On isomorphisms between Hardy spaces on complex balls

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1. Introduction

Let B be the unit ball in $\mathbf{C}^n$ and D the unit disc in $\mathbf{C}$. The aim of this paper is to present a new proof that the Hardy spaces $H^1(B)$ and $H^1(D)$ are isomorphic. This became necessary after the discovery that Wojtaszczyk's paper [18] contains a mistake. Part of Wojtaszczyk's proof was based on the intuition that nonisotropic distances on the unit sphere in $\mathbf{C}^n$ and on $\mathbf{R}^{2n-1}$ give "locally similar" metric spaces. This, as we show in Section 5 is not true, hence the argument behind Proposition B of [18] is not valid. Our proof is based on the probabilistic approach developed by Maurey in [10]. We show that if the standard Brownian motion is replaced with the diffusion corresponding to the invariant Laplacian in B, then Maurey's method gives an isomorphic embedding of $H^1(B)$ onto a complemented subspace of a martingale H^1-space which, in turn, is isomorphic to a complemented subspace of $H^1(D)$. Then we can use the first part of Wojtaszczyk's proof, which established that $H^1(D)$ is isometric to a complemented subspace of $H^1(B)$ (see also [1]) and the isomorphism follows by the decomposition principle. The last part of [18], where the atomic H^1-space on $\mathbf{R}^m$ with a nonisotropic distance is studied, is of independent interest and in Section 5 we use it to show that, in spite of the mistaken proof, Proposition B of [18] is true.

The case of $1<p<\infty$ was solved in [1, 18] where it was shown that $H^p(B)$ is isomorphic to $L^p([0, 1])$, hence to $H^p(D)$.

In order to make the paper reasonably self-contained we included some arguments which are fairly strightforward modifications of proofs which can be found in the literature.

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2. Preliminaries

2.1. Spaces of analytic functions

Throughout the paper we will assume that $1 \leqq p < \infty$.

Let B be the unit ball in $\mathbf{C}^n$, S the unit sphere and σ the normalized, rotation-invariant measure on S.

$H^p(B)$ will denote the space of all analytic functions in B such that

$$\|f\|_{H^p} = \sup_{r<1} \left(\int_S |f(rz)|^p \, d\sigma(z)\right)^{1/p} < \infty.$$

$A(B)$ will denote the ball algebra of all functions continuous in $\bar{B}$ and analytic in B. Rudin's book [13] is an excellent reference for the theory of $H^p(B)$.

If $f \in H^p(B)$ then $\tilde{f}(z) = \lim_{r\to 1} f(rz)$ exists for almost every $z \in S$ and $\|f\|_{H^p} = \|\tilde{f}\|_{L^p(S,\sigma)}$. f can be recovered from $\tilde{f}$ by the means of the Poisson formula

$$f(w) = \int_S \tilde{f}(z) P(w, z) \, d\sigma(z) \tag{1}$$

where $w \in B$ and $P(w, z) = (1-|w|^2)^n / |1-\langle w, z\rangle|^{2n}$ is the invariant Poisson kernel.

For $\alpha > 1$, $z \in S$ and $f \in H^p(B)$ let

$$D_\alpha(z) = \left\{w \in B\colon\ |1-\langle w, z\rangle| < \frac{\alpha}{2}(1-|w|^2)\right\}$$

and

$$(M_\alpha f)(z) = \sup\{|f(w)|\colon\ w \in D_\alpha(z)\}.$$

Then, by [13, Theorem 5.6.5]

$$\|M_\alpha f\|_{L^p(S)} \leqq A(\alpha)^{1/p} \|f\|_{H^p}. \tag{2}$$

For $z, w \in \bar{B}$ let $d(z, w) = |1-\langle z, w\rangle|^{1/2}$. Then $d(z, w)$ satisfies the triangle inequality in $\bar{B}$ and restricted to S is a distance ([13, Proposition 5.1.2]). If $w \in S$ then $Q(w, r)$ will denote the d-ball in S.

The space BMOA is defined as consisting of all $f \in H^2(B)$ such that

$$\|f\|^2_{\mathrm{BMO}} = \max\left\{|f(0)|^2,\ \sup\left\{\frac{1}{\sigma(Q)}\int_Q \left|\tilde{f} - \frac{1}{\sigma(Q)}\int_Q \tilde{f}\, d\sigma\right|^2 d\sigma\colon Q = Q(z, r)\right\}\right\} < \infty.$$

BMOA can be identified with the dual space of $H^1(B)$ in the sense that every continuous functional φ on $H^1(B)$, restricted to the dense subspace $H^2(B)$ is given in the form $\varphi(f) = \int \tilde{f}\bar{\tilde{g}}\, d\sigma$ for some $g \in$ BMOA and $\|\varphi\|_{(H^1)^*}$ and $\|g\|_{\mathrm{BMO}}$ are equivalent (cf. [2]).

2.2. Martingale spaces

We will also consider the martingale analogues of $H^p(B)$ (for details we refer to [9]). If $(\Omega, \mathcal{M}, P)$ is a probability space, E the corresponding expectation and $\mathcal{F}=(\mathcal{F}_m)_0^\infty$, an increasing sequence of sub-σ-fields of $\mathcal{M}$ then $H^p(\mathcal{F})$ is defined as consisting of all $\mathcal{F}$-martingales (f_m), for which

$$\|f\|_{H^p(\mathcal{F})} = \big(E(\sup_m |f_m|^p)\big)^{1/p} < \infty.$$

Let $\mathcal{F}_\infty = [(\mathcal{F}_m)_m]$ be the smallest σ-field containing all $\mathcal{F}_m$. If $(f_m)\in H^p(\mathcal{F})$ then f_m converges almost surely and in L^p to an $\mathcal{F}_\infty$-measurable random variable f_∞ and $f_m = E(f_\infty|\mathcal{F}_m)$. Hence $H^p(\mathcal{F})$ may be identified with a space of $\mathcal{F}_\infty$-measurable functions. The martingale version of BMO consists of all $f\in L^2(\Omega, \mathcal{M}, P)$ such that

$$\|f\|_{\mathrm{BMO}(\mathcal{F})} = \max\big\{|Ef|, \sup_m \big\|E(|f-E(f|\mathcal{F}_{m-1})|^2|\mathcal{F}_m)\big\|_{L^\infty(\Omega)}^{1/2}\big\} < \infty.$$

As before BMO $(\mathcal{F})$ can be identified with the dual of $H^1(\mathcal{F})$. We will also need the inequality $\|f\|_{\mathrm{BMO}(\mathcal{F})} \leqq 2\|f\|_{H^2(\mathcal{F})}$.

If $\mathcal{A}_m \subset \Omega$ then by $[(\mathcal{A}_m)_m]$ we denote the σ-field generated by $(\mathcal{A}_m)$, similarly for random variables X_m, $[(X_m)_m] = [\{\{X_m\in A\}: A \text{ is a Borel set}\}]$.

2.3. Diffusion in B

Let $\tilde{\Delta}$ denote the invariant Laplacian in B (cf. [13]) and $(X_t^z(\omega))$ the corresponding analogue of the Brownian motion — the diffusion in B with the infinitesimal generator $1/2\tilde{\Delta}$ (cf. [5], [11, p. 90]). It was proved in [5] that (X_t^z) has almost surely infinite life-time and for every $z\in B$ the process (X_t^z), starting at z, converges almost surely to a random variable X_∞^z taking values in S. Below we list the main properties of (X_t^z).

(A) X_t^z is Aut (B)-invariant, i.e. for every biholomorphic automorphism Φ of B, $(\Phi\circ X_t^z)$ is identical to $(X_t^{\Phi(z)})$. Because of this we may assume that $X_t^z = \Phi_z\circ X_t^0$ for some $\Phi_z\in \operatorname{Aut} B$ satisfying $\Phi_z(0)=z$. For convenience we will denote X_t^0 by X_t.

(B) If ζ is an X_t^z-stopping time then X_ζ^z will denote the variable $X_{\zeta(\omega)}^z(\omega)$. The process X_t^z has the so-called strong Markov property, i.e. conditional to $\zeta<\infty$ and $X_\zeta^z = w$, $X_{\zeta+t}^z$ is independent of the past and identical to X_t^w (cf. [11, pp. 90—91]). As a consequence we get, for example, that conditional to $(\zeta<\infty, X_\zeta^z = w)$ X_∞^z has the same distribution as X_∞^w.

(C) If $f\in C^2(B)$ and $\tilde{\Delta}f=0$ then for every finite X_t^z-stopping time ζ,

$$E\big(f(X_\zeta^z)\big) = f(z)$$

(cf. [7, 16, 17). In particular the above equality holds for all analytic functions in B.

(D) Trajectories $t\mapsto X_t^z(\omega)$ are almost surely continuous on $\mathbf{R}_+$.

As the first application of (A) we get

Lemma 1. *Let μ_z denote the distribution of X_∞^z on S. Then*

$$d\mu_z(w) = P(z, w)\, d\mu(w).$$

Proof. μ_0 is obviously rotation-invariant, hence $\mu_0=\sigma$. Since $X_t^z=\Phi_z \circ X_t$ and Φ_z is continuous in $\bar{B}$, we have $X_\infty^z=\Phi_z \circ X_\infty$. In particular for every $\varphi \in C(S)$

$$\int_S \varphi(w)\, d\mu_z(w) = \int_\Omega \varphi(X_\infty^z)\, dP = \int_\Omega \varphi(\Phi_z(X_\infty))\, dP$$

$$= \int_S \varphi(\Phi_z(w))\, d\mu_0(w) = \int_S \varphi(\Phi_z(w))\, d\sigma(w).$$

By formula (3.3.5) of [13] we get

$$\int_S \varphi(\Phi_z(w))\, d\sigma(w) = \int_S \varphi(w) P(\Phi_z(0), w)\, d\sigma(w) = \int_S \varphi(w) P(z, w)\, d\sigma(w)$$

which proves the claim.

As we observed above, the distribution μ_0 of X_∞ is equal to σ. If we assume that the process (X_t) is defined on a complete probability space then for every Lebesgue-measurable function g on S the composition $g \circ X_\infty$ is measurable and

$$E(g \circ X_\infty) = \int_S g\, d\sigma.$$

We will use this fact without further comment.

3. Probabilistic properties of analytic functions

In the sequel $(\Omega, \mathcal{M}, P)$ denotes the probability space corresponding to the process (X_t) defined in the previous section.

For $f \in H^p(B)$ define

$$f^*(\omega) = \sup_{t \geqq 0} |f(X_t(\omega))|.$$

Proposition 2. *To every $1 \leqq p < \infty$ corresponds a constant A_p such that for every $f \in H^p(B)$*

$$\|f\|_{H^p} \leqq \|f^*\|_{L^p(\Omega)} \leqq A_p \|f\|_{H^p}.$$

Proof. We begin with the right-hand inequality. Fix $\alpha > 2$. By (2) it is enough to show that

$$\|f^*\|_{L^p(\Omega)} \leqq A_{p,\alpha} \|M_\alpha f\|_{L^p(S)}. \tag{3}$$

This inequality is essentially due to Debiard but in his paper [4] it is formulated and proved only for the unbounded realization of the ball. Here we give a direct proof for the ball generalizing the one-dimensional case presented in [12]. Inequality (3) will immediately follow from:

(4) $$\forall_{\lambda\geqq 0}\forall_{f\in C(B)} P(f^* > \lambda) \leqq C_\alpha \sigma\{z\in S\colon (M_\alpha f)(z) > \lambda\}.$$

To prove (4) we define

$$U_\lambda = \{w\in S\colon (M_\alpha f)(w) > \lambda\};$$
$$V_\lambda = \bigcup_{w\in S\setminus U_\lambda} D_\alpha(w);$$
$$G_\lambda = B\setminus V_\lambda.$$

Notice that V_λ is open in B, U_λ is open in S, and $U_\lambda\cap\bar{V}_\lambda=\emptyset$. Since $\sigma(U_\lambda)=P(X_\infty\in U_\lambda)$ we have to prove that

(5) $$P(f^* > \lambda) \leqq C_\alpha P(X_\infty\in U_\lambda).$$

If $U_\lambda=S$ then (5) holds if $C_\alpha\geqq 1$ so we may assume that $U_\lambda\neq S$.

Obviously for $z\in V_\lambda$, $|f(z)|<\lambda$, hence if $f^*(\omega)>\lambda$ we must have $X_t(\omega)\in G_\lambda$ for some t. It follows that if ζ is the time of the first entrance to G_λ then

$$P(f^* > \lambda) \leqq P(\zeta < \infty).$$

Thus it is sufficient to prove the following:

If $U\subsetneqq S$ is open, $V=\bigcup_{w\in S\setminus U} D_\alpha(w)$, $G=B\setminus V$ and ζ is the time of the first entrance to G then

(6) $$P(\zeta < \infty) \leqq C_\alpha P(X_\infty\in U).$$

Since $U\cap\bar{V}=\emptyset$ and for all ω, $X_0(\omega)=0\in V$, if $X_\infty(\omega)\in U$ we must have $X_t(\omega)\in G$ for some t, or equivalently $\zeta(\omega)<\infty$. It follows that $P(X_\infty\in U|\zeta=\infty)=0$ and so

$$P(X_\infty\in U) = P(X_\infty\in U|\zeta < \infty)P(\zeta < \infty).$$

Hence it is enough to show that $P(X_\infty\in U|\zeta<\infty)\geqq k_\alpha>0$. But

$$P(X_\infty\in U|\zeta < \infty) = \int_B P(X_\infty\in U|\zeta < \infty, X_\zeta = z)\, dP(X_\zeta = z|\zeta < \infty)$$

(7) $$= \int_G P(X_\infty\in U|\zeta < \infty, X_\zeta = z)\, dP(X_\zeta = z|\zeta < \infty)$$

by (B) $$= \int_G P(X^z_\infty\in U)\, dP(X_\zeta = z|\zeta < \infty).$$

We will now find a lower bound for $P(X^z_\infty\in U)$ when $z\in G$. By Lemma 1,

$$P(X^z_\infty\in U) = \int_U P(z, w)\, d\sigma(w).$$

For $z\in B$ let

$$E_\alpha(z) = \left\{w\in S\colon |1-\langle z, w\rangle| < \frac{\alpha}{2}(1-|z|^2)\right\} = \{w\in S\colon z\in D_\alpha(w)\}.$$

It is easily seen that for $z\in G$, $E_\alpha(z)\subset U$. In addition, if $w\in E_\alpha(z)$ then

$$(8) \qquad P(z, w) = \frac{(1-|z|^2)^n}{|1-\langle z, w\rangle|^{2n}} > (2/\alpha)^{2n}\frac{1}{(1-|z|^2)^n}.$$

Now for $z\in B$ let $w=z/|z|$. Then $d(z, w)=(1-|z|)^{1/2}<(1-|z|^2)^{1/2}$. Hence if $w'\in S$ is such that $d(w, w')<\big((\sqrt{\alpha/2})-1\big)(1-|z|^2)^{1/2}$ then $d(z, w')\leqq d(z, w)+d(w, w')<\big((\alpha/2)(1-|z|^2)\big)^{1/2}$. This proves that

$$(9) \qquad Q\left(\frac{z}{|z|}, \left(\sqrt{\frac{\alpha}{2}}-1\right)(1-|z|^2)^{1/2}\right)\subset E_\alpha(z).$$

Hence if $z\in G$ we get

$$(10) \qquad \begin{aligned} P(X_\infty^z\in U) &= \int_U P(z, w)\, d\sigma(w) \geqq \int_{E_\alpha(z)} P(z, w)\, d\sigma(w) \\ \text{by (8)} \qquad &\geqq \left(\frac{2}{\alpha}\right)^{2n}\frac{1}{(1-|z|^2)^n}\,\sigma\big(E_\alpha(z)\big). \end{aligned}$$

By [13, Proposition 5.1.4], $\sigma\big(Q(w, r)\big)\geqq 2^{-n}r^{2n}$ so (9, 10) imply

$$P(X_\infty^z\in U) \geqq \left(\frac{2}{\alpha}\right)^{2n}\frac{1}{(1-|z|^2)^n}\,2^{-n}\,\frac{(\sqrt{\alpha}-\sqrt{2})^{2n}}{2^n}(1-|z|^2)^n = \left(\frac{\sqrt{\alpha}-\sqrt{2}}{\alpha}\right)^{2n} = k_\alpha > 0.$$

Putting this into (7) we obtain

$$P(X_\infty\in U|\zeta<\infty) \geqq k_\alpha\int_G dP(X_\zeta = z|\zeta<\infty) = k_\alpha P(X_\zeta\in G|\zeta<\infty) = k_\alpha$$

which proves (6), hence also (4).

Now the left-hand inequality follows easily. If $f\in A(B)$ then $f(X_t)\to\tilde f(X_\infty)$ a.s. hence $|\tilde f(X_\infty)|\leqq f^*$ a.s. and

$$\|f\|_{H^p} = \|\tilde f\|_{L^p(S)} = \|\tilde f(X_\infty)\|_{L^p(\Omega)} \leqq \|f^*\|_{L^p(\Omega)},$$

which proves the inequality for $f\in A(B)$, and by density it extends to the whole $H^p(B)$.

Lemma 3. *Let* $\mathscr{F}_t^z=[X_s^z, s\leqq t]$. *If* $f\in H^1(B)$ *then* $\big(f(X_t^z)\big)_{t\geqq 0}$ *forms a uniformly integrable* $(\mathscr{F}_t^z)$*-martingale and as* $t\to\infty$, $f(X_t^z)$ *converges to* $\tilde f(X_\infty^z)$ *almost surely and in* L^1.

Proof. The martingale property of $\big(f(X_t^z)\big)$ is well known and easily follows from (B) and (C) (see also [17, III.22.6]).

Next observe that for $f\in H^1(B)$ and $\Phi\in\operatorname{Aut}(B)$, $f\circ\Phi\in H^1(B)$ and $(f\circ\Phi)^\sim = \tilde f\circ\Phi$. This and (A) allows us to consider only $z=0$. Uniform integrability follows then from the L^1-boundedness of the maximal function f^* (Proposition 2). Now by the martingale convergence theorem, $(f(X_t))$ converges to a random variable F and $|F|\leqq f^*$. Hence, by Proposition 2, we have

$$\|F\|_{L^1(\Omega)} \leqq A_1 \|f\|_{H^1}. \tag{11}$$

If $f\in A(B)$ then $F=\tilde{f}\circ X_\infty$. For a general $f\in H^1(B)$ let $f_m\in A(B)$, $f_m\to f$ in $H^1(B)$. Putting f_m-f into (11) we get that $\tilde{f}_m(X_\infty)-F\to 0$ in $L^1(\mathbf{R})$, but we also have

$$\|f_m-f\|_{H^1} = \|\tilde{f}_m-\tilde{f}\|_{L^1(S)} = \|\tilde{f}_m\circ X_\infty-\tilde{f}\circ X_\infty\|_{L^1},$$

hence $\tilde{f}_m(X_\infty)-f(X_\infty)\to 0$ and so $F=f(X_\infty)$ a.s.

Proposition 4. *There is a constant C such that for every $f\in$ BMOA and every finite (X_t)-stopping time ζ*

$$\left\|E(|f(X_\infty)-f(X_\zeta)|^2|X_\zeta)\right\|^{1/2}_{L^\infty(\Omega)} \leqq C\|f\|_{\mathrm{BMO}}.$$

Proof. Let ν_ζ be the distribution of X_ζ. Using (B) and Lemma 1 we get

$$\left\|E(|\tilde{f}(X_\infty)-f(X_\zeta)|^2|X_\zeta)\right\|_{L^\infty(\Omega)} = \left\|E(|\tilde{f}(X_\infty)-f(X_\zeta)|^2|X_\zeta = z)\right\|_{L^\infty(\nu_\zeta)}$$
$$= \left\|E(|f(X^z_\infty)-f(z)|^2)\right\|_{L^\infty(\nu_\zeta)} \leqq \sup_{z\in B}\left\{\int_S |f(w)-f(z)|^2\, d\mu_z(w)\right\}$$
$$= \sup_{z\in B}\left\{\int_S |f(w)-f(z)|^2 P(z,w)\, d\sigma(w)\right\} \leqq C^2\|f\|^2_{\mathrm{BMO}}.$$

The last inequality is a consequence of the well known equivalence of the BMO-norm and the Garsia norm (cf. [14, Proposition 2.7]).

4. Main result

In this section we prove:

Theorem 5. *$H^1(B)$ is isomorphic to $H^1(D)$.*

By [1, 18] $H^1(D)$ is isomorphic to a complemented subspace of $H^1(B)$. We will prove that also $H^1(B)$ is isomorphic to a complemented subspace of $H^1(D)$. This, as shown in [18, Section 1], is sufficient to prove our Theorem 5.

We will use the following result of Maurey:

Proposition 6 ([10, Proposition 4.15]). *For every increasing sequence of finite σ-fields $(\mathcal{F}_m)$, $H^1(\mathcal{F})$ is isomorphic to a complemented subspace of $H^1(D)$.*

Thus the proof of Theorem 5 can be reduced to the next proposition.

Proposition 7. *There is an increasing sequence $(\mathcal{F}_m)$ of finite sub-σ-fields of $\mathcal{M}$ such that $H^1(B)$ is isomorphic to a complemented subspace of $H^1(\mathcal{F})$.*

We will closely follow the proof of Proposition 3.7 in [10]. We give first the construction we will need and the rest of the proof will be divided into the following sequence of lemmas.

Fix $\varepsilon < 1/2$. It easily follows from (1) and obvious properties of the kernel $P(z, w)$ that for every $z \in B$ we can find an open ball $B_z = B(z, r_z)$ such that

(i) $\bar{B}_z \subset B$.

(ii) If $x, y \in B_z$ and $f \in H^1(B)$ then

$$|f(x) - f(y)| \leqq \varepsilon(1-|z|)\|f\|_{H^1}.$$

(iii) If $\varrho < 1$ then $\inf_{|z|<\varrho} r_z > 0$.

By $1/2B_z$ we will denote the ball $B(z, r_z/2)$.

As in [10], in addition to $(\mathscr{F}_m)$, we also construct a sequence ζ_m of (X_t)-stopping times and a sequence (Z_m) of $\mathscr{F}_m$-measurable random variables approximating the variables X_{ζ_m}.

We begin with

$$\mathscr{F}_0 = \{\emptyset, \Omega\}, \quad \zeta_0 = 0, \quad Z_0 = 0.$$

Next we define $\zeta_1 = \zeta_{B_0}$ — the time of the first exit from the ball B_0.

Let $z_1, \ldots, z_k \in \partial B_0$, $A_1, \ldots, A_k \subset \partial B_0$ be such that A_i are Borel, disjoint subsets of ∂B_0, $\bigcup_{i=1}^k A_i = \partial B_0$ and $A_i \subset 1/2B_{z_i}$. Define

$$\mathscr{A}_i = \{X_{\zeta_i} \in A_i\},$$

$$\mathscr{F}_1 = [(\mathscr{A}_i)_1^k],$$

$$Z_1 = \sum_{i=1}^k z_i \chi_{\mathscr{A}_i}.$$

In the next step ζ_2 is defined as follows: if $\omega \in A_i$ then $X_{\zeta_1}(\omega) \in A_i \subset B_{z_i}$ and

$$\zeta_2(\omega) \overset{\text{def}}{=} \inf\{t \geqq \zeta_1(\omega) \colon X_t(\omega) \notin B_{z_i}\}.$$

$\mathscr{F}_2$ and Z_2 are then constructed with the same procedure which was used for $\mathscr{F}_1$ and Z_1: for each i we find $z_{i,j} \in \partial B_{z_i}$, $A_{i,j} \subset \partial B_{z_i}$ such that $(A_{i,j})_j$ are Borel, disjoint, cover ∂B_{z_i} and $A_{i,j} \subset 1/2B_{z_{i,j}}$. Next we define

$$\mathscr{A}_{i,j} = \{\omega \in \mathscr{A}_i \colon X_{\zeta_2}(\omega) \in A_{i,j}\},$$

$$\mathscr{F}_2 = [(\mathscr{A}_{i,j})_{i,j}],$$

$$Z_2 = \sum_{i,j} z_{i,j} \chi_{\mathscr{A}_{i,j}},$$

and we continue in the obvious way.

Lemma 8. (a) *If* $\zeta_m(\omega) \leqq t \leqq \zeta_{m+1}(\omega)$ *then* $X_t(\omega) \in B_{Z_m(\omega)}$, *in particular for every* $f \in H^1(B)$

$$|f(X_{\zeta_m}(\omega)) - f(X_t(\omega))| < \varepsilon \|f\|_{H^1}, \tag{12}$$

(b) $\zeta_m \to \infty$ *as* $m \to \infty$,

(c) $|Z_m| \to 1$ *a.s. as* $m \to \infty$,

(d) $\mathscr{F}_m \subset \mathscr{N}_m \overset{\text{def}}{=} [X_{\zeta_k}\colon k \leqq m]$.

Proof. (a) and (d) are obvious by construction. To prove (b) and (c) let $\omega \in \Omega$. Since $\zeta_m(\omega)$ increase, $\zeta_m(\omega) \to t_0 \in [0, \infty]$. Then, by continuity of $X_t(\omega)$ on $[0, \infty]$, $X_{\zeta_m}(\omega) \to X_{t_0}(\omega)$. Since $X_{\zeta_m} \in 1/2B_{Z_m}$ and $X_{\zeta_{m+1}} \in \partial B_{Z_m}$ we get

$$|Z_m(\omega) - X_{\zeta_m}(\omega)| \leqq \tfrac{1}{2} r_{Z_m(\omega)} \leqq |X_{\zeta_m}(\omega) - X_{\zeta_{m+1}}(\omega)|. \tag{13}$$

Hence $r_{Z_m(\omega)} \to 0$ and thus, by (iii), $|Z_m(\omega)| \to 1$. But, by (13), we also have $Z_m(\omega) \to X_{t_0}(\omega)$ so $|X_{t_0}(\omega)| = 1$ which is only possible if $t_0 = \infty$.

Now for $f \in H^1(B)$ let

$$f_m = E(\tilde{f} \circ X_\infty | \mathscr{F}_m).$$

Lemma 9. (a) f_m *converge to* $\tilde{f} \circ X_\infty$ *a.s.*

(b) *For every* m

$$\|f_m - f \circ X_{\zeta_m}\|_{L^\infty(\Omega)} \leqq \varepsilon \|f\|_{H^1}. \tag{14}$$

Proof. Since $\mathscr{F}_m \subset \mathscr{N}_m$, we have $f_m = E\big(E(\tilde{f}(X_\infty)|\mathscr{N}_m)|\mathscr{F}_m\big)$. By the optional stopping theorem ([17, Theorem II.53.1], [6, Theorem 10, Ch. VI]) $E(\tilde{f}(X_\infty)|\mathscr{N}_m) = f(X_{\zeta_m})$ so $f_m = E(f(X_{\zeta_m})|\mathscr{F}_m)$. Let $\mathscr{A}$ be an atom in $\mathscr{F}_m$ and $\omega \in \mathscr{A}$ be fixed. Then it follows that

$$f_m(\omega) = \frac{1}{P(\mathscr{A})} \int_{\mathscr{A}} f(X_{\zeta_m})\, dP$$

(we disregard atoms of probability zero). In particular $|f_m(\omega) - f(X_{\zeta_m}(\omega))|$ is bounded by the oscillation of $f \circ X_{\zeta_m}$ on $\mathscr{A}$. But, by Lemma 8 (a), $X_{\zeta_m}(\mathscr{A}) \subset B_{Z_m(\omega)}$ and, by (ii), the oscillation of f on $B_{Z_m(\omega)}$ is bounded by $\varepsilon(1 - |Z_m(\omega)|)\|f\|_{H^1}$. Hence

$$|f_m(\omega) - f(X_{\zeta_m}(\omega))| \leqq \varepsilon(1 - |Z_m(\omega)|)\|f\|_{H^1}, \tag{15}$$

and as $|Z_m| \to 1$ and $f(X_{\zeta_m}) \to \tilde{f}(X_\infty)$, (15) implies both (14) and the fact that f_m converge to $\tilde{f}(X_\infty)$.

Now we define

$$Tf = \tilde{f} \circ X_\infty.$$

Lemma 10. (a) *For every* $p \geqq 1$, T *is an isomorphic embedding of* $H^p(B)$ *into* $H^p(\mathscr{F})$.

(b) T *is bounded from* BMOA *to* BMO $(\mathscr{F})$.

Proof. We begin with (a). Since $\zeta_m \to \infty$ we have

$$f^* = \sup_{t \geqq 0} |f(X_t)| = \sup_m \sup_{\zeta_m \leqq t \leqq \zeta_{m+1}} |f(X_t)|.$$

By Lemma 8 (a), for $\zeta_m \leqq t \leqq \zeta_{m+1}$, $|f(X_t) - f(X_{\zeta_m})| \leqq \varepsilon \|f\|_{H^1}$, hence

$$\Big\| \sup_m |f(X_{\zeta_m})| - f^* \Big\|_{L^\infty(\Omega)} \leqq \varepsilon \|f\|_{H^1}.$$

Comparing this with (14) we get

$$\Big\| \sup_m |f_m| - f^* \Big\|_{L^\infty(\Omega)} \leqq 2\varepsilon \|f\|_{H^1},$$

so

$$\Big| \Big\| \sup_m |f_m| \Big\|_{L^p(\Omega)} - \|f^*\|_{L^p(\Omega)} \Big| \leqq 2\varepsilon \|f\|_{H^1}, \tag{16}$$

and since $\|\sup_m |f_m|\|_{L^p(\Omega)} = \|Tf\|_{H^p(\mathscr{F})}$ we get, by Proposition 2 and (16),

$$(1-2\varepsilon)\|f\|_{H^p} \leqq \|f^*\|_{L^p(\Omega)} - 2\varepsilon \|f\|_{H^1} \leqq \|Tf\|_{H^p(\mathscr{F})} \leqq 2\varepsilon \|f\|_{H^1} + \|f^*\|_{L^p(\Omega)}$$
$$\leqq 2\varepsilon \|f\|_{H^p} + \|f^*\|_{L^p(\Omega)} \leqq (1+2\varepsilon) A_p \|f\|_{H^p}.$$

To prove (b) we notice that

$$\big(E(|\tilde{f}(X_\infty) - f_{m-1}|^2 | \mathscr{F}_m)\big)^{1/2} \leqq \big(E(|\tilde{f}(X_\infty) - f(X_{\zeta_m})|^2 | \mathscr{F}_m)\big)^{1/2} + \big(E(|f(X_{\zeta_m}) - f_{m-1}|^2 | \mathscr{F}_m)\big)^{1/2}.$$

Since $\mathscr{F}_m \subset \mathscr{N}_m$, we have

$$\Big\| E\big(|\tilde{f}(X_\infty) - f(X_{\zeta_m})|^2 | \mathscr{F}_m\big) \Big\|_{L^\infty(\Omega)} \leqq \Big\| E\big(|\tilde{f}(X_\infty) - f(X_{\zeta_m})|^2 | \mathscr{N}_m\big) \Big\|_{L^\infty(\Omega)}$$

by (B) $= \Big\| E\big(|\tilde{f}(X_\infty) - f(X_{\zeta_m})|^2 | X_{\zeta_m}\big) \Big\|_{L^\infty(\Omega)}$ by Proposition 4 $\leqq C^2 \|f\|^2_{\mathrm{BMO}}$.

By (12), we get

$$\|f(X_{\zeta_m}) - f(X_{\zeta_{m-1}})\|_{L^\infty(\Omega)} \leqq \varepsilon \|f\|_{H^1},$$

and this, together with (14), gives

$$\|f(X_{\zeta_m}) - f_{m-1}\|_{L^\infty(\Omega)} \leqq 2\varepsilon \|f\|_{H^1} \leqq 4\varepsilon \|f\|_{\mathrm{BMO}}$$

so

$$\Big\| E\big(|f(X_{\zeta_m}) - f_{m-1}|^2 | \mathscr{F}_m\big) \Big\|_{L^\infty(\Omega)} \leqq 16\varepsilon^2 \|f\|^2_{\mathrm{BMO}}.$$

Remark. Using duality and a slightly more careful calculation in the proof of Proposition 4, one can prove that T is also bounded below on BMOA, however this is not necessary for the proof of Theorem 5.

Now we can finish the proof of Proposition 7 exactly the same way as in [10]. It is well known (cf. [9]) that $H^2(\mathscr{F}) = L^2(\Omega, \mathscr{F}_\infty, P)$, because of this and the preceding lemma, $T(H^2(B))$ is a closed subspace of $L^2(\Omega, \mathscr{F}_\infty, P)$. Let Q be the orthogonal projection from $L^2(\Omega, \mathcal{F}_\infty, P)$ onto $T(H^2(B))$. We intend to show that Q is bounded in $H^1(\mathcal{F})$-norm, this will mean that Q can be extended to a bounded projection from the $H^1(\mathcal{F})$-closure of $L^2(\Omega, \mathcal{F}_\infty, P)$ onto the closure of $T(H^2(B))$, but these closures are respectively $H^1(\mathcal{F})$ and $T(H^1(B))$ so this extension will give the desired projection.

Below we will denote various constants by C. They may change from line to line but do not depend on particular functions.

Let $F\in L^2(\Omega, \mathcal{F}_\infty, P)$. Then $QF\in T(H^2(B))$ so $QF=Th=\tilde{h}\circ X_\infty$ for some $h\in H^2(B)$. Hence

$$\|QF\|_{H^1(\mathcal{F})} = \|Th\|_{H^1(\mathcal{F})} \leqq C\|h\|_{H^1} \leqq C\sup\left\{\left|\int_S \tilde{h}\bar{\tilde{g}}\, d\sigma\right|: g\in \mathrm{BMOA},\ \|g\|_{\mathrm{BMO}} \leqq 1\right\}$$

$$= C\sup\left\{\left|E\left(\tilde{h}(X_\infty)\overline{\tilde{g}(X_\infty)}\right)\right|: g\in \mathrm{BMOA},\ \|g\|_{\mathrm{BMO}} \leqq 1\right\}$$

$$= C\sup\left\{\left|E\left((QF)\overline{\tilde{g}(X_\infty)}\right)\right|: g\in \mathrm{BMOA},\ \|g\|_{\mathrm{BMO}} \leqq 1\right\}.$$

If $g\in\mathrm{BMOA}$ then $g\in H^2(B)$ so $\tilde{g}(X_\infty)\in T(H^2(B))$, and as Q is an orthogonal projection we get

$$E\left((QF)\overline{\tilde{g}(X_\infty)}\right) = E\left(F\overline{\tilde{g}(X_\infty)}\right).$$

We also have, by Lemma 10 (b),

$$\left\{\overline{\tilde{g}(X_\infty)}: g\in \mathrm{BMOA},\ \|g\|_{\mathrm{BMO}} \leqq 1\right\}\subset\left\{G\in \mathrm{BMO}(\mathcal{F}): \|G\|_{\mathrm{BMO}(\mathcal{F})} \leqq C\right\},$$

so

$$\sup\left\{\left|E\left((QF)\overline{\tilde{g}(X_\infty)}\right)\right|: g\in \mathrm{BMOA},\ \|g\|_{\mathrm{BMO}} \leqq 1\right\}$$

$$\leqq \sup\left\{|E(FG)|: G\in \mathrm{BMO}(\mathcal{F}),\ \|G\|_{\mathrm{BMO}(\mathcal{F})} \leqq C\right\} \leqq C\|F\|_{H^1(\mathcal{F})},$$

and it follows that

$$\|QF\|_{H^1(\mathcal{F})} \leqq C\|F\|_{H^1(\mathcal{F})},$$

which ends the proof.

5. Nonisotropic distances and atomic H^1

Following [18] we define a nonisotropic distance Δ on $\mathbf{R}^m$,

$$\Delta(x, y) = \max\left\{|x_1-y_1|, \ldots, |x_{m-1}-y_{m-1}|, |x_m-y_m|^{1/2}\right\}.$$

We will show that in spite of a certain local similarity between (S, d) and $(\mathbf{R}^{2n-1}, \Delta)$, they are not locally equivalent as metric spaces. For this purpose we examine Lipschitz functions from (S, d) to $(\mathbf{R}, |\cdot|^\alpha)$, $\alpha<1$. When $\alpha=1$ there are many examples of Lipschitz functions, this is no longer true for $\alpha<1$.

Theorem 11. *Let U be an open and connected subset of S and let $0<\alpha<1$. Suppose $\Psi\colon U\to\mathbf{R}$ satisfies*

$$|\Psi(z)-\Psi(w)|^{\alpha}\leqq Kd(z,w)$$

for all $z, w\in U$. Then Ψ is constant.

To begin with we introduce some notation. At each point $z\in S$ the tangent space T_zS to S contains the vector iz. The complex tangent space $T_z^{\mathbf{C}}S$ is defined as the orthogonal complement to iz in T_zS and consists of all vectors $\Theta\in\mathbf{C}^n$ for which $\langle z,\Theta\rangle=0$. The module of all C^∞ vector fields on U will be denoted by $\Gamma(TU)$ and the submodule consisting of these with values in $T^{\mathbf{C}}S$, by $\Gamma(T^{\mathbf{C}}U)$. For $X, Y\in\Gamma(TU)$, $[X, Y]$ denotes their (real) Lie bracket. If γ is a curve in S then $\dot\gamma$ will denote its tangent vector.

For convenience we will assume that $U\subset\{|z_n|<1\}$ and $K=1$.

Lemma 12. *Let γ be a C^2 curve in U such that $\dot\gamma(t)\in T^{\mathbf{C}}U$ for all t. Then $\Psi\circ\gamma$ is constant.*

Proof. Since $\langle\dot\gamma(t),\gamma(t)\rangle=0$ we get

$$d(\gamma(t+s),\gamma(t))^2=|\langle\gamma(t+s)-\gamma(t),\gamma(t)\rangle|=|\langle\gamma(t+s)-\gamma(t)-s\dot\gamma(t),\gamma(t)\rangle+s\langle\dot\gamma(t),\gamma(t)\rangle|$$
$$\leqq|\gamma(t+s)-\gamma(t)-s\dot\gamma(t)|=O(s^2).$$

Hence

$$\left|\Psi(\gamma(t+s))-\Psi(\gamma(t))\right|\leqq d(\gamma(t+s),\gamma(t))^{1/\alpha}=O(s^{1/\alpha})$$

so $\frac{d}{ds}(\Psi\circ\gamma)(s)|_{s=t}=0$ and we are done.

Let $X, Y\in\Gamma(TU)$ and let $\varphi(z,t)$, $\psi(z,t)$ be their integral curves. Fix $z_0\in U$ and define

$$\gamma(t)=\psi(\varphi(\psi(\varphi(z_0,\sqrt{t}),\sqrt{t}),-\sqrt{t}),-\sqrt{t}).$$

Then γ is a smooth curve and, by [15, Theorem 6, Ch. 5]

$$\dot\gamma(0)=[X,Y](z_0).$$

If $X, Y\in\Gamma(T^{\mathbf{C}}U)$ then, by Lemma 12, Ψ is constant along their integral curves and it follows that $\Psi\circ\gamma$ is also constant so we get

Corollary 13. *If $X, Y\in\Gamma(T^{\mathbf{C}}U)$ then for each $z_0\in U$ there is a C^2 curve γ such that $\gamma(0)=z_0$, $\dot\gamma(0)=[X, Y](z_0)$ and $\Psi\circ\gamma$ is constant.*

Lemma 14. *Let $z_0\in U$ and $\Theta\in T_{z_0}S$. Then there are $X, Y\in\Gamma(T^{\mathbf{C}}U)$ such that $[X, Y](z_0)=\Theta$.*

The proof easily follows from the observation that if I, E_n are vector fields defined on $\mathbf{C}^n$ by

$$I(z) = z,$$
$$E_n(z) = e_n = (0, \ldots, 1),$$

and $X_0=E_n-\langle E_n, I\rangle I$, $Y_0=iX_0$ then, restricted to S, X_0, $Y_0\in\Gamma(T^{\mathbf{C}}U)$ and $[X_0, Y_0]$ does not vanish on U and is orthogonal to $T^{\mathbf{C}}U$.

Now we can finish the proof of Theorem 11. Let η be a C^2 curve in U. We will show that Ψ is constant along η.

Fix t. Then, by Lemma 14, there are $X, Y\in\Gamma(T^{\mathbf{C}}U)$ such that

$$[X, Y](\eta(t)) = \dot{\eta}(t).$$

By Corollary 13, there is a C^2 curve γ such that $\gamma(0)=\eta(t)$, $\dot{\gamma}(0)=\dot{\eta}(t)$ and $\Psi\circ\gamma$ is constant. The first two conditions imply that $\eta(t+s)-\gamma(s)=O(s^2)$, and it follows that $\Psi(\eta(t+s))-\Psi(\gamma(s))=O(s^{1/\alpha})$. Since $\Psi(\eta(t)=\Psi(\gamma(0))=\Psi(\gamma(s))$, we get $\Psi(\eta(t+s))-\Psi(\eta(t))=O(s^{1/\alpha})$, so $d/ds(\Psi\circ\eta)(s)|_{s=t}=0$, and as t was arbitrary, $\Psi\circ\eta$ is constant.

Corollary 15. *If U is an open and connected subset of S and*

$$\Psi = (\Psi_1, \ldots, \Psi_{2n-1})\colon (U, d) \to (\mathbf{R}^{2n-1}, \Delta)$$

is a Lipschitz map, then Ψ_{2n-1} is constant.

Let λ be the Lebesgue measure on $\mathbf{R}^m$. Then $([0, 1]^m, \Delta, \lambda)$ and (S, d, σ) are examples of spaces of homogeneous type in the sense of [3], and one can define atomic H^1-spaces $H^1_{\mathrm{at}}(S, d)$, $H^1_{\mathrm{at}}([0, 1]^m, \Delta)$. Corollary 15 shows that a direct comparison of these spaces, as in [18], is not possible. However, it is still true that they are isomorphic, in particular Proposition B of [18] is true.

Theorem 16. *$H^1_{\mathrm{at}}([0, 1]^m, \Delta)$ and $H^1_{\mathrm{at}}(S, d)$ are isomorphic to $H^1(D)$.*

The first part is due to Wojtaszczyk [18], the second part generalizes Theorem 5, since $H^1(B)$ is complemented in $H^1_{\mathrm{at}}(S, d)$ ([2]). The proof is exactly the same as that of Theorem 5, once one establishes Proposition 2 for $H^1_{\mathrm{at}}(S, d)$. In this case the left-hand inequality in Proposition 2 is much harder as the $L^1(S, \sigma)$-norm is not equivalent to the norm of $H^1_{\mathrm{at}}(S, d)$. Instead one has to consider the L^1-norm of the maximal function $M_\alpha f$ ([8]) and the proof of Proposition 2 in this case needs the inequality

$$\|M_\alpha f\|_{L^p(S)} \leqq A_{p,\alpha}\|f^*\|_{L^p(\Omega)}.$$

The corresponding inequality for the "half-plane" realization of the unit ball appears in [4] and the proof can be adapted to the case of the ball itself.

Remark. After the preprint of this paper had been distributed I learned from Steven Krantz that he had been aware of the nonequivalence of $(\mathbf{R}^{2n-1}, \Delta)$ and (S, d), but I do not know of any published proof of this fact.

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