Note on a matroid with parity conditon

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1. Introduction

This paper presents a theorem concerning a matroid with the parity condition. The theorem provides matroid generalizations of graph-theoretic results of Lewin [3] and Gallai [1].

Let $M=(E, F)$ be a matroid, where E is a finite set of elements and F is the family of independent sets of M (in this paper, we presuppose a knowledge of matroid theory; our standard reference is Welsh [4]). Assume that E is of even cardinality and let P be a partition of E into disjoint blocks, each of size 2. A triple $M=(E, F, P)$ is called a matroid with the parity condition. One of two elements of each block of P is called the mate of the other. The mate of $e\in E$ is denoted by $\bar{e}$. A set $X\subseteq E$ is called a parity set, if for each $e\in X$, $\bar{e}\in X$. The matroid parity problem is to find a maximum independent parity set (an independent parity set with maximum number of elements). It is known that the matching problem of graphs is a special case of the matroid parity problem [2].

The purpose of this paper is to present a theorem on the maximal independent parity set of matroids with parity conditions. It is shown that the theorem provides matroid generalizations of results on the maximum matching of graphs by Lewin [3] and Gallai [1].

2. The main results

For any independent parity set I of $M=(E, F, P)$, we define the deficiency of I by

$$\delta(M, I) = \mathrm{rg}(E)-|I|,$$

where rg is the rank function of M. The deficiency of a maximum independent parity set of M is called the deficiency of M and is denoted by $\delta(M)$. $M^*=(E, F^*, P)$ is the dual matroid of $M=(E, F, P)$.

For a family Z of sets, we say that $X \in Z$ is maximal (minimal) in Z, if there is no $Y \in Z$, such that $X \subset Y$ $(Y \subset X)$. Note that a maximal (minimal) set is not necessarily unique nor does it necessarily have maximum (minimum) cardinality.

Theorem 1. *Let I be a maximal independent parity set of $M=(E, F, P)$. For any maximal independent parity set J^* of $M^* \cdot (E-I)$,*

$$\delta(M, I) = \delta(M^*, J^*),$$

where $M \cdot T$ $(T \subseteqq E)$ denotes the restriction of M to T.

Proof. Since $\text{rg}(E)=\text{rg}(E-J^*)$, there exists a base B of M, such that $I \subseteqq B \subseteqq E-J^*$. Obviously, $B^*=E-B$ is a base of M^* which contains J^*. Since I and J^* are maximal independent parity sets of M and $M^* \cdot (E-I)$, respectively, $e \in B-I$ if and only if $\bar{e} \in B^*-J^*$. Hence, there exists a one to one correspondence between $B-I$ and B^*-J^*.

Therefore, $|B|-|I|=|B^*|-|J^*|$. Since $|B|=\text{rg}(E)$ and $|B^*|=\text{rg}^*(E)$, $\delta(M, I)=\delta(M^*, J^*)$, where rg^* is the rank function of M^*.

Corollary 1.1. $\delta(M)=\delta(M^*)$.

Proof. For any maximum independent parity set I of M, there exists an independent parity set J^* of M^*, whose deficiency is equal to that of I, by Theorem 1. Hence, $\delta(M) \geqq \delta(M^*)$. Similarly, we can show that $\delta(M) \leqq \delta(M^*)$.

Corollary 1.2. *A maximal independent parity set I of M is maximum if and only if there exists a maximum independent parity set of M^* which is included in $E-I$.*

Proof. Theorem 1 and Corollary 1.1 insist that if $\delta(M, I)=\delta(M)$, there exists an independent parity set J^* of $M^* \cdot (E-I)$, such that $\delta(M^*, J^*)=\delta(M^*)$ and that if $\delta(M, I)>\delta(M)$, then for any independent parity set J^* of $M^* \cdot (E-I)$, $\delta(M^*, J^*)>\delta(M^*)$.

3. Derivation of theorems of Lewin and Gallai

Let $G=(V, A)$ be a graph, where V and A are vertex set and edge set of G, respectively. Subdivide the graph to obtain a graph $H=(U, E)$, in which each edge e' of G is replaced by a series connection of edges e and $\bar{e}$. Let $M=(E, F)$ be a partition matroid induced by incidences of edges in H on the original set of vertices $V \subseteqq U$.

Let P be a partition of E into disjoint blocks, each of size 2, such that e and $\bar{e}$ are the mates of each other. $M=(E, F, P)$ is said to be the matroid with the parity condition derived from G. It is clear that there is a one to one correspondence between independent parity sets of M and matchings in G. It is easy to see that if $M=(E, F, P)$ is the matroid derived from G, then the complement of an independent parity set of $M^*=(E, F^*, P)$ corresponds to a covering of G and vice versa. The following theorem is a direct consequence of Theorem 1.

Theorem 2. *Let n be the number of vertices of a connected graph $G=(V, A)$.*

(i) *Suppose that X is a maximal matching of G. If Y is a covering of G which contains X and is minimal in the coverings containing X, then*

$$|X|+|Y| = n.$$

(ii) *Suppose that Y is a minimal covering of G. If X is a matching of G which is contained in Y and maximal in the matchings contained in Y, then*

$$|X|+|Y| = n.$$

Proof. We will show (i). (ii) is the dual statement of (i). Let m be the number of edges of G, and $M=(E, F, P)$ be the matroid derived from G. Let $\bar{X}\subseteqq E$ be a parity set of M corresponding to $X\subseteqq A$. Then, $\bar{X}$ is a maximal independent parity set of M. Similarly, let $\bar{Y}$ be a parity set of M^* corresponding to $Y\subseteqq A$. Then, $E-\bar{Y}$ is a maximal independent parity set of $M^*\cdot(E-\bar{X})$. Since

$$\mathrm{rg}(E) = n, \quad \mathrm{rg}^*(E) = 2m-n,$$

we have

$$\delta(M, \bar{X}) = n-|\bar{X}| = n-2|X|$$

and

$$\delta(M^*, E-\bar{Y}) = 2m-n-|E-\bar{Y}| = 2m-n-2(m-|Y|) = 2|Y|-n.$$

From Theorem 1, we have

$$\delta(M, \bar{X}) = \delta(M^*, E-\bar{Y}).$$

Therefore

$$n-2|X| = 2|Y|-n,$$

and the theorem is proved.

The following facts, i.e. Corollary 2.1 and Corollary 2.2, which are the results of Lewin and Gallai, are direct consequences of Corollary 1.1 and Corollary 1.2, respectively.

Corollary 2.1 (Gallai [1]). *Let X and Y be a maximum matching and minimum covering of a connected graph, respectively. Then, $|X|+|Y|=n$.*

Corollary 2.2 (Lewin [3]). (i) *A minimal covering of a connected graph G is minimum if and only if it contains a maximum matching of G.*

(ii) *A maximal matching of a connected graph G is maximum if and only if it is contained in a minimum covering of G.*

References

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