulting 6×6 system has determinant equal to

$$\alpha^2\beta\tau^2\big((\tau^2-\psi)^2-(\alpha^{-1}-\beta^{-1})^2(\xi_1^2+\xi_2^2)^2/4\big), \tag{1.11}$$

where

$$\psi = (\alpha^{-1}+\beta^{-1})(\xi_1^2+\xi_2^2)/2+\alpha^{-1}\xi_3^2.$$

Clearly, when $\tau^2\neq 0$, (1.11) satisfies (1.1)—(1.4). In fact, (1.3) and (1.4) are satisfied trivially. By choosing

$$\begin{cases} \eta_0 = \tau^2-\psi \\ \eta_j = \xi_j, \quad j>0, \end{cases}$$

as new local coordinates when $\tau\neq 0$, we find $\Sigma\cap\{\tau\neq 0\}=S_1\cup S_2$ where

$$S_j = \{\eta_0 = (-1)^j(\alpha^{-1}-\beta^{-1})(\eta_1^2+\eta_2^2)/2\}.$$

These are non-radial, and tangent of order 2 at $\Sigma_2\cap\{\tau\neq 0\}=\{\eta_0=\eta_1=\eta_2=0\}$, which is involutive of codimension 3.

2. Reduction to a first order system

We assume $P\in\Psi^m_{phg}(X)$ satisfies (1.1)—(1.4) microlocally near $w\in\Sigma_2$. Since the result is local and we have invariance of the conditions, we may assume $X=\mathbf{R}^n$. Because Σ_2 is involutive and $\Pi(\Sigma_2)=X$, we may choose symplectic, homogeneous coordinates $(x, \xi)\in T^*\mathbf{R}^n$ near $w\in\Sigma_2$ so that $w=(0; (0, \dots, 1))$ and

$$\Sigma_2 = \{(x, \xi)\in T^*\mathbf{R}^n\colon\ \xi' = 0\}, \tag{2.1}$$

where $\xi=(\xi', \xi'')\in\mathbf{R}^{d_0}\times\mathbf{R}^{n-d_0}$. We may also assume

$$S_1 = \{(x, \xi)\in T^*\mathbf{R}^n\colon\ \xi_1 = 0\}, \tag{2.2}$$

near w. In fact, $\Sigma_2\subset S_1$ implies $S_1=f^{-1}(0)$ with real homogeneous $f=\langle a, \xi'\rangle$, $a\neq 0$ near w. Since Σ_2 is involutive, H_f is tangent to Σ_2. If we assume $a_1\neq 0$, then H_f is non-characteristic to $\{x_1=0\}$ at w. Thus we may complete $f=\eta_1$ to a symplectic coordinate system so that $\eta_j=\xi_j$ when $x_1=0$, $j>1$, and clearly $\eta'=0$ at Σ_2.

Now we rename $x_1=t$, $(x_2, ..., x_{d_0})=x'$ and $(x_{d_0+1}, ..., x_n)=x''$. Since S_j is tangent to S_1 at Σ_2, we obtain

$$S_j = \{(t, x; \tau, \xi)\in T^*(\mathbf{R}\times\mathbf{R}^{n-1}) \colon \tau+\beta_j(t, x, \xi) = 0\}, \tag{2.3}$$

with β_j real and homogeneous of degree 1 in ξ, $\beta_1\equiv 0$, and

$$c|\xi'|^{k_0+1}/|\xi|^{k_0} \leqq |\beta_j-\beta_k| \leqq C|\xi'|^{k_0+1}/|\xi|^{k_0}, \quad j\neq k, \tag{2.4}$$

in a conical neighborhood of w. By taking $k=1$, we obtain

$$c|\xi'|^{k_0+1}/|\xi|^{k_0} \leqq |\beta_j| \leqq C|\xi'|^{k_0+1}/|\xi|^{k_0}, \quad j = 2, ..., r_0,$$

so β_j vanishes of exactly order k_0+1 at $\{\xi'=0\}$.

Next, we prepare $P\in\Psi^m_{phg}(X)$. Assume P to be given by the expansion $p+p_{m-1}+p_{m-2}+\dots$, where $p=\sigma(P)$ and $p_j\in S^j$. Conditions (1.3) (with $\varphi_1=t$) and (1.4) give $\partial^j_\tau p=0$ at Σ_2 when $j<m_0$, and $\partial^{m_0}_\tau p\neq 0$, near $w\in\Sigma_2$. Thus Malgrange's preparation theorem gives, by homogeneity (see [5, Th. 7.5.5]),

$$p = c\sum\nolimits_{j=0}^{m_0} a_{m_0-j}\tau^j \quad \text{near} \quad w\in\Sigma_2,$$

where $0\neq c\in S^{m-m_0}$, $a_j\in C^\infty(\mathbf{R}, S^j)$ are homogeneous in ξ, $a_0\equiv 1$ and $a_j=0$ at Σ_2, $j>0$. By multiplication with an elliptic pseudo-differential operator we may assume $m=m_0$ and $c\equiv 1$. By Malgrange's preparation theorem, we have (see [5, Th. 7.5.6])

$$p_{m_0-1} = c_{-1}p+\sum\nolimits_{j=0}^{m_0-1} b_{m_0-j-1}\tau^j, \quad \text{near} \quad w\in\Sigma_2,$$

where $c_{-1}\in S^{-1}$ and $b_j\in C^\infty(\mathbf{R}, S^j)$ are homogeneous in ξ. By multiplication with $1-c_{-1}\in S^0$ we may assume $c_{-1}\equiv 0$. In this way, we obtain by induction

$$P \cong \sum\nolimits_{j=0}^{m_0} A_{m_0-j}D_t^j, \quad \text{microlocally near} \quad w, \tag{2.5}$$

where $A_j\in C^\infty(\mathbf{R}, \Psi^j_{phg})$ and $A_0\equiv 1$. Now (1.3) gives more information about A_j, but we first have to introduce some symbol classes corresponding to the β_j's.

Let

$$m(\xi) = 1+|\xi'|^{k_0+1}\langle\xi\rangle^{-k_0}, \tag{2.6}$$

where $\langle\xi\rangle=(1+|\xi|^2)^{1/2}$, thus $m\approx 1+|\beta_j|$. Put

$$g(dx, d\xi) = |dx|^2+|d\xi'|^2/(\langle\xi\rangle^\mu+|\xi'|)^2+|d\xi''|^2/\langle\xi\rangle^2 \quad \text{at} \quad (x, \xi), \tag{2.7}$$

where $\mu=k_0/(k_0+1)$, which gives $h^2=\sup g/g^\sigma=(\langle\xi\rangle^\mu+|\xi'|)^{-2}\leqq 1$. It is easy to see that g is σ temperate. In fact,

$$|\xi'-\eta'|(\langle\xi\rangle^\mu+|\xi'|)^{-1}+|\xi''-\eta''|\langle\xi\rangle^{-1} \leqq \varepsilon$$

implies $\langle\xi\rangle \leqq C\langle\eta\rangle$ and

$$(1-\varepsilon)|\xi'| \leqq |\eta'|+\varepsilon\langle\xi\rangle^{\mu} \leqq |\eta'|+C\varepsilon\langle\eta\rangle^{\mu},$$

which gives the slow variation. Since $g^{\sigma} \geqq h^{-2}g \geqq \langle\xi\rangle^{2\mu}g$, the metric is σ temperate, and $m \approx \langle\xi\rangle^{-k_0}h^{-k_0-1}$ is a weight for g. We shall denote by $S(mh^j, g)$ the symbol classes in (x, ξ) of weight mh^j, $j \in \mathbf{Z}$, depending C^{∞} on t, and $\operatorname{Op} S(mh^j, g)$ the corresponding (classical) pseudo-differential operators. (Thus we shall suppress the t dependence.) The reason for using these classes is that $\beta_j \in S(m, g)$. In fact, Taylor's formula gives

$$\beta_j = \sum\nolimits_{|\alpha|=k_0+1} a_j^{\alpha}\xi'^{\alpha}, \tag{2.8}$$

where $a_j^{\alpha} \in S^{-k_0}$ are homogeneous in ξ. Thus we get

$$\langle\xi\rangle^{|\gamma''|}|\partial_x^{\alpha}\partial_{\xi'}^{\gamma'}\partial_{\xi''}^{\gamma''}\beta_j| \leqq C_{j\alpha\gamma}\langle\xi\rangle^{-k_0-(|\gamma'|-k_0-1)_+}|\xi'|^{(k_0+1-|\gamma'|)_+} \leqq C'_{j\alpha\gamma}mh^{|\gamma'|},$$

by considering the cases $|\xi'| \gtrless \langle\xi\rangle^{\mu}$. Similarly, we obtain that if $a(t, x, \xi)$ is homogeneous of degree j in ξ and $|a| \leqq cm^k$, then $a \in S(m^j, g)$. In fact, if $k<j$, then $a \equiv 0$, otherwise a vanishes of order $\geqq j(k_0+1)$ at Σ_2. This we will use together with (1.3) to prove the following preparation result.

Lemma 2.1. *Assume that P is given by* (2.5) *and satisfies* (1.1)—(1.4) *with* $m=m_0$ *and* $S_1=\{\tau=0\}$, *near* $w \in \Sigma_2$. *Then* $A_i \in \operatorname{Op} S(m^i, g)$ *and*

$$b_j = e^{-i\varphi_j}P(e^{i\varphi_j}a) \in S(m^{m_0+r-m_j}, g) \quad \textit{near} \quad (t_0, x_0, \xi_0), \tag{2.9}$$

for all $a \in S(m^r, g)$, *if* $\varphi_j(t, x, \xi)$ *is homogeneous of degree* 1 *in* ξ, $(t, x, d_{t,x}\varphi_j) \in S_j$ *near* (t_0, x_0, ξ_0), $(t_0, x_0, d_{t,x}\varphi_j(t_0, x_0, \xi_0))=w$, *and* $(t, x, d_{t,x}\varphi_j) \in \Sigma_2$ *when* $\xi'=0$.

Proof. First we observe that by solving

$$\begin{cases}\partial_t\varphi_j+\beta_j(t, x, d_{t,x}\varphi_j) = 0\\ \varphi_j|_{t=t_0} = \langle x, \xi\rangle,\end{cases}$$

near (t_0, x_0, ξ_0), by Hamilton–Jacobi, we obtain φ_j satisfying the conditions in the lemma. In fact, by Lemma 3.1 we have $\partial_t\partial_{x'}\varphi_j \equiv 0$ when $\xi'=0$, and $\partial_{x'}\varphi_j=\xi'$ at $t=t_0$. Let P have symbol expansion $p+p_{m_0-1}+p_{m_0-2}+\dots$, where $p=\sigma(P)$ and $p_j \in S^j$. To compute (2.9) for homogeneous a, we may use the formal expansion in Lemma A.1 and homogeneity to get

$$b_j \cong \sum\nolimits_{k\geqq 0} L_k(P, \varphi_j)a \mod S^{-\infty},$$

since $h \leqq \langle\xi\rangle^{-\mu}$. Here $L_k(P, s\varphi_j)=s^{m_0-k}L_k(P, \varphi_j)$ is differential operator of order k in (t, x), with principal symbol

$$\sigma(L_k(P, \varphi_j))(\varrho, \eta) = \sum\nolimits_{|\alpha|=k}(\partial_{\tau,\xi}^{\alpha}p)(t, x, d_{t,x}\varphi_j)(\varrho, \eta)^{\alpha}/k!.$$

Applying this to $a\in S(1, g)$, homogeneous of degree 0 in ξ, (1.3) gives that $L_k(P, \varphi_j)\equiv 0$ when $k<m_j$, and that all coefficients of $L_k(P, \varphi_j)$ are bounded by $cm^{m_0-m_j}$ when $k\geqq m_j$ (since $m=m_0$). By homogeneity, all coefficients of $L_k(P, \varphi_j)$ are in $S(m^{m_0-k}, g)$ when $k\geqq m_j$. Observe that this implies that p vanishes of order m_j at S_j, and

$$\partial^\alpha_{\tau,\xi}p|_{S_j}\in S(m^{m_0-|\alpha|}, g),\quad |\alpha|\geqq m_j,$$

near w. In fact, the proof of Lemma A.1 shows that the mapping $(x, \xi)\to(x, d_x\varphi_j)$ preserves $S(m^i, g)$, $\forall i$.

Now by induction we obtain that p_{m_0-i} vanishes of order $(m_j-i)_+$ at S_j, and that

$$\partial^\alpha_{\tau,\xi}p_{m_0-i}|_{S_j}\in S(m^{m_0-i-|\alpha|}, g),\quad |\alpha|\geqq m_j-i. \tag{2.10}$$

In fact, the term of order k in $L_{i+k}(P, \varphi_j)$, is equal to

$$\sum\nolimits_{|\alpha|=k}\partial^\alpha_{\tau,\xi}p_{m_0-i}(t, x, d_{t,x}\varphi_j)(\varrho, \eta)^\alpha/k!$$

modulo terms with coefficients on the form

$$c_{\alpha\beta}\prod\nolimits_{1\leqq\nu\leqq\mu}\partial^{\alpha_\nu}_{x}\varphi_j(t, x, \eta)\partial^\beta_{\tau,\xi}p_{m_0-j}(t, x, d_{t,x}\varphi_j),\quad j<i, \tag{2.11}$$

where (by homogeneity) $m_0-j-|\beta|+\mu=m_0-i-k$. By the induction hypothesis and (3.5), (2.11) is bounded in $S(m^\mu m^{m_0-j-|\beta|}, g)=S(m^{m_0-i-|\alpha|}, g)$, which gives (2.10). By using the expansion (A.6) for general a, we get (2.9). In order to prove $A_i\in\operatorname{Op} S(m^i, g)$, we observe that when $j=1$, (2.10) gives

$$\partial^k_\tau p_{m_0-i}|_{\tau=0}\in S(m^{m_0-i-k}, g),\quad \forall i, k.$$

Since the symbol of P is a polynomial in τ, we obtain the result, by considering the τ derivatives at $\{\tau=0\}$.

Lemma 2.2. *Assume that P satisfies the conditions in Lemma 2.1. Then we can find A, $A_I\in\operatorname{Op} S(1, g)$, $I=(i_1, \dots, i_{r_0})\in\mathbf{N}^{r_0}$, so that $\sigma(A)\equiv 1$ and*

$$P=A\prod\nolimits_{j=1}^{r_0}Q_j^{m_j}+\sum\nolimits_{\substack{|I|<m_0\\ i_j\leqq m_j}}A_I\prod\nolimits_{j=1}^{r_0}Q_j^{i_j}, \tag{2.12}$$

microlocally near $w\in\Sigma_2$. Here $Q_j=D_t+B_j$, $B_j\in\operatorname{Op} S(m, g)$ and $\sigma(B_j)=\beta_j$.

Observe that the products in (2.12) are commutative modulo lower order terms. In fact, we obtain from (2.4) and (2.8)

$$\{q_j, q_k\}=H_{q_j}q_k=c_1^{jk}(q_j-q_k)+c_0^{jk},\quad c_i^{jk}\in S(1, g),\quad \forall jk, \tag{2.13}$$

where $q_j=\sigma(Q_j)$, by considering $|\xi'|\gtrless c\langle\xi\rangle^\mu$. Changing lower order terms in Q_j only changes A_I in (2.12). It is also clear that all terms in (2.12) satisfy (2.9).

Proof. First we observe that $\sigma(P)=p=\prod q_j^{m_j}$, since it is a monic polynomial of degree m_0 in τ, vanishing of order m_j at $\tau=-\beta_j$. We shall consider the cases $|\xi'|\gtrless c\langle\xi\rangle^\mu$, by using a partition of unity in $S(1, g)$. When $|\xi'|\leqq c\langle\xi\rangle^\mu$, we find $S(m^k, g)\subset S(1, g)$, $\forall k$. Replacing D_t^k by $\prod Q_j^{k_j}$, where $\sum k_j=k$ and $k_j\leqq m_j$, only changes terms of lower order in D_t. Thus we only have to consider $|\xi'|\geqq c\langle\xi\rangle^\mu$. Let P have the expansion $p+p_{m_0-1}+\dots$. The result will follow if we can write p_{m_0-k}, $k>0$, in the form

$$p_{m_0-k}=\sum_{\substack{0\leqq i_j\leqq m_j\\ |I|<m_0}} a_I^k \prod_j q_j^{i_j},\quad a_I^k\in S(1, g), \tag{2.14}$$

when $|\xi'|\geqq c\langle\xi\rangle^\mu$. In fact, if $\chi\in S(1, g)$ is a cut-off function, we find that $\sum \chi a_I^k \prod q_j^{i_j}$ may be written in the form (2.12), since $h\leqq m^{-1}$.

The proof of Lemma 2.1 implies that p_{m_0-k} vanishes of order $(m_j-k)_+$ at $\{\tau=-\beta_j\}$. Since p_{m_0-k} is polynomial in τ of degree m_0, we find

$$p_{m_0-k}=q_j^{(m_j-k)_+}r_k^j,\quad \forall j,$$

where r_k^j is polynomial in τ of order $m_0-(m_j-k)_+$, and satisfies

$$\partial_\tau^i r_k^j|_{\tau=-\beta_j}\in S(m^{m_0-\max(m_j,k)-i}, g),\quad \forall i, j,$$

according to (2.10). Since p_{m_0-k}/p is rational in τ, residue calculus gives

$$p_{m_0-k}/p=\sum_{\substack{1\leqq i\leqq \min(m_j,k)\\ j}} a_{ki}^j(q_j)^{-i}.$$

Here $a_{ki}^j\in S(1, g)$ in $\{|\xi'|\geqq c\langle\xi\rangle^\mu\}$, where it is essentially given by the τ derivative of order $\min(m_j, k)-i$ of

$$q_j^{\min(m_j,k)}p_{m_0-k}/p=r_k^j\prod_{i\neq j} q_i^{-m_i}\quad \text{at}\quad \{\tau=-\beta_j\}.$$

Observe that, since $|\xi'|\geqq c\langle\xi\rangle^\mu$, we find $q_i^{-1}|_{\tau=-\beta_j}=(\beta_i-\beta_j)^{-1}\in S(m^{-1}, g)$, according to (2.4). This proves (2.14) and the lemma.

Now it is simple to reduce (2.12) to a first order diagonal system. We shall follow Morimoto [11]. We rename the factors by letting

$$P_k=Q_j\quad \text{for}\quad \sum_{i<j} m_i<k\leqq\sum_{i\leqq j} m_i,\quad 1\leqq j\leqq r_0.$$

Then we may rewrite (2.12) as

$$P=P_1P_2\cdots P_{m_0}+\sum_{|J|<m_0} A^J \prod_{j\in J} P_j, \tag{2.15}$$

with $A^J\in \mathrm{Op}\, S(1, g)$.

Assume $Pu=f$, $u\in\mathscr{D}'(\mathbf{R}^n)$. Let $U={}^t(u_\emptyset, u_{(1)}, \dots)={}^t(u_J)_{|J|<m_0}$ where $u_\emptyset=u$, and $u_J=P_Ju=P_{j_1}\dots P_{j_r}u$, for $J=(j_1, \dots, j_r)\in\mathbf{N}^r$, with $1\leqq j_k\leqq m_0$, $j_i\neq j_k$ when $i\neq k$. Then U has values in $\mathbf{C}^{N_0}$, $N_0=\sum_{j=1}^{m_0} m_0!/j!$, and we shall construct a first order $N_0\times N_0$ system for U. First we observe that when $|J|=m_0$, we can find

$B_J^I \in \operatorname{Op} S(1, g)$ such that

$$P_J = P_1 P_2 \cdots P_{m_0} + \sum_{|I|<m_0} B_J^I P_I, \tag{2.16}$$

because of (2.13). Since there are many relations between the components of U, we make the following choice in order to get a diagonal system. If $|J|=r<m_0-1$, we take the largest $i \in \mathbf{Z}$, so that $i \leqq m_0$ and $i \neq j_k$, $1 \leqq k \leqq r$, then

$$P_i u_J = u_{J'} \quad \text{where} \quad J' = (i, j_1, \dots, j_r). \tag{2.17}$$

If $|J|=m_0-1$, we take $i \in \mathbf{Z}$ so that $1 \leqq i \leqq m_0$ and $i \neq j_k$, $1 \leqq k \leqq m_0-1$, then

$$P_i u_J = f + \sum_{|I|<m_0} (B_{J'}^I - A^I) u_I \quad \text{with} \quad J' = (i, j_1, \dots, j_{m_0-1}) \tag{2.18}$$

follows from (2.13) and (2.16). Clearly (2.17) and (2.18) form a first order $N_0 \times N_0$ system for U, with principal symbol being diagonal matrix with elements $\tau + \beta_j = \sigma(Q_j)$ (repeated several times). Summing up, we obtain the following result.

Proposition 2.3. *Assume that* $P \in \Psi_{phg}^m$ *satisfies* (1.1)—(1.4). *Then, by conjugation with elliptic Fourier integral operators and multiplication by elliptic pseudo-differential operators, the equation* $Pu=f$, $u \in \mathscr{D}'(X)$, *can be reduced to the* $N_0 \times N_0$ *system*

$$D_t U + K(t, x, D_x) U = F, \tag{2.19}$$

microlocally near $w \in \Sigma_2$. *Here* $\mathrm{WF}\, F = \mathrm{WF}\, f$, $\mathrm{WF}\, U = \mathrm{WF}\, u$, $N_0 = \sum_{j=1}^{m_0} m_0!/j!$, *and* $K \in \operatorname{Op} S(m, g)$ *with principal symbol*

$$k_1(t, x, \xi) = (\delta_{jk} \beta_{i_k})_{j, k=1, \dots, N_0} \tag{2.20}$$

being diagonal matrix, with real eigenvalues $\beta_i \in S(m, g)$ *homogeneous of degree* 1 *in* ξ, *satisfying* (2.4), *and* $\beta_1 \equiv 0$.

3. The Cauchy Problem

We shall study the Cauchy problem for the $N_0 \times N_0$ system

$$P = D_t\, Id_{N_0} + K(t, x, D_x)$$

having the properties in Proposition 2.3. As in Section 2, we shall suppress the t dependence and we shall use the notation of that section. Thus $K \in \operatorname{Op} S(m, g)$ has principal symbol $k(t, x, \xi)$ satisfying

(3.1) k is diagonizable in $S(1, g)$, with real eigenvalues $\{\beta_j\}_{j=1, \dots, r_0}$ homogeneous of degree 1 in ξ, satisfying (2.4), and $\beta_1 \equiv 0$.

Thus there exists a base of left (right) eigenvectors in $S(1, g)$. Since their eigenvalues are C^∞ functions, the dimension of the eigenspace corresponding to β_j is constant outside $\Sigma_2=\{\xi'=0\}$. Let $\pi_j(t, x, \xi)\in S(1, g)$ be the projection on the eigenvectors corresponding to the eigenvalue β_j along the others when $\xi'\neq 0$, and extended by continuity. Then we have

$$k = \sum_{j=1}^{r_0} \beta_j \pi_j, \tag{3.2}$$

and k is symmetrizable with symmetrizer $M=\sum \pi_j^* \pi_j$, i.e. $M>0$ and Mk is symmetric.

Now we are going to solve the microlocal Cauchy problem

$$\begin{cases} D_t E + KE \cong 0 \\ E|_{t=0} \cong Id_{N_0} \end{cases} \tag{3.3}$$

microlocally near $(0, (0, \xi_0), (0, \xi_0))$, $\xi_0'=0$, with E: $\mathcal{E}'(\mathbf{R}^{n-1})\to\mathcal{D}'(\mathbf{R}^n)$. We shall use Lax' method of oscillatory solutions. In order to do that, we must solve the eiconal equations

$$\begin{cases} \partial_t \Phi_j + \beta_j(t, x, d_x \Phi_j) = 0 \\ \Phi_j(0, x, \eta) = \langle x, \eta\rangle \end{cases} \quad \text{for} \quad j = 1, \ldots, r_0. \tag{3.4}$$

By Hamilton–Jacobi, this has a unique local solution, homogeneous of degree 1 in η.

Lemma 3.1. *Let Φ_j solve* (3.4) *with $\{\beta_j\}$ satisfying the conditions in* (3.1). *Then we find that $\varphi_j(t, x, \eta)=\Phi_j(t, x, \eta)-\langle x, \eta\rangle$ satisfies*

$$\partial_{\eta'}^{\gamma} \varphi_j \equiv 0 \quad \textit{when} \quad \eta' = 0, \quad |\gamma| \leqq k_0, \quad \forall j. \tag{3.5}$$

Proof. By (2.8), the eiconal equation gives

$$\partial_t \varphi_j + \sum_{|\alpha|=k_0+1} a_j^\alpha(t, x, \eta + d_x \varphi_j)(\eta' + d_{x'} \varphi_j)^\alpha = 0, \tag{3.6}$$

and $\varphi_j(0, x, \eta)\equiv 0$. When $\eta'=0$ we get

$$\partial_t \varphi_j + \sum_{|\alpha|=k_0+1} a_j^\alpha(t, x, \eta + d_x \varphi_j)(d_{x'} \varphi_j)^\alpha = 0,$$

so uniqueness gives $\varphi_j\equiv 0$ when $\eta'=0$. By differentiating (3.6) we find that $\partial_t \partial_{\eta'} \varphi_j \equiv 0$ when $\eta'=0$, since $d_{x'}\varphi_j\equiv 0$ then. By induction over $|\gamma|\leqq k_0$ we obtain the result.

Now we define E_j: $\mathcal{E}'(\mathbf{R}^{n-1})\to\mathcal{D}'(\mathbf{R}^n)$, $j=1, \ldots, r_0$, as oscillatory integrals

$$E_j u(t, x) = (2\pi)^{1-n} \iint e^{i(\Phi_j(t,x,\eta)-\langle y, \eta\rangle)} a_j(t, x, \eta) u(y)\, dy\, d\eta, \tag{3.7}$$

with $a_j\in S(1, g)$. Assume that a_j is supported in a conical neighborhood of $\{\eta'=0\}$.

By Lemma A.1 in the appendix, we get

(3.8) $$PE_j u(t,x) = (2\pi)^{1-n} \iint e^{i(\Phi_j(t,x,\eta)-\langle y,\eta\rangle)} b_j(t,x,\eta) u(y)\, dy\, d\eta,$$

where

(3.9) $$b_j(t,x,\eta) = \big(\partial_t \Phi_j \operatorname{Id}_{N_0} + k(t,x,d_x\Phi_j)\big) a_j + L_j a_j + R_j a_j,$$

R_j is continuous $S(m^i h^l, g) \to S(m^i h^{l+1}, g)$, $\forall i, j, l$, and

$$L_j a_j = D_t a_j + \textstyle\sum_i (\partial_{\xi_i} k)(t,x,d_x\Phi_j) D_{x_i} a_j + M_j a_j,$$

with $M_j \in S(1,g)$. In general, we cannot find homogeneous a_j making $b_j \in S^{-\infty}$. However, we have the following result.

Lemma 3.2. *Assuming* (3.1), *we can find* $a_j \in S(1,g)$ *such that* $b_j \in S(m^{-N}, g)$, $\forall N$, *in* (3.9), $j=1, \ldots, r_0$, *and*

(3.10) $$\textstyle\sum_j a_j|_{t=0} \equiv \operatorname{Id}_{N_0}.$$

Proof. Let $a_j \sim a_j^0 + a_j^{-1} + \ldots$, where $a_j^{-k} \in S(m^{-k}, g)$. The dominant term in (3.9) is

$$\big(\partial_t \Phi_j \operatorname{Id}_{N_0} + (k(t,x,d_x\Phi_j)\big) a_j^0 = \textstyle\sum_i \Phi_j^*\big((\beta_i - \beta_j)\pi_i\big) a_j^0$$

where $\Phi_j^* f = f(t,x,d_x\Phi_j)$, so necessarily $a_j^0 \in \operatorname{Im} \Phi_j^* \pi_j = \bigcap_{i \neq j} \operatorname{Ker} \Phi_j^* \pi_i$. If we take $a_j^0 = \pi_j(0,x,\eta)$ at $t=0$, we obtain $\sum a_j^0|_{t=0} = \operatorname{Id}_{N_0}$. The term in $S(m^{-r}, g)$, $r \geqq 0$, in the expansion (3.9), is given by

$$\textstyle\sum_i \Phi_j^*\big((\beta_i - \beta_j)\pi_i\big) a_j^{-r-1} + L_j a_j^{-r} + R_j a_j^{1-r},$$

since $h \leqq m^{-1}$ ($a_j^1 \equiv 0$). Now $\Phi_j^*(\beta_i - \beta_j) \in S(m, g)$ is invertible modulo $S(m^{-\infty}, g)$ according to (2.4) when $j \neq i$, since $d_{x'}\Phi_j = \mathcal{O}(|\eta'|)$ by (3.5). Thus, it suffices to solve successively, with suitable initial data,

(3.11) $$(\Phi_j^* \pi_j)(L_j a_j^{-r} + \tilde{R}_j a_j^{1-r}) = 0, \quad r \geqq 0,$$

where $a_j^1 \equiv 0$, and $(\operatorname{Id}_{N_0} - \Phi_j^* \pi_j) a_j^{-r}$ has been determined in the previous step. Here $\tilde{R}_j$ is continuous $S(m^i, g) \to S(m^{i-1}, g)$, $\forall i$.

Now, let $\{v_j^i\}_i \in S(1,g)$ be a base for $\operatorname{Im} \Phi_j^* \pi_j$, and consider $\sum_i \alpha_i v_j^i$, $\alpha_i \in S(m^{-r}, g)$. (Such a base exists, since it follows from the proof of Lemma A.1 that Φ_j^* preserves the metric g.) Since $\pi_j \pi_l = \delta_{jl} \pi_l$, we get

$$(\partial \pi_j)\pi_l + \pi_j \partial \pi_l = \delta_{jl} \partial \pi_l \quad \forall ijl,$$

which gives

$$(\Phi_j^* \pi_j) \Phi_j^*(\partial \pi_l) v_j^i = \delta_{jl} \Phi_j^*(\partial \pi_l) v_j^i - \Phi_j^*(\partial \pi_j) \Phi_j^*(\pi_l) v_j^i = 0 \quad \forall i, j, l.$$

Since $\partial k = \sum_i ((\partial\beta_i)\pi_i + \beta_i\partial\pi_i)$, we obtain

$$(\Phi_j^*\pi_j)L_j \sum_i \alpha_i v_j^i = \sum_i \gamma_i v_j^i,$$

where

$$\gamma_i = D_t\alpha_i + \sum_l \Phi_j^*(\partial_{\xi_l}\beta_j)D_{x_l}\alpha_i + \sum_l \mu_i^l \alpha_l \in S(1, g), \tag{3.12}$$

with $\mu_i^l \in S(1, g)$. If we introduce local g orthogonal coordinates, then $\sum_l \Phi_j^*(\partial_{\xi_l}\beta_j)D_{x_l}$ transforms into a uniformly bounded C^∞ vector field. Thus, by adding a suitable linear combination of v_j^i to each column of a_j^{-r} we may solve (3.11) for all $1 \leqq j \leqq r_0$, with initial data making

$$\pi_l \sum_j a_j^{-r} = \begin{cases} \pi_l & r = 0 \\ 0 & r > 0 \end{cases} \quad \forall l, \quad \text{at} \quad t = 0.$$

If we do this recursively for $r > 0$, we obtain the lemma.

Now the symbols in $\bigcap_N S(m^{-N}, g)$ are integrable in η'. We obtain new symbol classes after integrating (3.8), according to the following lemma.

Lemma 3.3. *If* $a(t, x, \eta) \in \bigcap_N S(m^{-N}, g)$ *has support where* $|\eta'| \leqq c|\eta''|$, *and* $\varphi(t, x, \eta)$ *is homogeneous of degree* 1 *satisfying* (3.5), *then*

$$\tilde{a}(t, x, y', \eta'') = \int e^{i(\varphi(t,x,\eta) + \langle x' - y', \eta' \rangle)} a(t, x, \eta)\, d\eta' \in S^{\nu}_{1,\mu,0}, \tag{3.13}$$

where $\nu = \mu(d_0 - 1)$, $\mu = k_0/(k_0 + 1)$, $d_0 = \operatorname{codim} \Sigma_2$. *Here* $S^{\nu}_{1,\mu,0}$ *is defined by*

$$|D_t^k D_{x'}^{\alpha'} D_{x''}^{\alpha''} D_{y'}^{\beta'} D_{\eta''}^{\gamma''} b(t, x, y', \eta'')| \leqq C_{\alpha\beta\gamma k} \langle \eta'' \rangle^{\nu + \mu|\alpha' + \beta'| - |\gamma''|}. \tag{3.14}$$

Proof. If $N(k_0 + 1) \geqq d_0 + |\alpha|$, we obtain

$$\int_{|\eta'| \leqq c|\eta''|} \eta'^{\alpha}(1 + |\eta'|^{k_0+1}\langle\eta\rangle^{-k_0})^{-N}\, d\eta' \tag{3.15}$$
$$\leqq \langle\eta''\rangle^{(|\alpha| + d_0 - 1)\mu} \int \xi'^{\alpha}(1 + |\xi'|^{k_0+1})^{-N}\, d\xi' \leqq C_\alpha \langle\eta''\rangle^{(|\alpha| + d_0 - 1)\mu},$$

by putting $\xi' = \eta'/\langle\eta''\rangle^{\mu}$. This gives $|\tilde{a}| \leqq C\langle\eta''\rangle^{\nu}$. When differentiating (3.13), the derivatives falling on a give the right factors. The derivatives falling on the exponent give either η' factors, or factors

$$|\partial_t^k \partial_x^\alpha \partial_{\eta''}^{\gamma''} \varphi(t, x, \eta)| \leqq C_{k\alpha\gamma''} \langle\eta\rangle^{-|\gamma''|} m,$$

by (3.5) and homogeneity. The η' factors give only $\langle\eta''\rangle^{\mu}$ factors by (3.15), and the m factors are harmless since $a \in S(m^{-N}, g)$, $\forall N$. This completes the proof.

The lemma gives

$$\begin{cases} PE_j u = (2\pi)^{d_0 - n} \iint e^{i\langle x'' - y'', \eta'' \rangle} r_j(t, x, y', \eta'')\, u(y)\, dy\, d\eta'' \\ \sum_j E_j u|_{t=0} \equiv u, \end{cases} \tag{3.16}$$

where $r_j \in S^{\nu}_{1,\mu,0}$, $j=1, \dots, r_0$. We shall compensate for these terms by adding E_0: $\mathscr{E}'(\mathbf{R}^{n-1}) \to \mathscr{D}'(\mathbf{R}^n)$ defined by

$$E_0 u(t, x) = (2\pi)^{d_0 - n} \iint e^{i\langle x'' - y'', \eta'' \rangle} a_0(t, x, y', \eta'') u(y)\, dy\, d\eta'', \tag{3.17}$$

with $a_0 \in S^{\nu}_{1,\mu,0}$. By Lemma A.2 in the appendix, we have

$$P E_0 u(t, x) = (2\pi)^{d_0 - n} \iint e^{i\langle x'' - y'', \eta'' \rangle} b_0(t, x, y', \eta'') u(y)\, dy\, d\eta'', \tag{3.18}$$

where $b_0 \in S^{\nu}_{1,\mu,0}$ is given by

$$b_0 = D_t a_0 + e^{i\langle D_y, D_\xi \rangle} \tilde{k}(t, x, \xi)\, a_0(t, y, z', \eta'')\big|_{\substack{y=x \\ \xi=(0,\eta'')}} \tag{3.19}$$

if $\tilde{k}$ is the full symbol of K. In the next section, we shall study this type of equation (which is a microlocal type of pseudo-differential operator). By using Proposition 4.1, we may solve

$$\begin{cases} b_0 + \sum r_j \cong 0, & 0 < t < c, \\ a_0|_{t=0} \cong 0, \end{cases} \tag{3.20}$$

modulo $S^{-\infty}$. Anticipating this result, we obtain the solution to the Cauchy problem (3.3). Naturally, this can be done with t replaced by $t-s$, for small s, which gives the following

Proposition 3.4. *Let $K(t, x, D_x) \in \operatorname{Op} S(m, g)$ be an $N_0 \times N_0$ system with principal symbol $k(t, x, \xi)$ satisfying* (3.1). *Then the Cauchy problem for $|s| < \varepsilon$*

$$\begin{cases} D_t E^{(s)} + K(t, x, D_x) E^{(s)} \cong 0, & t > s, \\ E^{(s)}|_{t=s} \cong \mathrm{Id}_{N_0}, \end{cases} \tag{3.21}$$

microlocally near $(0, (0, \xi_0), (0, \xi_0))$, $\xi_0' = 0$, has a solution $E^{(s)}$: $\mathscr{E}'(\mathbf{R}^{n-1}) \to \mathscr{D}'(\mathbf{R}^n)$ in the form

$$E^{(s)} = \sum_{j=0}^{r_0} E_j^{(s)}.$$

Here

$$E_j^{(s)} u(t, x) = (2\pi)^{1-n} \iint e^{i(\Phi_j(t,x,\eta) - \langle y, \eta \rangle)} a_j(t, x, \eta) u(y)\, dy\, d\eta, \quad j \geqq 1,$$

Φ_j solves (3.4), *$a_j \in S(1, g)$, and*

$$E_0^{(s)} u(t, x) = (2\pi)^{d_0 - n} \iint e^{i\langle x'' - y'', \eta'' \rangle} a_0(t, x, y', \eta'') u(y)\, dy\, d\eta'',$$

where $a_0 \in S^{\nu}_{1,\mu,0}$, $\nu = \mu(d_0 - 1)$, $\mu = k_0/(k_0+1)$, $d_0 = \operatorname{codim} \Sigma_2$.

4. The micro-local pseudo-differential operator

We are going to study the system

$$(4.1)\qquad \begin{cases} D_t f + e^{i\langle D_y, D_\xi\rangle} k(t, x, \xi) f(t, y, z', \eta'')\big|_{\substack{y=x \\ \xi=(0,\eta'')}} \cong r(t, x, z', \eta''), \quad t>0, \\ f(0, x, z', \eta'') \cong f_0(x, z', \eta''), \end{cases}$$

modulo $S^{-\infty}$, where f_0, $r \in S^{\nu}_{1,\mu,0}$ have values in $\mathbf{C}^{N_0}$, and $k \in S(m, g)$ is $N_0 \times N_0$ system (see Section 3). By Lemma A.2 in the appendix, we have $r \in S^{\nu}_{1,\mu,0}$ if $f \in S^{\nu}_{1,\mu,0}$. We shall also assume that k is symmetrizable, i.e. $\exists$ symmetric $N_0 \times N_0$ system $M(t, x, \xi) \in S(1, g)$ such that $0 < c \leqq M$ and $Mk - (Mk)^* \in S(1, g)$.

Proposition 4.1. *Assume that* $k(t, x, \xi) \in S(m, g)$ *is a symmetrizable* $N_0 \times N_0$ *system. Then, for every* f_0, $r \in S^{\nu}_{1,\mu,0}$, *the equation* (4.1) *has a solution* $f \in S^{\nu}_{1,\mu,0}$ *in a conical neighborhood of* $(0, 0, (0, \eta_0'')) \in \mathbf{R} \times \mathbf{R}^{2d_0-2} \times T^*\mathbf{R}^{n-d_0}$.

Proof. We shall solve (4.1) by iteration, modulo $S^{\nu-\mu}_{1,\mu,0}$, $\mu = k_0/(k_0+1) < 1$. By Lemma A.2, we have

$$(4.2)\qquad e^{i\langle D_y, D_\xi\rangle} k(t, x, \xi) f(t, y, z', \eta'')\big|_{\substack{y=x \\ \xi=(0,\eta'')}}$$

$$\cong e^{i\langle D_{y'}, D_{\xi'}\rangle} k(t, x, \xi', \eta'') f(t, y', x'', z', \eta'')\big|_{\substack{y'=x' \\ \xi'=0}} = k(t, x, D_{x'}, \eta'') f(t, x, z', \eta''),$$

modulo terms in $S^{\nu-1}_{1,\mu,0}$. Also, we may assume k supported where $|\xi - (0, \eta'')| < \varepsilon \langle \eta'' \rangle$ and $|t| < c$. By cutting off, we may assume k, f supported where $\langle \eta'' \rangle \approx \langle \eta_0'' \rangle$, and f_0, r having compact support. Put $\lambda = \langle \eta_0'' \rangle^{-\mu} \leqq 1$, and let

$$(4.3)\qquad w = (x'', \lambda^{-1} z', \lambda^{1/\mu} \eta'').$$

Then (4.1) becomes, by (4.2),

$$(4.4)\qquad \begin{cases} D_t f(t, x', w) + k_\lambda(t, x', w, D_{x'}) f(t, x', w) \cong r(t, x', w), \quad t>0, \\ f(0, x', w) \cong f_0(x', w), \end{cases}$$

mod $S(\lambda^{(1-\nu)/\mu}, \lambda^{-2}|dx'|^2 + |dw|^2)$, where f_0, $r \in S(\lambda^{-\nu/\mu}, \lambda^{-2}|dx'|^2 + |dw|^2)$, and

$$k_\lambda \in S\big(1 + |\xi'|^{k_0+1} \lambda^{k_0+1}, \ |dx|^2 + |dw|^2 + |d\xi'|^2/(\lambda^{-1} + |\xi'|)^2\big).$$

Clearly, we may assume $\nu = 0$. If we make the symplectic dilation

$$(4.5)\qquad \begin{cases} y = \lambda^{-1} x' \\ \eta = \lambda \xi', \end{cases}$$

then it suffices to solve the system

$$(4.6)\qquad \begin{cases} D_t f(t, y, w) + k_\lambda(t, y, w, D_y) f(t, y, w) \cong r(t, y, w), \quad t>0, \\ f(0, y, w) \cong f_0(y, w), \end{cases}$$

modulo $S(\lambda, |dw|^2+|dy|^2)$, where $k_\lambda(t, y, w, \eta)\in S(\langle\eta\rangle^{k_0+1}, g_\lambda)$,

$$g_\lambda = \lambda^2|dy|^2+|dw|^2+|d\eta|^2/\langle\eta\rangle^2,$$

and f_0, $r\in S(1, |dw|^2+|dy|^2)$ uniformly in λ. By assumption, there exists a symmetric $N_0\times N_0$ system $0<c\leqq M_\lambda(t, y, w, \eta)\in S(1, g_\lambda)$, such that $M_\lambda k_\lambda$ is symmetric modulo $S(1, g_\lambda)$. To complete the proof we need to solve (4.6) with $f\in S(1, |dw|^2+|dy|^2)$ uniformly in λ. Going back, we obtain a solution in $S^{\nu}_{1,\mu,0}$ to (4.1) modulo $S^{\nu-\mu}_{1,\mu,0}$.

Choose a partition of unity $\{\chi_j(y, w)\}\in S(1, |dw|^2+|dy|^2)$, such that there is a fixed bound of the diameter of the supports of χ_j, and on the number of overlapping supports. Replacing f_0, r with $\chi_j f_0$, $\chi_j r$, and translating in y, it suffices to solve (4.6) with $f\in\mathcal{S}$ uniformly, when f_0, $r\in C_0^\infty$ uniformly with fixed support. Since

$$\lambda^{-1}(k_\lambda(t, y, w, \eta)-k_\lambda(t, 0, w, \eta))\in S(\langle y\rangle\langle\eta\rangle^{k_0+1}, g_1), \quad \lambda\leqq 1,$$

uniformly, we can replace $k_\lambda(t, y, w, D_y)$ by $k_\lambda(t, w, D_y)=k_\lambda(t, 0, w, D_y)$ in the system (4.6). By taking $M_\lambda(t, w, \eta)=M_\lambda(t, 0, w, \eta)$ we obtain that $M_\lambda k_\lambda$ is symmetric, mod $S(1, g_\lambda)$.

Now taking the Fourier transform with respect to y in (4.6), we want to solve

$$\text{(4.7)} \qquad \begin{cases} D_t\hat{f}(t, \eta, w)+k_\lambda(t, w, \eta)\hat{f}(t, \eta, w) = \hat{r}(t, \eta, w), \quad t>0,\\ \hat{f}(0, \eta, w) = \hat{f}_0(\eta, w).\end{cases}$$

The unique temperate solution to (4.7) is given by

$$\text{(4.8)} \qquad \hat{f}(t, \eta, w) = F_\lambda(t, \eta, w)\left(\hat{f}_0(\eta, w)+i\int_0^t F_\lambda^{-1}(s, \eta, w)\hat{r}(s, \eta, w)\,ds\right),$$

if $F_\lambda(t, \eta, w)$ is temperate solution to

$$\text{(4.9)} \qquad \begin{cases} D_tF_\lambda(t, \eta, w)+k_\lambda(t, w, \eta)F_\lambda(t, \eta, w) = 0, \quad t>0,\\ F_\lambda(0, \eta, w) = \mathrm{Id}_{N_0}.\end{cases}$$

Thus the proof is completed by showing that $f\in\mathcal{S}$ uniformly, which is done in the following

Lemma 4.2. *Let F_λ satisfy* (4.9). *Then the mapping $\mathcal{S}\times\mathcal{S}\ni(f_0, r)\to f\in\mathcal{S}$ defined by* (4.8) *is continuous, uniformly with respect to λ.*

Proof. Since Fourier transformation and integration are continuous in $\mathcal{S}$, it remains only to prove that multiplication with $F_\lambda^{\pm 1}$ is uniformly continuous. This will follow from

$$\text{(4.10)} \qquad 0<c\leqq|F_\lambda(t, \eta, w)|\leqq C$$

$$\text{(4.11)} \qquad |D_t^jD_\eta^\alpha D_w^\beta F_\lambda(t, \eta, w)|\leqq C_{j\alpha\beta}\langle\eta\rangle^{(j+|\beta|)(k_0+1)+|\alpha|k_0}.$$

To prove (4.10), we let

$$\|v\|_\lambda = \langle M_\lambda v, \bar{v}\rangle, \quad v\in \mathbf{C}^{N_0},$$

then

(4.12) $$c \leqq \|v\|_\lambda^2/|v|^2 \leqq C$$

uniformly. We obtain by (4.9) and (4.12)

$$|\partial_t \|F_\lambda v\|_\lambda^2| = |\langle(\partial_t M_\lambda)F_\lambda v, \overline{F_\lambda v}\rangle - i\langle M_\lambda k_\lambda F_\lambda v, \overline{F_\lambda v}\rangle + i\langle M_\lambda F_\lambda v, \overline{k_\lambda F_\lambda v}\rangle| \leqq C\|F_\lambda v\|_\lambda^2,$$

so Grönwall's lemma gives (4.10) by (4.12), since $F_\lambda|_{t=0}=\mathrm{Id}_{N_0}$. By differentiating (4.9), we get

(4.13) $$\begin{cases} D_t F^\beta_{\lambda,j\alpha} + k_\lambda(t,w,\eta)F^\beta_{\lambda,j\alpha} = -\sum \binom{j}{i}\binom{\alpha}{\delta}\binom{\beta}{\gamma}(D_t^{j-i}D_w^{\alpha-\delta}D_\eta^{\beta-\gamma}k_\lambda)F^\gamma_{\lambda,i\delta} \\ F^\beta_{\lambda,j\alpha}|_{t=0} = D_w^\alpha D_\eta^\beta \Gamma_j|_{t=0}, \end{cases}$$

where $F^\beta_{\lambda,j\alpha}=D_t^j D_w^\alpha D_\eta^\beta F_\lambda$, $\Gamma_1=-k$ and $\Gamma_{j+1}=[D_t,\Gamma_j]-\Gamma_j k$ is defined recursively. Thus $|\Gamma_j|\leqq C_j\langle\eta\rangle^{j(k_0+1)}$, and

$$D_t(F_\lambda^{-1}F^\beta_{\lambda,j\alpha}) = -F_\lambda^{-1}\sum \binom{j}{i}\binom{\alpha}{\delta}\binom{\beta}{\gamma}(D_t^{j-i}D_w^{\alpha-\delta}D_\eta^{\beta-\gamma}k_\lambda)F^\gamma_{\lambda,i\delta},$$

where the sum is taken over $i+|\delta+\gamma|<j+|\alpha+\beta|$. Since

$$|D_t^j D_w^\alpha D_\eta^\beta k_\lambda(t,w,\eta)| \leqq \begin{cases} c\langle\eta\rangle^{k_0} & \text{if } |\beta|>0 \\ c\langle\eta\rangle^{k_0+1} & \text{if } |\beta|=0, \end{cases}$$

we get (4.11) by induction. This completes the proof.

Remark 4.3. *The argument above shows that the unique $f\in\mathcal{S}$ solving* (4.7) *with f_0, $r\in\mathcal{S}$, gives a continuous map $B^\infty\times B^\infty\to B^\infty$, uniformly in λ. Here B^∞ is the set of C^∞ functions having L^∞ bounds on all derivatives.*

This follows easily by writing (4.8) as an oscillatory integral and integrating by parts, using (4.11). In fact, it suffices to estimate

$$\int e^{\langle x-y,\eta\rangle}a(y,\eta)\,dy\,d\eta$$

where, according to (4.11),

$$|D_y^\alpha D_\eta^\beta a(y,\eta)| \leqq C_{\alpha\beta}\langle\eta\rangle^{\varrho+|\beta|k_0},$$

and the constants depend on the B^∞ seminorms of f_0, r and the seminorms of F_λ. Integration by parts gives a convergent integral and the desired estimate.

5. The propagation of singularities

In this section, we shall construct a microlocal parametrix for the $N_0\times N_0$ system $P=D_t\,\mathrm{Id}_{N_0}+K(t,x,D_x)$, where $K\in\mathrm{Op}\,S(m,g)$ has principal symbol k satisfying (3.1), and study the propagation of singularities. This will be done by using Duhamel's principle and the parametrix for the Cauchy problem constructed by Proposition 3.4.

As before, it suffices to consider $w=(0,0,\eta_0'')\in\Sigma_2$. Let ϱ_s be the restriction to $\{t=s\}$ and $\varphi\in S^0_{1,0}$ have support in a conical neighborhood of w, such that $w\notin\mathrm{WF}\,(\varphi-1)$ and $N^*\{t=s\}\cap\mathrm{WF}\,\varphi=\emptyset$, $\forall s$, where N^* is the conormal bundle. Then the composition $\varrho_s\circ\varphi$ is well defined, and we put

$$Ef=\int_{-\varepsilon}^{t}E^{(s)}\circ\varrho_s\circ\varphi f\,ds,\quad f\in\mathscr{D}'(\mathbf{R}^n),\tag{5.1}$$

$t\in]-\varepsilon,\varepsilon[$, where $E^{(s)}$ is the solution to (3.21) for sufficiently small $\varepsilon>0$. Then E is a microlocal parametrix near w, since

$$PEf=E^{(t)}\circ\varrho_t\circ\varphi f+\int_{-\varepsilon}^{t}(PE^{(s)})\circ\varrho_s\circ\varphi f\,ds\cong\varphi f\mod C^\infty.$$

We shall study the singularities of this parametrix. Recall that $\sum=\bigcup_{j=1}^{r_0}S_j$, where S_j are non-radial hypersurfaces. Let $C_j\subset S_j\times S_j$ be the forward (in t) Hamilton flow on S_j, $j=1,\dots,r_0$, and Δ^* the diagonal in $T^*\mathbf{R}^n\times T^*\mathbf{R}^n$.

Proposition 5.1. *Let $P=D_t+K(t,x,D_x)$ be an $N_0\times N_0$ system, with $K\in\mathrm{Op}\,S(m,g)$ having principal symbol k satisfying* (3.1). *If E is the parametrix for P defined by* (5.1), *then* $\mathrm{WF}'\,E\subset\big(\bigcup_{j=1}^{r_0}C_j\big)\cup\Delta^*$, *microlocally near* $(w,w)\in\Sigma_2\times\Sigma_2$.

Proof. We have $\mathrm{WF}\,(\varrho_s\varphi f)=\pi(\mathrm{WF}\,(\varphi f))|_{t=s}$, where $\pi\colon(t,x;\tau,\xi)\to(t,x,\xi)$ is the projection. Thus, it suffices to show

$$\mathrm{WF}\,(E^{(s)}f_0)|_{t>s}\subset\bigcup_{j=1}^{r_0}C_j\circ\iota_s^{*^{-1}}(\mathrm{WF}\,f_0),\quad f_0\in\mathscr{D}'(\mathbf{R}^{n-1}),\tag{5.2}$$

where $\iota_s^*\colon T^*_{t=s}\mathbf{R}^n\to T^*\mathbf{R}^{n-1}$ is the dual to the inclusion of $\mathbf{R}^{n-1}$ as the surface $\{t=s\}$ in $\mathbf{R}^n$. Now, (5.2) holds for $E_j^{(s)}f_0$, $j>0$, since φ_j solves (3.4). It is clear that

$$\mathrm{WF}\,(E_0^{(s)}f_0)|_{t>s}\subset C_0\circ\iota_s^{*^{-1}}(\mathrm{WF}\,f_0),\quad f_0\in\mathscr{D}'(\mathbf{R}^{n-1}),$$

where $C_0\subset\Sigma_2\times\Sigma_2$ is the set of (w_1,w_2) such that w_1 and w_2 are in the same leaf of Σ_2 and $t(w_1)>t(w_2)$. Thus it suffices to prove that $E_0^{(s)}\in C^\infty$ microlocally near $(t,x,(0,\eta_0''),z,(0,\eta_0''))$ when $x'\neq z'$. By translation we may assume $s=0$.

Now applying P to $E_0^{(0)}$ we obtain by (3.17)—(3.20) and Lemma A.2 in the appendix

$$\begin{cases}D_ta_0+e^{i\langle D_{y'},D_{\xi'}\rangle}\tilde{k}(t,x,\xi',\eta'')a_0(t,y',x'',z',\eta'')\big|_{\substack{\xi'=0\\ y'=x'}}\cong R_0a_0,\quad t>0,\\ a_0(0,x,z',\eta'')\cong 0,\end{cases}\tag{5.3}$$

mod $S^{-\infty}$, microlocally when $|x'-z'|\geqq\varepsilon>0$, $\forall\varepsilon>0$. Here R_0: $S^{\nu}_{1,\mu,0}\to S^{\nu-1}_{1,\mu,0}$, $\forall\nu$, and $\tilde{k}$ is the full symbol of K. (This follows since (5.2) holds for $E_j^{(0)}$, $j>0$.) Also, (5.3) is determined mod $S^{-\infty}$ by the restriction of a_0 to $\{|y'-z'|>\varepsilon/2\}$, and $\tilde{k}$ to $\{|\xi'|\leqq C\langle\eta''\rangle\}$. We shall prove $a_0\in S^{-\infty}$ in $\{x'\neq z'\}$, by showing that $a_0\in S^{\nu}_{1,\mu,0}\Rightarrow a_0\in S^{\nu-\mu/2}_{1,\mu,0}$, $\forall\nu$, there.

Thus assume $a_0\in S^{\nu}_{1,\mu,0}$ near $(t_0, x_0, z_0', \eta_0'')$, $|x_0'-z_0'|\geqq\varrho>0$. By translation and localization, we may assume $x_0'=0$, $a_0\in S^{\nu}_{1,\mu,0}$ supported where $\langle\eta''\rangle\cong\langle\eta_0''\rangle$, and $\tilde{k}$ supported where $|\xi'|\leqq C\langle\eta''\rangle\cong C\langle\eta_0''\rangle$. Let $\lambda=\langle\eta_0''\rangle^{-\mu}$, and make the change of variables (4.3) and (4.5). Then $a_0(t, y, w)\in S(\lambda^{-\nu/\mu}, e)$, $\tilde{k}(t, y, w, \eta)\in S(\langle\eta\rangle^{k_0+1}, g_\lambda)$, where e is equal to the euclidean metric and we may assume $\nu=0$. Clearly $|w|>\varrho\lambda^{-1}$. Observe that (5.3) holds mod $S(\lambda^N, e)$, $\forall N$, when $|y|=|\lambda^{-1}x'|<\varrho\lambda^{-1}/2$. Choose $\Phi(s)\in C_0^\infty(\mathbf{R})$, such that $\Phi(s)=1$ when $|s|\leqq 1/2$, $\Phi(s)=0$ when $|s|>1$, and put

$$\chi(y, w)=\Phi(4\lambda|y|^2/\varrho^2+C\lambda^2|w|^2)\in S(1, \lambda|dy|^2+|dw|^2).$$

Then $b_0=\lambda^{-1/2}\chi a_0$ satisfies

$$\begin{cases} D_t b_0+\tilde{k}_0(t, w, D_y)b_0=r_1, & 0<t<\varepsilon, \\ b_0|_{t=0}=r_0, \end{cases} \tag{5.4}$$

where $\tilde{k}_0(t, w, \eta)=\tilde{k}(t, 0, w, \eta)$, and $r_j\in C_0^\infty$ are uniformly bounded in B^∞. In fact, $\chi a_0\in S(\lambda^N, e)$, $\forall N$, at $t=0$. Also, the calculus gives

$$\lambda^{-1/2}[\tilde{k}_0(t, w, D_y), \chi]\in \operatorname{Op} S(\langle\eta\rangle^{k_0}, \tilde{g}_\lambda),$$

and

$$\lambda^{-1/2}\chi(\tilde{k}(t, y, w, D_y)-\tilde{k}_0(t, w, D_y))\in \operatorname{Op} S(\langle\eta\rangle^{k_0+1}, \tilde{g}_\lambda),$$

where $\tilde{g}_\lambda=\lambda|dy|^2+|dw|^2+|d\eta|^2/\langle\eta\rangle^2$. Then Remark 4.3 gives that b_0 is uniformly in B^∞, $0\leqq t<\varepsilon$. Thus $\chi a_0\in S(\lambda^{1/2}, e)$, and since this is uniform in λ when $|x'-z'|\geqq\varrho>0$, we obtain the proposition.

Proof of Theorem 1.3. As mentioned before, we only have to consider $w\in\Sigma_2$. By Proposition 2.3 it suffices to prove the propagation of singularities for the system $P=D_t\,\mathrm{Id}_{N_0}+K(t, x, D_x)$ satisfying (3.1). The adjoint P^* satisfies the same conditions, so by Proposition 5.1 we can construct a parametrix E for P^* such that $\mathrm{WF}'\,E\subset(\cup\, C_j)\cup\Delta^*$, microlocally near $(w, w)\in\Sigma_2\times\Sigma_2$. Cutting off, we may assume $u\in\mathscr{E}'$ and $w\in\Sigma_2\setminus\mathrm{WF}\,Pu$. Then $u\cong E^*Pu$ modulo C^∞, and since we may change t to $-t$, this gives the result.

Appendix. Some calculus lemmas

We are going to study the composition of conormal distributions having non-standard symbols. Let $a_\varphi(x, D)\in\mathscr{D}'(\mathbf{R}^n\times\mathbf{R}^n)$ be given by

$$\text{(A.1)}\qquad a_\varphi(x, D)u(x) = (2\pi)^{-n}\int e^{i(\langle x-y,\eta\rangle+\varphi(x,\eta))}a(x,\eta)u(y)\,dy\,d\eta,$$

$u\in C_0^\infty(\mathbf{R}^n)$, where $a\in S(m^k, g)$, $\varphi(x,\eta)\in C^\infty(T^*\mathbf{R}^n\setminus 0)$ is homogeneous of degree 1 in the η variables and satisfies (3.5). Here g, m are defined by (2.6)—(2.7). The composition with $p(x, D)$ is given by

$$\text{(A.2)}\qquad p(x,D)\,a_\varphi(x,D)u(x) = (2\pi)^{-2n}\iint e^{i(\langle x-y,\xi\rangle+\langle y-z,\eta\rangle+\varphi(y,\eta))}p(x,\xi)$$
$$\times a(y,\eta)u(z)\,dz\,d\eta\,dy\,d\xi = b_\varphi(x,D)u(x),$$

if p, $a\in\mathscr{S}$, where

$$\text{(A.3)}\qquad b(x,\eta) = (2\pi)^{-n}\int e^{-iE}p(x,\xi)a(y,\eta)\,dy\,d\xi$$

and $E=\langle y-x, \xi-\eta\rangle-\varphi(y,\eta)+\varphi(x,\eta)=\langle y-x, \theta-\eta\rangle$, if we put

$$\text{(A.4)}\qquad \theta = \xi-\int_0^1\partial_x\varphi(x+s(y-x),\eta)\,ds.$$

Now $\chi\colon (x,\xi; y,\eta)\to(x,\theta; y,\eta)$ is a diffeomorphism. Thus if we let

$$f(x,\theta; y,\eta) = p(x,\xi)\,a(y,\eta),$$

we obtain

$$\text{(A.5)}\qquad b(x,\eta) = e^{i\langle D_y, D_\theta\rangle}f(x,\theta;\ y,\eta)\big|_{\substack{\theta=\eta\\ y=x}}$$

since $\left|\dfrac{d(y,\xi)}{d(y,\theta)}\right|\equiv 1$. This can be extended to general symbols by the following

Lemma A.1. *Assume $\varphi(x,\eta)\in C^\infty(T^*\mathbf{R}^n\setminus 0)$ is homogeneous of degree 1 in the η variables and satisfies (3.5). If $a\in S(m^k, g)$, $k\in\mathbf{Z}$, has support in a sufficiently small conical neighborhood of $\{\eta'=0\}$ and $p\in S(m, g)$, then the composition is given by (A.2) where $b\in S(m^{k+1}, g)$ satisfies (A.5), and has the expansion*

$$\text{(A.6)}\qquad b(x,\eta)\cong\sum_{j=0}^{N-1}\big(i\langle D_\xi, D_y-(\partial\theta/\partial y)D_\xi\rangle\big)^j p(x,\xi)\,a(y,\eta)/j!\big|_{\substack{y=x\\ \xi=\eta+d_x\varphi(x,\eta)}}$$

modulo $S(m^{k+1}h^N, g)$, with θ given by (A.4).

Proof. If $\varphi\equiv 0$ then (A.5)—(A.6) follows from the Weyl calculus, since $g(t,-\tau)=g(t,\tau)$ (see Th. 18.5.4 and 18.5.10 in [5]). Now $p(x,\xi)a(y,\eta)\in S(M, G)$ where $M(\xi,\eta)=m(\xi)m^k(\eta)$ is a weight for $G=g_{x,\xi}(dx, d\xi)+g_{y,\eta}(dy, d\eta)$. Thus, if we can prove that $\chi^*S(M, G)=S(M, G)$, then we would obtain (A.6) since $\partial_\xi\chi=(0, \mathrm{Id}; 0, 0)$ and $\partial_y\chi=(0, \partial\theta/\partial y; \mathrm{Id}, 0)$. Now we only have to consider the case when

$$(A.7) \qquad |\theta-\eta| \leqq \varepsilon(|\theta|^{\mu}+|\eta|^{\mu}),$$

since otherwise we may integrate by parts with respect to y in (A.3) to obtain $b\in S^{-\infty}$. In fact, (3.5) gives $|\theta-\xi|\leqq\varrho|\eta'|$, when η is in a small conical neighborhood of $\{\eta'=0\}$, and for small ϱ we find $|\theta|^{\mu}+|\eta|^{\mu}\geqq c(|\xi|^{\mu}+|\eta|^{\mu})$. When (A.7) holds for small ε, we obtain

$$1/c \leqq (\langle\theta\rangle^{\mu}+|\theta'|)/(\langle\eta\rangle^{\mu}+|\eta'|) \leqq c.$$

This gives $|\langle\theta\rangle-\langle\xi\rangle|\leqq\varrho|\eta'|\leqq C\varrho\langle\theta\rangle$, and

$$\langle\theta\rangle^{\mu}+|\theta'| \leqq C(\langle\xi\rangle^{\mu}+|\xi'|+\varrho^{\mu}(\langle\eta\rangle^{\mu}+|\eta'|)) \leqq C'(\langle\xi\rangle^{\mu}+|\xi'|+\varrho^{\mu}(\langle\theta\rangle^{\mu}+|\theta'|)).$$

For small ϱ we find $\langle\theta\rangle^{\mu}+|\theta'|\leqq C(\langle\xi\rangle^{\mu}+|\xi'|)$, and similarly $\langle\xi\rangle^{\mu}+|\xi'|\leqq C(\langle\theta\rangle^{\mu}+|\theta'|)$. Since $m(\xi)\approx h^{-k_0-1}(\xi)\langle\xi\rangle^{-k_0}$, we have $\chi^*M\approx M$ for η in a small conical neighborhood of $\{\eta'=0\}$. Clearly, $\partial\theta/\partial\eta''=\mathcal{O}((|\eta'|/|\eta|)^{k_0+1})$, $\partial\theta/\partial\eta'=\mathcal{O}(1)$ and $\partial\theta/\partial x=\mathcal{O}(|\eta'|)$, so $\chi^*G\approx G$ in a small conical neighborhood of $\{\eta'=0\}$. Thus by Lemma 8.2 in [4] we obtain $\chi^*S(M,G)=S(M,G)$ if

$$(A.8) \qquad G_{\chi(w)}(\chi^{(k)}(w;\ t_1,\dots,t_k)) \leqq C_k \Pi_{i=1}^{k} G_{\chi(w)}(\chi'(w,t_i))$$

for $k>1$, where χ^k is the k:th differential. This means that

$$(A.9) \qquad \begin{cases} |\partial_y^{\alpha}\partial_{\eta}^{\beta}\partial_x^{\gamma}\theta'(x,y,\eta)| \leqq C_{\alpha\beta\gamma}\langle\eta\rangle^{-|\beta''|}(\langle\eta\rangle^{\mu}+|\eta'|)^{1-|\beta'|} \\ |\partial_y^{\alpha}\partial_{\eta}^{\beta}\partial_x^{\gamma}\theta''(x,y,\eta)| \leqq C_{\alpha\beta\gamma}\langle\eta\rangle^{1-|\beta''|}(\langle\eta\rangle^{\mu}+|\eta'|)^{-|\beta'|}, \end{cases}$$

for $|\alpha|+|\beta|+|\gamma|>1$, where θ is given by (A.4). Since θ is homogeneous of degree 1, the second inequality follows from $\langle\eta\rangle^{-1}\leqq 2(\langle\eta\rangle^{\mu}+|\eta'|)^{-1}$. Similarly, we get the first when $|\beta'|>0$, and otherwise

$$|\partial_y^{\alpha}\partial_{\eta''}^{\beta''}\partial_x^{\gamma}\theta'(x,\eta)| \leqq C_{\alpha\beta''\gamma}|\eta'|\langle\eta\rangle^{-|\beta''|},$$

according to (3.5), which proves (A.9) and the lemma.

Next, let $S_{1,\mu,0}^{\nu}$ be the symbol classes defined by (3.14), $\mu=k_0/(k_0+1)$, $\nu=\mu(d_0-1)$ and $d_0=\operatorname{codim}\Sigma_2$. For $a\in S_{1,\mu,0}^{\nu}$ we define $a(x,D'')\in\mathcal{D}'(\mathbf{R}^n\times\mathbf{R}^n)$ by

$$(A.10) \qquad a(x,D'')u(x) = (2\pi)^{d_0-n}\iint e^{i\langle x''-y'',\eta''\rangle}a(x,y',\eta'')u(y)\,dy\,d\eta'',$$

$u\in C_0^{\infty}(\mathbf{R}^n)$. If p, $a\in\mathcal{S}$, then the composition is given by

$$(A.11) \qquad p(x,D)a(x,D'')u(x) = (2\pi)^{d_0-2n}\iint e^{i(\langle x-y,\xi\rangle+\langle y''-z'',\eta''\rangle)}p(x,\xi) \times a(y,z',\eta'')u(z)\,dz\,d\eta''\,dy\,d\xi = b(x,D'')u(x)$$

where

(A.12) $$b(x, z', \eta'') = (2\pi)^{-n} \iint e^{i\langle x-y, \xi-(0,\eta'')\rangle} p(x, \xi) a(y, z', \eta'')\, dy\, d\xi$$
$$= e^{i\langle D_y, D_\xi\rangle} p(x, \xi) a(y, z', \eta'')\big|_{\substack{y=x \\ \xi=(0,\eta'')}}.$$

For more general symbols we obtain the following lemma.

Lemma A.2. *If* $p\in S(m, g)$ *and* $a\in S^{\nu}_{1,\mu,0}$, *then the composition is given by* (A.11) *where* $b\in S^{\nu}_{1,\mu,0}$ *satisfies* (A.12) *and*

(A.13) $$b(x, z', \eta'') = e^{i\langle D_{y'}, D_{\xi'}\rangle} p(x, \xi', \eta'') a(y', x'', z', \eta'')\big|_{\substack{\xi'=0 \\ y'=x'}} + Ra,$$

where $R\colon S^{\nu}_{1,\mu,0}\to S^{\nu-1}_{1,\mu,0}$ *is continuous. Also, b and Ra are determined modulo* $S^{-\infty}$ *by the restriction of a to* $\{|y-x|<\varepsilon\}$, *and p to* $\{|\xi-(0,\eta'')|<\varepsilon\langle\eta''\rangle\}$, $\forall\varepsilon>0$.

Proof. Let

$$G_{(x,\xi,y,z',\eta'')} = |dx|^2 + |d\xi'|^2/(\langle\xi\rangle^{\mu}+|\xi'|)^2 + |d\xi''|^2/\langle\xi\rangle^2$$
$$+|dy'|^2\langle\eta''\rangle^{2\mu} + |dy''|^2 + |dz'|^2\langle\eta''\rangle^{2\mu} + |d\eta''|^2/\langle\eta''\rangle^2,$$

and $A(x, \xi, y, z', \eta'') = \langle y, \xi\rangle$. Then the dual metric is given by

$$G^A_{(x,\xi,y,z',\eta'')}(0, d\xi, dy, 0) = |d\xi'|^2/\langle\eta''\rangle^{2\mu} + |d\xi''|^2 + |dy'|^2(\langle\xi\rangle^{\mu}+|\xi'|)^2 + |dy''|^2\langle\xi\rangle^2,$$

and equal to $+\infty$ otherwise. We have $p(x, \xi)a(y, z', \eta'')\in S(M, G)$ where $M(\xi, \eta'') = m(\xi)\langle\eta''\rangle^{\nu}$. In the following we shall suppress the z' variables, which are not important.

Now, G is slowly varying, $G\leqq G^A$ at $\Delta=\{\xi=(0,\eta'')\wedge y=x\}$ and G is A temperate with respect to Δ, i.e.

$$G_{(x,\xi,y,\eta'')} \leqq C G_{(x,(0,\eta''),x,\eta'')}\big(1+G^A_{(x,\xi,y,\eta'')}(0, \xi-(0,\eta''), y-x, 0)\big)^N.$$

This follows since

$$\langle\eta''\rangle^{\mu}/(\langle\xi\rangle^{\mu}+|\xi'|) + \langle\eta''\rangle/\langle\xi\rangle \leqq C(1+|\xi''-\eta''|),$$

and similarly M is A, G temperate with respect to Δ, since

$$M(\xi, \eta'')/M((0, \eta''), \eta'') = 1+(|\xi'|/\langle\xi\rangle^{\mu})^{k_0+1}$$

and

$$|\xi'|/\langle\xi\rangle^{\mu} = (|\xi'|/\langle\eta''\rangle^{\mu})(\langle\eta''\rangle/\langle\xi\rangle)^{\mu} \leqq C(1+|\xi'|/\langle\eta''\rangle^{\mu}+|\xi''-\eta''|)^2.$$

By Theorem 18.4.10′ in [5] we obtain that $b\in S^{\nu}_{1,\mu,0}$ satisfies (A.12).

In order to prove (A.13), we observe that

$$e^{i\langle D_y, D_\xi\rangle} = e^{i\langle D_{y'}, D_{\xi'}\rangle} \circ e^{i\langle D_{y''}, D_{\xi''}\rangle}.$$

If $A''(x, \xi, y, \eta'') = \langle y'', \xi''\rangle$ we obtain

$$G^{A''}(0, d\xi'', 0, dy'', 0) = G^A(0, d\xi'', 0, dy'', 0),$$

and equal to $+\infty$ otherwise. We have $G \leqq \langle \eta'' \rangle^{-2} G^{A'}$ at $\Delta'' = \{\xi'' = \eta'' \wedge y'' = x''\}$ and G is A'' temperate with respect to Δ'', since

$$(\langle (\xi', \eta'') \rangle^{\mu} + |\xi'|)/(\langle \xi \rangle^{\mu} + |\xi'|) + \langle (\xi', \eta'') \rangle / \langle \xi \rangle \leqq C(1 + |\xi'' - \eta''|).$$

Similarly, M is A'', G temperate with respect to Δ'', so Theorem 18.4.11 in [5] gives

$$c = e^{i\langle D_{y''}, D_{\xi''} \rangle} p(x, \xi) a(y, \eta'')|_{\Delta''} \in S(\tilde{M}, \tilde{G}),$$

where $\tilde{G}$, $\tilde{M}$ are the restrictions of G, M to Δ''. Here $c \cong p(x, \xi) a(y, \eta'')|_{\Delta''}$ modulo $S(\tilde{M}_1, \tilde{G})$, where $\tilde{M}_1 = \tilde{M} \langle \eta'' \rangle^{-1}$.

If $A'(x, \xi, y, \eta'') = \langle y', \xi' \rangle$, we get

$$\tilde{G}^{A'}(0, d\xi', dy', 0) = G^{A}(0, d\xi', dy', 0)|_{\Delta''},$$

and equal to $+\infty$ otherwise. Then $\tilde{G} \leqq \tilde{G}^{A'}$ at $\Delta' = \{\xi' = 0 \wedge y' = x'\}$ in Δ'', and $\tilde{G}$ is clearly A' temperate with respect to Δ', since it is the restriction of an A temperate metric. Similarly, $\tilde{M}$ is A', $\tilde{G}$ temperate with respect to Δ', since it is the restriction of an A, G temperate weight. As before, we obtain that

$$b(x, \eta'') = e^{i\langle D_{y'}, D_{\xi'} \rangle} c(x, \xi', y', \eta'')|_{\Delta'} \in S^{\nu}_{1,\mu,0}$$

satisfies (A.13), since

$$\tilde{G}|_{\Delta'} = G|_{\Delta} \approx |dx'|^2 \langle \eta'' \rangle^{2\mu} + |dx''|^2 + |dz'|^2 \langle \eta'' \rangle^{2\mu} + |d\eta''|^2 / \langle \eta'' \rangle^2.$$

Outside the support of the integrand in (A.12), the symbol decays as any power of the G^A distance to the support (see [5, Section 18.4]). Thus, the last statement follows from the fact that

$$G^{A}(y-x, \xi-(0, \eta'')) \geqq |y-x|^2 \langle \eta'' \rangle^{2\mu} + |\xi - (0, \eta'')|^2 \langle \eta'' \rangle^{-2\mu},$$

at $(x, (0, \eta''), x, \eta'')$, where $0 < \mu < 1$. This completes the proof of the lemma.

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