On an extremal configuration for capacity

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1. Main theorem

It is well known that the capacity of a closed set E on the (unit) circle is decreased by circular symmetrization [1, pp. 31—36]. Thus, if the length mE of E is L, we have the estimate $\operatorname{cap} E \geqq \sin(L/4)$ [1, p. 35]. How large can $\operatorname{cap} E$ be, if $mE=L$ and E consists of a given number of arcs, n arcs? The maximal configuration is given by a set E^* of n arcs of equal length, L/n, "regularly" or "symmetrically" distributed around the circle.

Theorem 1. *Let* $E^* = \bigcup_{k=0}^{n-1} \{\exp(i\vartheta)\colon -L/2n \leqq \vartheta - 2\pi k/n \leqq L/2n\}$ *and let E be a union of n arcs on the unit circle of total length L. Then*

$$\operatorname{cap} E \leqq \operatorname{cap} E^* = (\sin(L/4))^{1/n}, \tag{1}$$

with equality for $E=E^*$.

A proof of this theorem follows from work of Dubinin's [2]. He proved a conjecture by Gončar for harmonic measure by introducing a process called desymmetrization, which can also be used for transforming E^* to E and for comparing the capacities of these sets.

In terms of equivalent characteristics of E and E^*, the inequality in Theorem 1 can be stated for *Robin constants,* (see [1, p. 30]), γ of E and γ^* of E^*, as

$$\gamma^* \leqq \gamma, \tag{1'}$$

and for *reduced extremal distances,* (see [1, pp. 78—80]), as

$$\delta(0, E^*) \leqq \delta(0, E). \tag{1''}$$

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In fact, by the references just cited,

$$\gamma = -\log \operatorname{cap} E = \pi\delta(0, E).$$

The last equality is valid for sets on the unit circle. The inequality (1) was conjectured by the author after considering relations between harmonic measure and reduced extremal length, cf. [4, Cor. 1, p. 6].

2. Proof

Desymmetrization. We shall apply a variation of Dubinin's desymmetrization procedure, so as to transform E^* to E and $\mathscr{C}E^*$ to $\mathscr{C}E$, $\mathscr{C}$ denoting complements w.r.t. the unit circle. His procedure consists of dividing the complex plane $\mathbf{C}$ into a finite number of suitable angles and rotating these in an appropriate manner. Let $L_k^* = \{z\colon \arg z = 2\pi k/n\}$ and $L_k = \{z\colon \arg z = \alpha_k\}$, $k=0, 1, \ldots, n-1$. The set $\{z\colon \vartheta_1 < \arg z < \vartheta_2\}$ is called an angle. *Dubinin's Lemma* 1 [2, p. 273] contains the following statements a), b), c).

There exist a finite number of pairwise disjoint angles P_k and rotations $\lambda_k(z) = z\exp(i\vartheta_k)$ (ϑ_k real), $k=0, 1, \ldots, N-1$, having the following properties:

a) $\bigcup_{k=0}^{N-1} \bar{P}_k = \mathbf{C}$, $\bigcup_{k=0}^{N-1} \bar{S}_k = \mathbf{C}$, $S_k = \lambda_k(P_k)$;

b) the ray L_k^* is the bisectrix of the angle P_k and $\lambda_k(L_k^*) = L_k$, $k=0, 1, \ldots, n-1$;

c) if $\bar{S}_k \cap \bar{S}_{k'} \neq \emptyset$, then the common boundary ray of the angles S_k and S'_k is the image under the mappings λ_k and $\lambda_{k'}$ of two boundary rays which can be obtained from one another by a finite number of reflections with respect to straight lines through the origin, forming angles which are integer multiples of π/n with the real axis.

Dubinin [2, p. 273] calls a domain D *symmetric* if it is symmetric with respect to the rays $\{z\colon \arg z = \pi k/n\}$, $k=0, 1, \ldots, 2n-1$. A function $u(z)$ defined in a symmetric domain D is called symmetric if the sets $\{z\colon u(z)=a\}$ are symmetric. The *desymmetrization* $\tilde{D}$ of a symmetric domain D is defined by $\tilde{D} = \bigcup_{k=0}^{N-1} \lambda_k(D \cap \bar{P}_k)$ and the *desymmetrization* $\tilde{u}$ of u is defined by $\tilde{u}(z) = u(z\exp(-i\vartheta_k))$, $z \in \tilde{D} \cap \bar{S}_k$, $k=0, 1, \ldots, N-1$, using the notation of Lemma 1. The desymmetrization procedure is such that the boundary rays fit together so as to preserve (Lipschitz) continuity of the desymmetrization of a symmetric (Lipschitz) continuous function in a symmetric domain [2, Lemma 2, p. 274].

The angles between the rays L_k^* are equivalent in Dubinin's procedure. We now consider two separate sets of angles, corresponding to E^* and to $\mathscr{C}E^*$. A desymmetrization procedure is applied to each set of angles in such a way that the common boundary rays fit together. We indicate the beginning of the procedure.

The set E^* on the unit circle is defined by the union U^* of n angles in the plane

$\{z=r\exp(i\vartheta)\colon |\vartheta-2\pi k/n|<L/2n,\ r\in R^+\}$, $k=0, 1, \ldots, n-1$. The set E is defined by the union U of n angles $\{z=r\exp(i\vartheta)\colon |\vartheta-\psi_k|<\varphi_k/2\}$, $k=0, 1, \ldots, n-1$. (E and E^* are closed sets, the angles are open.) Let $\varphi=\min\varphi_k$, $k=0, 1, \ldots, n-1$. We make the following definitions:

$$P_k=\{r\exp(i\vartheta)\colon (L/2n)-\varphi/2<|\vartheta-2\pi k/n|<L/2n\},\quad k=0,1,\ldots,n-1,$$
$$S_k=\{r\exp(i\vartheta)\colon (\varphi_k-\varphi)/2<|\vartheta-\psi_k|<\varphi_k/2\},\quad k=0,1,\ldots,n-1.$$

Thus P_k and S_k, $k=0, 1, \ldots, n-1$, consist of two angles adjoining to rays whose intersections with the unit circle are endpoints of arcs in E^* and E, whereas in Dubinin's initial step P_k and S_k consist of one angle each. Let

$$P_k=P_k'\cup P_k'',\ S_k=S_k'\cup S_k'',$$

where

$$P_k'\cap P_k''=S_k'\cap S_k''=\emptyset,\quad k=0,1,\ldots,n-1.$$

We now define two rotations λ_k' and λ_k'' for each k, $k=0, 1, \ldots, n-1$, such that

$$\lambda_k'(P_k')=S_k' \quad\text{and}\quad \lambda_k''(P_k'')=S_k''.$$

Let

$$A_n=\bigcup_{k=0}^{n-1}\bar{P}_k,\ B_n=\bigcup_{k=0}^{n-1}\bar{S}_k.$$

The number of angles in $U\setminus B_n$ is $n_1<n$, and the number of angles in $U^*\setminus A_n$ is n (unless $n_1=0$). We next choose P_n as one of the remaining angles in $U^*\setminus A_n$ and define a rotation λ_n such that $\lambda_n(P_n)=S_n$, where S_n adjoins B_n and is contained in a largest remaining angle in $U\setminus B_n$. (It is possible to find a suitable S_n since $n_1<n$ and $L-n\varphi=m(U^*\setminus A_n)=m(U\setminus B_n)$, m denoting angular measure. Thus there is at least one angle greater than $L/n-\varphi=m(P_n)$ in $U\setminus B_n$.) The number of angles in $U^*\setminus A_{n+1}$ is now $n-1$; the number of angles in $U\setminus B_{n+1}$ is n_1. If $n_1<n-1$, P_{n+1} is defined in an analogous manner to the definition of P_n. (Now we have $(n-1)(L/n-\varphi)=(n-1)m(P_{n+1})=m(U\setminus B_{n+1})$ and thus there is at least one angle greater than P_{n+1} in $U\setminus B_{n+1}$.)

One can thus choose $n-n_1$ angles $P_n, \ldots, P_{2n-n_1-1}$ and corresponding angles $S_n, \ldots, S_{2n-n_1-1}$, such that $m(U^*\setminus A_{2n-n_1})=m(U\setminus B_{2n-n_1})$ and $U^*\setminus A_{2n-n_1}$ and $U\setminus B_{2n-n_1}$ each consist of n_1 intervals. Moreover, one can define rotations λ_k such that $\lambda_k(P_k)=S_k$, $k=n, \ldots, 2n-n_1-1$.

Now we can return to the initial step, determine the least value $\varphi^{(1)}$ in $U\setminus B_{2n-n_1}$ and proceed as above, etc. We finally obtain a desymmetrization of the angles corresponding to E^* and then deal with the angles corresponding to $\mathscr{C}E^*$ in the same way. The common boundary rays fit together. Let Δ denote the unit disk. We have described a total *desymmetrization* procedure for the whole configuration $\Delta\cup E^*\cup \mathscr{C}E^*$ and hence also for a symmetric function defined in $\Delta\cup E^*\cup\mathscr{C}E^*$, cf. p. 98, so that E^* corresponds to E and $\mathscr{C}E^*$ to $\mathscr{C}E$.

Proof of Theorem 1. Let F denote a closed set on the unit circle and $u(z)$ a function harmonic in the unit disk Δ, with $u(0)=1$ and $\limsup u(z) \leqq 0$, as z approaches F. Let $g(z)$ denote the restriction to Δ of the Green function of the complement (w.r.t. $\mathbf{C}$) of F and let $\gamma = -\log \operatorname{cap} F$, [1, p. 30]. The Dirichlet integral over Δ, $D(u)$, satisfies

$$D(u) \geqq \pi/\gamma = D(g/\gamma), \tag{2}$$

with equality for $u=g/\gamma$, [1, p. 30].

We start from this result for $F=E^*$ and the corresponding g^* (Green function) and γ^* (Robin constant). Let $u^*=g^*/\gamma$. Then

$$D(u^*) = \pi/\gamma^*. \tag{3}$$

To our function u^* in Δ with boundary values 0 on E^* we define, by the total desymmetrization procedure of $\Delta \cup E^* \cup \mathscr{C}E^*$ as described above, a total desymmetrization $\tilde{u}$ in Δ with boundary values 0 on E. By considering the Dirichlet integral of u^* as a finite sum of integrals over sectors defined by the procedure, it is seen that (cf. [2, p. 275])

$$D(u^*) = D(\tilde{u}). \tag{4}$$

Now let u denote the Poisson integral of the values of $\tilde{u}$ on the unit circle. By Dirichlet's principle, cf. [2, p. 275]:

$$D(\tilde{u}) \geqq D(u). \tag{5}$$

Since $u^*(0)=1$ and u^* and $\tilde{u}$ are equimeasurable on the unit circle it follows that $u(0)=1$. Since u as a Poisson integral is harmonic in the unit disk and $\limsup u(z) \leqq 0$ as z approaches E, we have by (2), γ denoting $-\log \operatorname{cap} E$,

$$D(u) \geqq \pi/\gamma. \tag{6}$$

From (3,) (4), (5) and (6) we obtain that

$$\pi/\gamma^* \geqq \pi/\gamma$$

and thus (1′), (1″) and the inequality in (1) follow.

The explicit value of $\operatorname{cap} E^*$ follows from a theorem by Fekete, for which we refer to [3, Thm. 2, p. 299]. According to Fekete's theorem $\operatorname{cap} E^* = (\operatorname{cap} F)^{1/n}$, where $F=\{z^n\colon z \in E^*\}$, that is $F=\{\exp(i\vartheta)\colon |\vartheta| \leqq L/2\}$. However, $\operatorname{cap} F = \sin(L/4)$ [1, p. 35]. Hence $\operatorname{cap} E^* = (\sin(L/4))^{1/n}$.

Corollary 1. *Let D be a Jordan domain with $\vartheta \subset \partial D$. Let $\omega(z, \vartheta, D)$ denote the harmonic measure at z of ϑ w.r.t. D and let $\delta(z, \vartheta, D)$ denote the reduced extremal distance between z and ϑ w.r.t. D. Let ϑ consist of n boundary arcs. Then*

$$\arcsin \exp(-n\pi\delta(z, \vartheta, D)) \leqq \pi\omega(z, \vartheta, D)/2 \leqq \arcsin \exp(-\pi\delta(z, \vartheta, D)). \tag{7}$$

Proof. The right-hand inequality was stated in [4, p. 3]. Map D conformally onto the unit disk Δ, so that z goes to the origin and ϑ onto E of length L on the unit circle. By conformal invariance of ω we have

(8) $$\pi\omega(z, \vartheta, D)/2 = \pi\omega(0, E, \Delta)/2 = L.$$

By Theorem 1 and the well-known estimate of $\operatorname{cap} E$ from below ([1, p. 35])

(9) $$\sin(L/4) \leqq \operatorname{cap} E \leqq (\sin(L/4))^{1/n}.$$

However, for a set E on the unit circle, by [1, p. 80],

(10) $$\operatorname{cap} E = \exp\big(-\pi\delta(0, E, \Delta)\big).$$

By conformal invariance of ω and δ we obtain (7) from (8), (9) and (10).

3. Concluding comments

Remark 1. An inequality for reduced extremal distance in Dubinin's configuration.

Let, for a fixed r, $0<r<1$,

$$D_\alpha = \{|z| < 1\}\setminus\alpha = \{|z| < 1\}\setminus\bigcup_{k=0}^{n-1} l_k,$$

where

$$l_k = \{z\colon \arg z = \alpha_k,\ r \leqq |z| < 1\},\quad k = 0, 1, \ldots, n-1,$$

and

$$D_{\alpha^*} = \{|z| < 1\}\setminus\alpha^* = \{|z| < 1\}\setminus\bigcup_{k=0}^{n-1} l_k^*,$$

where

$$l_k^* = \{z\colon \arg z = 2\pi k/n,\ r \leqq |z| < 1\},\quad k = 0, 1, \ldots, n-1.$$

Then the following inequality holds for reduced extremal distances with respect to D_α and D_{α^*}:

(11) $$\delta(0, \alpha) \geqq \delta(0, \alpha^*),$$

that is, *desymmetrization increases* δ.

In fact, Dubinin proves, for harmonic measure (in standard notation) that

(12) $$\omega(0, \alpha, D_\alpha) \leqq \omega(0, \alpha^*, D_{\alpha^*}),$$

[2, Theorem A, p. 272]. For that purpose he maps D_α and D_{α^*} conformally onto the unit disk so that the origin goes onto the origin. We denote the images of α and α^* on the unit circle by E' and E^*. By conformal invariance, $\delta(0, \alpha)=\delta(0, E')$ where the last reduced extremal distance is taken with respect to the unit disk. By [1, p. 80] we have that $\pi\delta(0, E')=-\log\operatorname{cap} E'$. Now let E be a desymmetrization of E^* that contains E'. E can be found since, by Dubinin's theorem, $mE'<mE^*$,

unless $\alpha^*=\alpha$. Since capacity increases when the set increases it follows that $\operatorname{cap} E' \leqq \operatorname{cap} E$, which however by Theorem 1 is $\leqq \operatorname{cap} E^*$. Thus

$$\exp(-\pi\delta(0,\alpha)) = \operatorname{cap} E' \leqq \operatorname{cap} E^* = \exp(-\pi\delta(0,\alpha^*)),$$

and (11) is proved.

The inequality (11) can be written in terms of Robin's constants (relative to the origin) for α and ∂D_α w.r.t. D_α. By [1, p. 79] the inequality (11) is equivalent to

$$\gamma(\alpha^*)-\gamma(\partial D_{\alpha^*}) \leqq \gamma(\alpha)-\gamma(\partial D_\alpha).$$

However, by [2, Corollary, p. 275] it follows that $\gamma(\partial D_{\alpha^*}) \leqq \gamma(\partial D_\alpha)$. Thus, we note that the quantity $\gamma(\alpha^*)$ is actually increased more by desymmetrization than $\gamma(\partial D_{\alpha^*})$.

Remark 2. Conjectures. Let $g(\cdot, 0)$ denote the Green function of D_α with pole at the origin and define g^* in an analogous manner w.r.t. D_{α^*}, in the notation of Remark 1. Let $y=(y_1, y_2)$. We conjecture that

$$\iint_{D_{\alpha^*}} g^*(y,0)\,dy \leqq \iint_{D_\alpha} g(y,0)\,dy. \tag{13}$$

An intuitive reason for this conjecture is given by a probabilistic interpretation as

$$E_0\tau^* \leqq E_0\tau,$$

where τ (alt. τ^*) denotes the exit time from D_α (alt. D_{α^*}) for a Brownian motion starting at the origin and E_0 stands for expectation (w.r.t. start at the origin) [5, p. 309].

A stronger conjecture is, in fact, that, for every $\lambda>0$,

$$P_0(\tau^* > \lambda) \leqq P_0(\tau > \lambda)$$

for the actual probabilities for τ^* and τ. In connection with (13) one can also ask whether, for $y=re^{i\vartheta}$, the inequality

$$\int_{|y|=r} g^*(y,0)\,d\vartheta \leqq \int_{|y|=r} g(y,0)\,d\vartheta,$$

holds for $0<r<1$.

We note that Dubinin's proof of Gončar's conjecture (12) uses conformal mapping onto the unit disk of D_α and D_{α^*}. Let us change the definition of α and l_k in Remark 1 by taking $l_k=\{z\colon \arg z=\alpha_k,\ r\leqq|z|\leqq r_1<1\}$, $k=0, 1, \dots, n-1$, so that D_α now denotes a multiply connected domain with n slits. One conjectures that (12) remains true.

Remark 3. An alternative proof of Theorem 1. Consider $G_R=\{z\colon |z|<R\}\setminus E^*$ and the Dirichlet integral of u_R^*, harmonic in G_R, with boundary values 1 on $\{z\colon |z|=R\}$ and 0 on E^* (for large R). We can now use the following characteriza-

tion of capacity:

(14) $$\log \operatorname{cap} E^* = \lim_{R\to\infty} \left(\log R - 2\pi/D(u_R^*)\right).$$

This can be stated in terms of outer conformal radius for a continuum E^* (cf. [3, p. 314]), in terms of extremal length etc. (We have referred to reduced extremal length earlier in this paper; therefore the similar argument in [1, p. 79] can be referred to here.) (An analogue of (14) for inner radius can be used for a short proof of Dubinin's Corollary [p. 275, 2].)

Using (14) rather than (3) one can apply the desymmetrization procedure in an analogous manner to the proof of Theorem 1.

Remark 4. Further examples. I. The approach in Remark 3 is also applicable to a proof for the inequality

$$\operatorname{cap} F \leqq \operatorname{cap} F^*,$$

where

$$F = \bigcup_{k=0}^{n-1} \{z\colon \arg z = \alpha_k,\ r \leqq |z| \leqq r_1\}$$

and

$$F^* = \bigcup_{k=0}^{n-1} \{z\colon \arg z = 2\pi k/n,\ r \leqq |z| \leqq r_1\}.$$

In fact Dubinin's original desymmetrization applies to this configuration.

II. An inverted Gončar configuration is

$$F = \bigcup_{k=0}^{n-1} \{z\colon \arg z = \alpha_k,\ r \leqq |z| \leqq r_1\} \cup \{z\colon |z| \leqq r\}.$$

This "sun" — for $r=0$ a "star" — has maximal outer radius/capacity when the rays are equidistributed.

III. Let α and α^* denote the images of the sets E and E^* in Theorem 1 under the mapping $z \to rz$, r fixed, $0<r<1$. Let $D_\alpha = \Delta \setminus \alpha$ and $D_{\alpha^*} = \Delta \setminus \alpha^*$ (as earlier, Δ denotes the unit disk). Then an analogue of (12) is true:

$$\omega(0, \alpha, D_\alpha) \leqq \omega(0, \alpha^*, D_{\alpha^*}).$$

Let $g(x, y)$ denote the Green function of the unit disk Δ. The Green capacity for α^* equals the conformal capacity or $D(u_1^*)/2\pi$, in the notation of Remark 3, [6, p. 309]. The conformal capacity is decreased by a desymmetrization, as previously. In terms of Green capacities or equilibrium measures μ^* and μ (cf. [6, p. 309]) this decrease implies that

$$\mu^* \geqq \mu.$$

By comparing boundary values one sees that $\omega(0, \alpha^*, D_{\alpha^*})$ equals the equilibrium

Green potential of α^* (cf. [6, p. 309—310])

$$\omega(0, \alpha^*, D_{\alpha^*}) = \int_{\Delta} g(0, y)\, d\mu^*(y).$$

Since μ^* lies on α^* and $g(0, y) = -\log r$ on α^* and α, we obtain

$$\omega(0, \alpha^*, D_{\alpha^*}) = (-\log r)\mu^* \geqq (-\log r)\mu = \omega(0, \alpha, D_\alpha)$$

and thus the desired analogue of (12).

Remark 5. Extremality of symmetric configurations. Consider a set F containing at most a *given number, n,* of circular arcs, rays etc. of *given measure,* such that F can be viewed as the result of a suitable desymmetrization of a symmetric set F^* (in Dubinin's sense). There are various quantities $Q = Q(F)$, connected with suitable Dirichlet integrals, that can be shown (or conjectured) to vary with F in the following manner: $Q(F)$ varies between a) one extremal value for *one* arc, ray etc. and b) the opposite extremal value for F^*. For the case a) various standard symmetrization techniques have long been available; now Dubinin's method can be applied to the case b).

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Added in proof. The conjecture at the end of Remark 2, p. 102, for the multiply connected case with slits on n rays, has been proved for $n \leqq 3$ by A. Baernstein, On the harmonic measure of slit domains, *Complex Variables* **9** (1987), 131—142. See also A. Baernstein, *Dubinin's symmetrization theorem,* Complex Analysis I, Springer Lect. Notes Math. **1275** (1987), 23—30.

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