Regularity of averages over hypersurfaces

Lennart Börjeson

Abstract. Averages over smooth measures on smooth compact hypersurfaces in $\mathbf{R}^n$ are studied. With assumptions on the decay of the Fourier transform of the measure we obtain mixed norm estimates for these means, for example L^p estimates of multiparameter maximal functions over compact hypersurfaces.

1. Introduction

Let S be a smooth compact hypersurface (possibly with boundary) in $\mathbf{R}^n$, $d\sigma$ the induced Lebesgue measure on S and μ a smooth mass distribution which is vanishing near the boundary. Set $\psi(t, y)=(\psi_1(t)y_1, \ldots, \psi_n(t)y_n)$, where $\psi_i \in C_0^\infty(\mathbf{R}^m)$, $i=1, \ldots, n$, and $y\in\mathbf{R}^n$. We define the average

$$F_x(t) = \int_S f(x-\psi(t, y))\,\mu(y)\,d\sigma(y)$$

for $f\in L^1_{\text{loc}}(\mathbf{R}^n)$. It follows from Fubini's theorem that for every $t\in\mathbf{R}^m$ $F_x(t)$ is well-defined for almost all x. If $S=S^{n-1}$, the unit sphere in $\mathbf{R}^n$, $m=1$, $\psi(t, y)=ty$ and $\mu\equiv 1$, then $F_x(t)$ becomes the ordinary spherical mean. In this case J. Bourgain [B1]—[B3], $n=2$, and E. M. Stein [St2], [SWa], $n\geqq 3$, showed that the corresponding maximal function $\sup_{t>0}|F_x(t)|$ is bounded on $L^p(\mathbf{R}^n)$ if $p>\dfrac{n}{n-1}$. For $n\geqq 3$, this was extended to more general hypersurfaces by M. Cowling and G. Mauceri [CM], A. Greenleaf [G] and J. L. Rubio de Francia [R]. Greenleaf assumes that S has a fixed number of principal curvatures different from zero while the other authors have (weaker) assumptions on the decay of the Fourier transform of the measure $\mu\,d\sigma$. The lower limit of p then depends on these assumptions. In [G] and [R] it is also shown that without loss of generality one can replace $\psi(t, y)=ty$ by the non-isotropic dilation $\psi(t, y)=(t^{\lambda_1}y_1, \ldots, t^{\lambda_n}y_n)$, $\lambda_i>0$, $i=1, \ldots, n$. The results in this note are closely related to these. C. D. Sogge [So] considers hypersurfaces which depend on both x and t, but have nowhere vanishing Gaussian curvature, and

L. Colzani [C] considers mixed norms of spherical means on compact symmetric spaces. See also [SS1] and [SS2]. In the main part of this note we study mixed norm estimates of the type considered by P. Sjölin and others in [Bö], [OS, Appendix], [PS], [S1]—[S5], where they obtain L^p estimates of L^q, BMO and Besov—Lipschitz norms in the parameter t of the spherical mean. Our results extend these to more general hypersurfaces and to averages depending on a multiparameter t. As a consequence we get L^p estimates of multiparameter maximal functions. In the last section we give further results and extensions of the main results in the third section.

2. Preliminaries

The mixed norms that we are going to use involve various function spaces. $B^\alpha_{pq}(\mathbf{R}^m)$ is the Besov space of tempered distributions with norm

$$\|\varphi\|_{B^\alpha_{pq}} = \|g*\varphi\|_p+\Big(\sum_{l=1}^\infty (2^{\alpha l}\|g_l*\varphi\|_p)^q\Big)^{1/q}, \quad 1 \leqq p,\, q \leqq \infty,\ \alpha\in\mathbf{R}.$$

Here $\{\hat{g}_l\}_{-\infty}^\infty$ is a dyadic partition of unity on $\mathbf{R}^m\setminus\{0\}$ and $\hat{g}=1-\sum_{l=1}^\infty \hat{g}_l$.

$H^\beta_p(\mathbf{R}^m)$ is the generalized Sobolev space of tempered distributions normed by

$$\|\varphi\|_{H^\beta_p} = \big\|((1+|\cdot|^2)^{\beta/2}\hat{\varphi})^\vee\big\|_p, \quad 1 \leqq p \leqq \infty,\ \beta\in\mathbf{R}.$$

More details of $B^\alpha_{pq}(\mathbf{R}^m)$ and $H^\beta_p(\mathbf{R}^m)$ are to be found in [BL], e.g. $B^\beta_{22}=H^\beta_p$.

BMO $(\mathbf{R}^m)$ is the space of functions of bounded mean oscillation normed by

$$\|\varphi\|_{\mathrm{BMO}} = \sup_Q \Big[|Q|^{-1}\int_Q \Big|\varphi(t)-|Q|^{-1}\int_Q \varphi(s)\,ds\Big|\,dt\Big],$$

where Q is any cube in $\mathbf{R}^m$. Cf. [St1, p. 164].

$\Lambda_\delta(\mathbf{R}^m)$, $\delta>0$, is the Lipschitz space with norm

$$\|\varphi\|_{\Lambda_\delta} = \|\varphi\|_\infty+\sup_{t,y} y^{k-\delta}\left|\frac{\partial^k u}{\partial y^k}(t, y)\right|$$

where $u(t, y)$, $t\in\mathbf{R}^m$, $y>0$, is the Poisson integral of φ and k is the smallest integer greater than δ. See [St1, Ch. V, § 4].

Throughout this paper we take the dimension n to be $\geqq 2$. Finally, we shall stick to the convention that C denote a constant which is not necessarily the same at each occurrence.

3. Main results

Let the Fourier transform of the measure defining the averages satisfy

$$|\widehat{\mu\, d\sigma}(\xi)| = \left|\int_S e^{-ix\cdot\xi}\mu(x)\,d\sigma(x)\right| \leqq C(1+|\xi|)^{-a},$$

for $a\geqq 0$. We note that if S has k principal curvatures different from zero and if μ vanishes near the boundary of S, then a result of W. Littman [L] shows that $a=\frac{k}{2}$ will suffice.

Theorem 1. *Let* $f\in L^p(\mathbf{R}^n)$ *and* $\frac{1}{p}+\frac{1}{p'}=1$. *Assume that*

a) $\psi_i\in C_0^\infty(\mathbf{R}^m)$, $i=1, \ldots, n$, *and that* $\varphi\in C_0^\infty(\mathbf{R}^m)$, *where*

$$\operatorname{supp}\varphi \subset \textstyle\bigcap_{i=1}^n \{t;\ \psi_i(t)\neq 0\}.$$

If $1\leqq p\leqq 2$ *and* $\alpha=\dfrac{2a}{p'}$, *then*

$$\Big(\int_{\mathbf{R}^n} \|\varphi F_x\|^p_{B^\alpha_{pp'}}\,dx\Big)^{1/p} \leqq C\,\|f\|_p. \tag{1}$$

If $2\leqq p\leqq \infty$ *and* $\alpha=\dfrac{2a}{p}$, *then*

$$\Big(\int_{\mathbf{R}^n} \|\varphi F_x\|^p_{B^\alpha_{pp}}\,dx\Big)^{1/p} \leqq C\,\|f\|_p. \tag{2}$$

Assume that

b) *S lies in the boundary of a set which is star-shaped with respect to the origin and that S does not possess a tangent plane containing the origin,* $\psi_i(t)=(\max(t, 0))^{\lambda_i}$, $\lambda_i>0$, $i=1, \ldots, n$, *and that* $\varphi\in C_0^\infty(\mathbf{R})$ *with* $\operatorname{supp}\varphi\subset(0, \infty)$.

If $1\leqq p\leqq 2$, $\alpha=\dfrac{2a}{p'}$ *and* $p\leqq r\leqq p'$, *then*

$$\Big(\int_{\mathbf{R}^n} \|\varphi F_x\|^r_{B^\alpha_{pp'}}\,dx\Big)^{1/r} \leqq C\,\|f\|_p. \tag{3}$$

By various continuous embeddings of B^α_{pq} in larger spaces we obtain a corollary.

Corollary. *Let* ψ_i *and* φ *satisfy* a) *and take* $f\in L^p(\mathbf{R}^n)$.

If $1\leqq p=r<1+\dfrac{m}{2a}\leqq 2$ *and* $\dfrac{1}{q}=\dfrac{1}{p}\Big(1+\dfrac{2a}{m}\Big)-\dfrac{2a}{m}$, *then*

$$\Big(\int_{\mathbf{R}^n} \|\varphi F_x\|^r_q\,dx\Big)^{1/r} \leqq C\,\|f\|_p. \tag{4}$$

If $p=r=1+\dfrac{m}{2a}\leqq 2$, *then*

$$\Big(\int_{\mathbf{R}^n} \|\varphi F_x\|^r_{\mathrm{BMO}}\,dx\Big)^{1/r} \leqq C\,\|f\|_p. \tag{5}$$

If $1+\dfrac{m}{2a}<p=r\leqq 2$ *and* $\delta=2a-\dfrac{m+2a}{p}$, *or if* $2\leqq p=r\leqq\infty$ *and* $\delta=\dfrac{2a-m}{p}$,

then

(6) $$\left(\int_{\mathbf{R}^n} \|\varphi F_x\|_{\Lambda_\delta}^r \, dx\right)^{1/r} \leqq C\|f\|_p.$$

If in the above conditions we replace a) *by* b), *take* $m=1$ *and allow* $p \leqq r \leqq p'$ *(for* $1 \leqq p \leqq 2$*), then* (4)—(6) *still hold.*

Remark. If $m=1$, $a=\frac{k}{2}$, $k\in\{0, 1, \ldots, n-1\}$, and $1\leqq p\leqq 2$, then the theorem and the corollary are best possible in the following sense.

It is possible to find S, $\mu\, d\sigma$, ψ and φ satisfying b) such that the following holds.

When $k\geqq 1$ and $1\leqq p<1+\frac{1}{2a}=1+\frac{1}{k}$, or when $k=0$ and $1\leqq p\leqq 2$, then (4) implies that $\frac{1}{q}\leqq\frac{1+2a}{p}-2a=\frac{k+1}{p}-k$.

If $k\geqq 1$ and $p=1+\frac{1}{2a}=1+\frac{1}{k}$, then we cannot replace the BMO-norm in (5) by the supremum-norm.

Assume that $k\geqq 2$ and $1+\frac{1}{2a}=1+\frac{1}{k}<p\leqq 2$ and that (6) holds, then $\delta\leqq 2a-\frac{1+2a}{p}=k-\frac{k+1}{p}$.

A consequence of these results is that $\alpha\leqq\frac{2a}{p'}=\frac{k}{p'}$ is necessary for (3), if $1\leqq p\leqq 2$, because the corollary follows from various imbeddings of $B^\alpha_{pp'}$, into L^q, BMO and Λ_δ. It is also necessary to have $p\leqq r\leqq p'$ in (1)—(6), if $1\leqq p\leqq 2$ (and $m=1$). The proofs are contained in the proof of the corollary. There are however hypersurfaces where one can obtain better estimates. For example, if we take

$$S=\left\{x\in\mathbf{R}^n;\ x_n=\prod_{i=1}^{n-1} x_i^{\lambda_i},\ 0<x_i<1,\ \lambda_i\in\mathbf{R},\ i=1, \ldots, n\right\}$$

and set

$$\psi_i(t)=\begin{cases} t_i, & \text{if } i=1, \ldots, n-1 \\ \prod_{i=1}^{n-1} t_i^{\lambda_i}, & \text{if } i=n, \end{cases}$$

and if μ vanishes near the boundary of S the corresponding maximal function

$$\sup\{|F_x(t)|;\ t_1, \ldots, t_{n-1}>0\}$$

then becomes bounded on $L^p(\mathbf{R}^n)$, for all $p>1$. This result of H. Carlsson, P. Sjögren and J.-O. Strömberg is contained in [CSS] and was extended in [CS]. See also [Wa] for a survey of the theory of averages and singular integrals over lower dimensional sets.

If b) holds, $a>\frac{1}{2}$ and $\psi(t,y)=ty$ the corollary contains the following estimates of the maximal function $\sup_{1<t<2}|F_x(t)|$ (since $\Lambda_\delta\subset L^\infty$).

$$\Big\|\sup_{1<t<2}|F(t)|\Big\|_r\leqq C\|f\|_p,$$

for $1+\dfrac{1}{2a}<p\leqq 2$ and $p\leqq r\leqq p'$. This is a weaker form of the following theorem.

Theorem 2. *Assume that* b) *holds and set* $\psi(t,y)=ty$. *If* $a>\dfrac{1}{2}$, $1+\dfrac{1}{2a}<p\leqq 2$ *and* $p\leqq r\leqq p'$, *then*

$$\Big\|\sup_{t>0}|t^{n((1/p)-(1/r))}F(t)|\Big\|_r\leqq C\|f\|_p.$$

For $r=p$ this theorem is contained in [R] and [CM], and is an extension of Stein's theorem on the continuity of the maximal spherical function.

Define the operator M_t^ϱ by

$$(M_t^\varrho f)^\wedge(\xi)=|t\xi|^{-(n/2)-\varrho+1}J_{(n/2)+\varrho-1}(|t\xi|)\hat{f}(\xi),$$

$t\in\mathbf{R}$, $\xi\in\mathbf{R}^n$. $J_{(n/2)+\varrho-1}$ is the Bessel function of order $\frac{n}{2}+\varrho-1$. For its definition and fundamental properties see [SWe] or [W]. A consequence of the proof of Theorem 2 is the following extension of part (a) of Theorem 2 in [St2].

Theorem 3. *If* $1<p\leqq 2$, $p\leqq r\leqq p'$ *and* $\varrho>1-\frac{n}{p'}$, *then*

$$\Big\|\sup_{t>0}|t^{n((1/p)-(1/r))}M_t^\varrho f|\Big\|_r\leqq C\|f\|_p.$$

$\varrho=0$ corresponds to the spherical mean. Theorem 3 also gives an estimate of $u(x,t)=CtM_t^{(3-n)/2}f(x)$, the solution of the wave equation with the boundary values $u(x,0)=0$, $\partial u/\partial t(x,0)=f(x)$. Theorem 3 is related to the estimates of $M_t^\varrho f(x)$ in [Bö].

4. Proofs

Proof of Theorem 1. Assume that $f\in C_0^\infty(\mathbf{R}^n)$. We start with the proof of the end-point estimate where $p=2$ and $\alpha=a$.

We compute the Fourier transform of $F_x(t)$, to get a multiplier, and obtain

$$\begin{aligned}\hat{F}_\xi(t)&=\int_{\mathbf{R}^n}e^{-ix\cdot\xi}F_x(t)\,dx\\&=\int_{\mathbf{R}^n}e^{-ix\cdot\xi}\int_S f(x-\psi(t,y))\,\mu(y)\,d\sigma(y)\,dx\\&=\int_S e^{-i\psi(t,y)\cdot\xi}\int_{\mathbf{R}^n}e^{-i(x-\psi(t,y))\cdot\xi}f(x-\psi(t,y))\,dx\,\mu(y)\,d\sigma(y)\\&=\int_S e^{-iy\cdot\psi(t,\xi)}\,\mu(y)\,d\sigma(y)\hat{f}(\xi)\\&=m(\psi(t,\xi))\hat{f}(\xi),\end{aligned}$$

where

$$\widehat{\mu\, d\sigma}(\xi) = m(\xi).$$

We consider the $L^2(\mathbf{R}^n)$-norm of $\|\varphi F_x\|_{H_2^N}$ for a non-negative integer N. The norm of $H_2^N(\mathbf{R}^m)$ is equivalent to the norm

$$\|g\|_2 + \sum_{k=1}^m \|D_k^N g\|_2, \quad \text{where} \quad D_k^N g = \frac{\partial^N g}{\partial t_k^N}.$$

Therefore,

$$\Big(\int_{\mathbf{R}^n} \|\varphi F_x\|_{H_2^N}^2 \,dx\Big)^{1/2} \leqq C\Big(\int_{\mathbf{R}^n} \big(\|\varphi F_x\|_2 + \sum_{k=1}^m \|D_k^N(\varphi F_x)\|_2\big)^2 \,dx\Big)^{1/2}$$
$$\leqq C\Big[\Big(\int_{\mathbf{R}^n} \|\varphi F_x\|_2^2 \,dx\Big)^{1/2} + \sum_{k=1}^m \Big(\int_{\mathbf{R}^n} \|D_k^N(\varphi F_x)\|_2^2 \,dx\Big)^{1/2}\Big]$$

and one of the terms in the last sum can be estimated using the Fubini and Plancherel theorems.

$$(7) \qquad \Big(\int_{\mathbf{R}^n} \|D_k^N(\varphi F_x)\|_2^2 \,dx\Big)^{1/2}$$
$$\leqq C\sum_{j=0}^N \Big(\int_{\mathbf{R}^n} \|D_k^{N-j}\varphi D_k^j F_x\|_2^2 \,dx\Big)^{1/2}$$
$$= C\sum_{j=0}^N \Big(\int_{\mathbf{R}^m} |D_k^{N-j}\varphi(t)|^2 \int_{\mathbf{R}^n} |D_k^j F_x(t)|^2 \,dx\,dt\Big)^{1/2}$$
$$= C\sum_{j=0}^N \Big(\int_{\mathbf{R}^m} |D_k^{N-j}\varphi(t)|^2 \int_{\mathbf{R}^n} |D_k^j \hat{F}_\xi(t)|^2 \,d\xi\,dt\Big)^{1/2}$$
$$= C\sum_{j=0}^N \Big(\int_{\mathbf{R}^m} |D_k^{N-j}\varphi(t)|^2 \int_{\mathbf{R}^n} \big|D_k^j\big(m(\psi(t,\xi))\big)\hat{f}(\xi)\big|^2 \,d\xi\,dt\Big)^{1/2}.$$

The differentiation of $m(\psi(t,\xi))$ gives

$$D_k^j\big(m(\psi(t,\xi))\big) = D_k^j \int_S e^{-ix\cdot\psi(t,\xi)}\,\mu(x)\,d\sigma(x)$$
$$= \int_S D_k^j\big(e^{-ix\cdot\psi(t,\xi)}\big)\,\mu(x)\,d\sigma(x)$$
$$= \int_S \sum_{|\gamma|\leqq j} P_k^\gamma(t)\,x^\gamma\,\xi^\gamma e^{-ix\cdot\psi(t,\xi)}\,\mu(x)\,d\sigma(x)$$
$$= \sum_{|\gamma|\leqq j} P_k^\gamma(t)\,\xi^\gamma \int_S e^{-ix\cdot\psi(t,\xi)}\,x^\gamma\mu(x)\,d\sigma(x).$$

Here P_k^γ are polynomials in $D_k^l\psi_i$, $l\leqq j$, of degree less than or equal to j, for multi-indices $\gamma=(\gamma_1, \ldots, \gamma_n)\in\mathbf{N}^n$, $\mathbf{N}=\{0, 1, 2, \ldots\}$, having length $|\gamma|=\gamma_1+\ldots+\gamma_n$, $k=1, \ldots, m$, $i=1, \ldots, n$, $j=1, \ldots, N$, and $x^\gamma = x_1^{\gamma_1}\cdot\ldots\cdot x_n^{\gamma_n}$.

We claim that the Fourier transform of the measure $x^\gamma\mu(x)\,d\sigma(x)$ is $\mathcal{O}(|\xi|^{-a})$ as $|\xi|\to\infty$. Take a function h in $C_0^\infty(\mathbf{R}^n)$ which is equal to 1 on S. For such h we have that

$$x^\gamma h(x)\,\mu(x)\,d\sigma(x) = x^\gamma\mu(x)\,d\sigma(x)$$

is a compactly supported distribution $\mu\, d\sigma$ multiplied by a compactly supported C^∞ function $x^\gamma h$, but the Fourier transform of this product gives a convolution of $\widehat{\mu\, d\sigma}(=\mathcal{O}(|\xi|^{-a}))$ with a rapidly decreasing C^∞ function. Thus, we have that the convolution is $\mathcal{O}(|\xi|^{-a})$, as $|\xi|\to\infty$, and since supp φ is contained in $\bigcap_{i=1}^n \{t;\ \psi_i(t)\neq 0\}$ we get that

$$\inf_{t\in \operatorname{supp}\varphi} |\psi(t,\xi)| \geqq \underbrace{\min_i \inf_{t\in\operatorname{supp}\varphi} |\psi_i(t)|}_{=C>0}\,|\xi| = C|\xi|,$$

and as a consequence, if $t\in\operatorname{supp}\varphi$,

$$\begin{aligned}\left|D_k^j(m(\psi(t,\xi)))\right| &\leqq \sum\nolimits_{|\gamma|\leqq j} |P_k^\gamma(t)\xi^\gamma| \left|\int_S e^{-ix\cdot\psi(t,\xi)} x^\gamma \mu(x)\, d\sigma(x)\right| \\ &\leqq C\sum\nolimits_{|\gamma|\leqq j} \frac{|\xi|^{|\gamma|}}{(1+|\psi(t,\xi)|)^\alpha} \\ &\leqq C\frac{(1+|\xi|)^j}{(1+|\xi|)^\alpha} = C(1+|\xi|)^{j-\alpha}.\end{aligned}$$

We apply this estimate to (7) and get

$$\begin{aligned}\Big(\int_{\mathbf{R}^n} \|D_k^N(\varphi F_x)\|_2^2\, dx\Big)^{1/2} &\leqq C\sum\nolimits_{j=0}^N \Big(\int_{\mathbf{R}^m} |D_k^{N-j}\varphi(t)|^2 \int_{\mathbf{R}^n} \left|(1+|\xi|)^{j-a}\hat f(\xi)\right|^2 d\xi\, dt\Big)^{1/2} \\ &\leqq C\Big(\int_{\mathbf{R}^n} \left|(1+|\xi|)^{N-a}\hat f(\xi)\right|^2 d\xi\Big)^{1/2}.\end{aligned}$$

Consider the $L^2(H_2^N)$-norm of φF.

$$\begin{aligned}(8)\quad \Big(\int_{\mathbf{R}^n} \|\varphi F_x\|_{H_2^N}^2\, dx\Big)^{1/2} &\leqq C\Big[\Big(\int_{\mathbf{R}^n} \|\varphi F_x\|_2^2\, dx\Big)^{1/2} + \sum\nolimits_{k=1}^m \Big(\int_{\mathbf{R}^n} \|D_k^N(\varphi F_x)\|_2^2\, dx\Big)^{1/2}\Big] \\ &\leqq C\Big[\Big(\int_{\mathbf{R}^n} \left|(1+|\xi|)^{-a}\hat f(\xi)\right|^2 d\xi\Big)^{1/2} + \sum\nolimits_{k=1}^m \Big(\int_{\mathbf{R}^n} \left|(1+|\xi|)^{N-a}\hat f(\xi)\right|^2 d\xi\Big)^{1/2}\Big] \\ &\leqq C\Big(\int_{\mathbf{R}^n} \left|(1+|\xi|)^{N-a}\hat f(\xi)\right|^2 d\xi\Big)^{1/2}.\end{aligned}$$

The estimate of the $L^2(L^2)$-norm above corresponds to the case $N=0$, i.e.

$$(9)\quad \Big(\int_{\mathbf{R}^n} \|\varphi F_x\|_{H_2^0}^2\, dx\Big)^{1/2} = C\Big(\int_{\mathbf{R}^n} \|\varphi F_x\|_2^2\, dx\Big)^{1/2} \leqq C\Big(\int_{\mathbf{R}^n} \left|(1+|\xi|)^{-a}\hat f(\xi)\right|^2 d\xi\Big)^{1/2}.$$

Choose N such that $N\geqq a$ and interpolate between (8) and (9), which gives (see [BL, pp. 17—18, 107, 152—153])

$$\Big(\int_{\mathbf{R}^n} \|\varphi F_x\|_{H_2^\eta}^2\, dx\Big)^{1/2} \leqq C\Big(\int_{\mathbf{R}^n} \left|(1+|\xi|)^{\eta-a}\hat f(\xi)\right|^2 d\xi\Big)^{1/2},$$

for $0\leqq\eta\leqq N$. Putting $\eta=a$ gives

$$(10)\qquad \Big(\int_{\mathbf{R}^n} \|\varphi F_x\|_{H_2^a}^2\, dx\Big)^{1/2} \leqq C\Big(\int_{\mathbf{R}^n} |\hat f(\xi)|^2\, d\xi\Big)^{1/2} = C\|f\|_2.$$

Now assume that $f\in L^2(\mathbf{R}^n)$ and take a sequence $\{f_l\}_1^\infty$ in $C_0^\infty(\mathbf{R}^n)$ converging to f in $L^2(\mathbf{R}^n)$ and set

$$F_x^l(t) = \int_S f_l(x-\psi(t, y))\mu(y)\,d\sigma(y).$$

Assume for a moment the following:

$$\Big(\int_{\mathbf{R}^n} |(\varphi(F_x-F_x^l))^\wedge(s)|^2\,dx\Big)^{1/2}\to 0, \quad \text{as} \quad l\to\infty, \quad \text{for all} \quad s\in\mathbf{R}^m. \tag{11}$$

Then

$$\lim_{l\to\infty}\Big(\int_{\mathbf{R}^n} |(\varphi(F_x^l))^\wedge(s)|^2\,dx\Big)^{1/2} = \Big(\int_{\mathbf{R}^n} |(\varphi(F_x))^\wedge(s)|^2\,dx\Big)^{1/2},$$

for all $s\in\mathbf{R}^m$, and by Fatou's lemma

$$\begin{aligned}
\Big(\int_{\mathbf{R}^n}\|\varphi F_x\|_{H_2^a}^2\,dx\Big)^{1/2} &= \Big(\int_{\mathbf{R}^n}\int_{\mathbf{R}^m}|(\varphi F_x)^\wedge(s)|^2(1+|s|^2)^a\,ds\,dx\Big)^{1/2}\\
&= \Big(\int_{\mathbf{R}^m}\int_{\mathbf{R}^n}|(\varphi F_x)^\wedge(s)|^2\,dx\,(1+|s|^2)^a\,ds\Big)^{1/2}\\
&= \Big(\int_{\mathbf{R}^m}\underline{\lim}\int_{\mathbf{R}^n}|(\varphi F_x^l)^\wedge(s)|^2\,dx\,(1+|s|^2)^a\,ds\Big)^{1/2}\\
&\leqq \underline{\lim}\Big(\int_{\mathbf{R}^m}\int_{\mathbf{R}^n}|(\varphi F_x^l)^\wedge(s)|^2\,dx\,(1+|s|^2)^a\,ds\Big)^{1/2}\\
&= \underline{\lim}\Big(\int_{\mathbf{R}^n}\|\varphi F_x^l\|_{H_2^a}^2\,dx\Big)^{1/2}\\
&\leqq \underline{\lim}\, C\|f_l\|_2 = C\|f\|_2.
\end{aligned}$$

In the proof of (11) we use Minkowski's inequality and some trivial estimates.

$$\begin{aligned}
&\Big(\int_{\mathbf{R}^n}|(\varphi(F_x-F_x^l))^\wedge(s)|^2\,dx\Big)^{1/2}\\
&= \Big(\int_{\mathbf{R}^n}\Big|\int_{\mathbf{R}^m}e^{-ist}\varphi(t)\int_S(f-f_l)(x-\psi(t, y))\,\mu(y)\,d\sigma(y)\,dt\Big|^2\,dx\Big)^{1/2}\\
&\leqq \Big(\int_{\mathbf{R}^n}\Big(\int_{\mathbf{R}^m}\int_S|\varphi(t)(f-f_l)(x-\psi(t, y))\,\mu(y)|\,d\sigma(y)\,dt\Big)^2dx\Big)^{1/2}\\
&\leqq \int_{\mathbf{R}^m}\int_S\Big(\int_{\mathbf{R}^n}|\varphi(t)(f-f_l)(x-\psi(t, y))\,\mu(y)|^2\,dx\Big)^{1/2}d\sigma(y)\,dt\\
&\leqq \int_{\mathbf{R}^m}|\varphi(t)|\int_S|\mu(y)|\Big(\int_{\mathbf{R}^n}|(f-f_l)(x-\psi(t, y))|^2\,dx\Big)^{1/2}d\sigma(y)\,dt\\
&= C\|\varphi\|_1\int_S|\mu(y)|\,d\sigma(y)\,\|f-f_l\|_2.
\end{aligned}$$

The right hand side tends to 0 as k tends to ∞ thus proving (11).

This proves Theorem 1 in the case $p=2$ and $\alpha=a$, but for the coming interpolation we also need that φF_x be strongly measurable with values in $B_{22}^a(\mathbf{R}^m)$ if $f\in L^2(\mathbf{R}^n)$. But this can be shown by the method applied in [S5, p. 156].

We continue with the other end-point estimates where $p=1$ or ∞ and $\alpha=0$. For $f\in L^1(\mathbf{R}^n)$, consider the $L^1(L^1)$-norm of φF.

$$\begin{aligned}\int_{\mathbf{R}^n} \|\varphi F\|_1\,dx &= \int_{\mathbf{R}^m}\int_{\mathbf{R}^n}\Big|\varphi(t)\int_S f(x-\psi(t,y))\,\mu(y)\,d\sigma(y)\Big|\,dx\,dt\\ &\leqq \int_{\mathbf{R}^m}|\varphi(t)|\int_S|\mu(y)|\int_{\mathbf{R}^n}\big|f(x-\psi(t,y))\big|\,dx\,d\sigma(y)\,dt\\ &= \|\varphi\|_1\int_S|\mu(y)|\,d\sigma(y)\,\|f\|_1\\ &= C\|f\|_1.\end{aligned}$$

Since $L^1(\mathbf{R}^n)$ is continuously embedded in $B^0_{1\infty}(\mathbf{R}^m)$ we also have

$$\int_{\mathbf{R}^n}\|\varphi F\|_{B^0_{1\infty}}\,dx \leqq C\|f\|_1, \tag{12}$$

which is Theorem 1 for $p=1$ and $\alpha=0$.

If $f\in L^1(\mathbf{R}^n)$ we claim that φF_x is a strongly measurable function of x with values in $B^0_{1\infty}(\mathbf{R}^m)$. It is enough to prove that φF_x is a strongly measurable function of x with values in $L^1(\mathbf{R}^m)$, since $L^1(\mathbf{R}^m)$ is continuously embedded in $B^0_{1\infty}(\mathbf{R}^m)$. But because $L^1(\mathbf{R}^m)$ is separable we only have to verify that φF_x is weakly measurable since then strong and weak measurability are equivalent notions. See [HP, p. 73]. Take therefore $g\in L^\infty(\mathbf{R}^m)$ and set

$$H(x) = \int_{\mathbf{R}^m}\varphi(t)F_x(t)g(t)\,dt = \int_{\mathbf{R}^m}\varphi(t)g(t)\int_S f(x-\psi(t,y))\,\mu(y)\,d\sigma(y)\,dt.$$

Then $H(x)$ becomes measurable since all functions involved are measurable. This proves the claim.

By an application of the interpolation theorem for vector-valued functions (see [BL, p. 107]) to (10) and (12), and the fact that

$$(B^0_{1\infty}, H^a_2)_{[\theta]} = (B^0_{1\infty}, B^a_{22})_{[\theta]} = B^\alpha_{pp'}$$

we get

$$\Big(\int_{\mathbf{R}^n}\|\varphi F_x\|^p_{B^\alpha_{pp'}}\,dx\Big)^{1/p} \leqq C\,\|f\|_p,$$

where $1\leqq p\leqq 2$ and $\alpha=\dfrac{2a}{p'}$. This is (1).

If $f\in L^\infty(\mathbf{R}^n)$ take a bounded $f_0\in L^\infty(\mathbf{R}^n)$ such that $f(x)=f_0(x)$ almost everywhere and define the mean F of f by

$$F_x(t) = \int_S f_0(x-\psi(t,y))\,\mu(y)\,d\sigma(y).$$

We get the trivial estimate

$$|F_x(t)| \leqq C\|f_0\|_\infty = C\|f\|_\infty$$

and also

$$\operatorname*{ess\,sup}_{x\in\mathbf{R}^n}\|\varphi F_x\|_{B^0_{\infty\infty}} \leqq C \operatorname*{ess\,sup}_{x\in\mathbf{R}^n}\|\varphi F_x\|_\infty \leqq C\|f\|_\infty .$$

Interpolating this with (10) gives (2).

(2) can also be obtained by a dual argument. Set $Tf(x,t)=\varphi(t)F_x(t)$ and consider the dual operator T^* of T applied to the function $g(x,t)$. For $g\in L^1(\mathbf{R}^{n+m})$ we get by definition

$$\begin{aligned}\langle T^*g, f\rangle &= \langle g, Tf\rangle\\ &= \int_{\mathbf{R}^{n+m}} g(x,t)\varphi(t)F_x(t)\,dx\,dt\\ &= \int_S\int_{\mathbf{R}^m}\varphi(t)\int_{\mathbf{R}^n} f(x-\psi(t,y))g(x,t)\,dx\,dt\,\mu(y)\,d\sigma(y)\\ &= \int_{\mathbf{R}^n}\underbrace{\int_{\mathbf{R}^m}\int_S \varphi(t)g(x+\psi(t,y),t)\,\mu(y)\,d\sigma(y)\,dt}_{=T^*g(x)}f(x)\,dx.\end{aligned}$$

Estimating the $L^1(\mathbf{R}^n)$-norm of T^*g gives

$$\begin{aligned}\|T^*g\|_1 &= \int_{\mathbf{R}^n}\left|\int_{\mathbf{R}^m}\int_S \varphi(t)g(x+\psi(t,y),t)\,\mu(y)\,d\sigma(y)\,dt\right|dx\\ &\leqq \int_S\int_{\mathbf{R}^m}|\varphi(t)|\int_{\mathbf{R}^n}\left|g(x+\psi(t,y),t)\right|dx\,dt\,|\mu(y)|\,d\sigma(y)\\ &\leqq \|\varphi\|_\infty\int_S\int_{\mathbf{R}^m}\int_{\mathbf{R}^n}|g(x,t)|\,dx\,dt\,|\mu(y)|\,d\sigma(y)\\ &= \|\varphi\|_\infty\int_S|\mu(y)|\,d\sigma(y)\int_{\mathbf{R}^n}\|g(x,\cdot)\|_1\,dx\end{aligned}$$

and by the continuous embedding $B^0_{11}(\mathbf{R}^m)\subset L^1(\mathbf{R}^m)$ also

$$\|T^*g\|_1 \leqq C\int_{\mathbf{R}^n}\|g(x,\cdot)\|_{B^0_{11}}\,dx,$$

for $g\in L^1(B^0_{11})$ (which makes T^*g measurable). Interpolating this with the dual estimate of (10)

$$\|T^*g\|_2 \leqq C\left(\int_{\mathbf{R}^n}\|g(x,\cdot)\|^2_{H_2^{-\alpha}}\,dx\right)^{1/2}$$

gives

$$\|T^*g\|_p \leqq C\left(\int_{\mathbf{R}^n}\|g(x,\cdot)\|^p_{B^{-\alpha}_{pp}}\,dx\right)^{1/p},$$

where $1<p<2$ and $\alpha=\dfrac{2a}{p'}$. The dual estimate of this is

$$\left(\int_{\mathbf{R}^n}\|Tf(x,\cdot)\|^p_{B^\alpha_{pp}}\,dx\right)^{1/p} \leqq C\|f\|_p,$$

if $2<p<\infty$ and $\alpha=\dfrac{2a}{p}$, which is (2).

Here we have used that

$$(L^p(B))^* = L^{p'}(B^*),$$

if $1<p<\infty$, for a reflexive Banach space B.

We continue with the proof of (3).

Let S, φ and ψ satisfy b), and therefore also a) with $m=1$. We claim that

$$\sup_{x\in\mathbf{R}^n} \|\varphi F_x\|_1 \leqq C\,\|f\|_1, \quad f\in L^1(\mathbf{R}^n). \tag{13}$$

From the proof of (1) we have

$$\int_{\mathbf{R}^n} \|\varphi F_x\|_1\, dx \leqq C\,\|f\|_1, \quad f\in L^1(\mathbf{R}^n). \tag{14}$$

Interpolation between (13) and (14) gives

$$\Big(\int_{\mathbf{R}^n} \|\varphi F_x\|_1^r\, dx\Big)^{1/r} \leqq C\,\|f\|_1,$$

for $1\leqq r\leqq\infty$ and $f\in L^1(\mathbf{R}^n)$. See [BP, p. 316]. Thus, by the embedding $L^1(\mathbf{R})\subset \subset B^0_{1\infty}(\mathbf{R})$ we get, as before, that

$$\Big(\int_{\mathbf{R}^n} \|\varphi F_x\|_{B^0_{1\infty}}^r\, dx\Big)^{1/r} \leqq C\,\|f\|_1, \quad f\in L^1(\mathbf{R}^n),$$

for $1\leqq r\leqq\infty$ and $f\in L^1(\mathbf{R}^n)$, and as in the proof of (1) we now obtain (3) by interpolation with the end-point $p=2$, $\alpha=a$, of (1). (In this interpolation the case $r=\infty$ needs some special care. The function space used for interpolation is $L_0^\infty(B^0_{1\infty})$, the completion in sup-norm of all simple functions on $\mathbf{R}^n$ with values in $B^0_{1\infty}$. But such an approximation is possible by the construction in [S5, p. 155] which also applies here.)

We now turn to the proof of the claim.

To emphasize that $\psi(t,y)$ is a non-isotropic dilation we adopt the usual convention and set $\delta_t(y)=\psi(t,y)$. By the assumption b) the map $h(y)=y/|y|$, $y\in S$, is a diffeomorphism from S onto its image $\bar S$ in S^{n-1}. We extend h to $\Omega=\{\delta_t(y);\, t>0, y\in S\}$ by $h(\delta_t(y))=th(y)$, and as a consequence, for $s>0$ and $\bar y\in\Omega$,

$$h(\delta_s(\bar y)) = h(\delta_s\delta_t(y)) = h(\delta_{st}(y)) = sth(y) = sh(\delta_t(y)) = sh(\bar y).$$

The extension h becomes a diffeomorphism from Ω onto its image $\bar\Omega=\{ty;\, t>0, y\in\bar S\}$. We also extend μ to Ω by $\mu(\delta_t(y))=\mu(y)$, $t>0$, $y\in S$.

We get that the condition

$$|h(y)|-1 = 0, \quad y\in\Omega,$$

becomes equivalent to $y\in S$. If u is the Dirac measure (see [GS, Ch. III]), then the average $F_x(t)$ can be written as follows:

$$F_x(t) = \int_\Omega f_x(\delta_t(y))\mu(y)u(1-|h(y)|)\,dy,$$

where $f_x=f(x-\cdot)$.

A change of variables, $z=h(y)$, gives

$$F_x(t) = \int_{\bar\Omega} f_x(\delta_t(h^{-1}(z)))\mu(h^{-1}(z))u(1-|z|)\,|(h^{-1})'(z)|\,dz,$$

where $(h^{-1})'$ is the Jacobian determinant of h^{-1}. But $\delta_t(h^{-1}(z))=h^{-1}(tz)$, because $h(\delta_t(h^{-1}(z)))=th(h^{-1}(z))=tz$.

Consequently,

$$\begin{aligned}|F_x(t)| &\leqq \int_{\bar\Omega} \left|f_x(h^{-1}(tz))\mu(h^{-1}(z))(h^{-1})'(z)\right| u(1-|z|)\,dz\\ &\leqq C\int_{\bar\Omega}\left|f_x(h^{-1}(tz))\right| u(1-|z|)\,dz\\ &= C\int_{\bar S}\left|f_x(h^{-1}(t\theta))\right| d\theta.\end{aligned}$$

This is because $\mu\circ h^{-1}$ and $(h^{-1})'$ are bounded on $\bar S\subset S^{n-1}$. Here $d\theta$ is the induced Lebesgue measure on S^{n-1}.

Let I be a closed bounded interval in $(0, \infty)$ containing $\operatorname{supp}\varphi$. We estimate the $L^1(\mathbf{R})$-norm of φF and obtain

$$\begin{aligned}\|\varphi F_x\|_{L^1(\mathbf{R})} &\leqq C\int_I\int_{\bar S}\left|\varphi(t)f_x(h^{-1}(t\theta))\right| d\theta\,dt\\ &= C\int_A\left|\varphi(|w|)|w|^{-n+1}f_x(h^{-1}(w))\right| dw\\ &\leqq C\int_A\left|f_x(h^{-1}(w))\right| dw,\end{aligned}$$

since $A=\{t\theta;\ t\in I,\ \theta\in\bar S\}$ lies in an annulus. With new variables, $y=h^{-1}(\omega)$, this becomes

$$\begin{aligned}\|\varphi F_x\|_{L^1(\mathbf{R})} &\leqq C\int_{h^{-1}(A)}|f_x(y)\,h'(y)|\,dy\\ &\leqq C\int_{h^{-1}(A)}|f_x(y)|\,dy\\ &\leqq C\,\|f_x\|_1 = C\,\|f\|_1,\end{aligned}$$

because $h^{-1}(A)$ is contained in an annulus where $h'(y)$ is bounded. This proves our claim and the theorem.

Proof of the Corollary. We start with the proof of (4) by showing that for certain values of p, $B^\alpha_{pp'}(\mathbf{R}^m)$ is continuously embedded in $L^q(\mathbf{R}^m)$.

We have that

$$B^\alpha_{pt}(\mathbf{R}^m)\subset H^\beta_t(\mathbf{R}^m), \tag{15}$$

if $1<p<t<\infty$ and $\alpha-\frac{m}{p}=\beta-\frac{m}{t}$, by [T, p. 206]. Hence $B^{\alpha}_{pq}\subset H^0_q=L^q$, if $\frac{2a}{p'}-\frac{m}{p}\left(=\alpha-\frac{m}{p}\right)=-\frac{m}{q}$ or (equivalently) $\frac{1}{q}=\frac{1}{p}\left(1+\frac{2a}{m}\right)-\frac{2a}{m}\,(>0)$. From the definition of B^{α}_{pq} it follows at once that $B^{\alpha}_{pp'}\subset B^{\alpha}_{pq}$, if $p'\leqq q$. But $p'\leqq q$, if $p\geqq 1+\frac{m}{m+2a}$. Thus $B^{\alpha}_{pp'}\subset L^q$, if $1+\frac{m}{m+2a}\leqq p<1+\frac{m}{2a}$. In the case where $1\leqq p<1+\frac{m}{m+2a}$ then (4) is obtained by interpolation between (4) in the case $p=1+\frac{m}{m+2a}$, $r=p$, and (4) in the case $r=p=1$.

If b) is assumed we use (4) in the case $p=1$ and $1\leqq r\leqq\infty$. The latter is included in the proof of Theorem 1. This finishes the proof of (4).

H^{β}_t is embedded in BMO, if $\beta=\frac{m}{t}$ (see [St1, p. 164]) and by (15) $B^{\alpha}_{pp'}$ embeds in BMO, if $t=p'$ and $\frac{2a}{p'}=\alpha=\frac{m}{p}$ but this means that $p=1+\frac{m}{2a}$. This shows (5).

(6) is a consequence of the following embeddings (see [BL, p. 153]).

$$\left.\begin{matrix} B^{\alpha}_{pp'}(\mathbf{R}^m) \\ B^{\alpha}_{pp}(\mathbf{R}^m)\end{matrix}\right\} \subset B^{\delta}_{\infty\infty}(\mathbf{R}^m) = \Lambda_{\delta}(\mathbf{R}^m).$$

Here $\alpha-\frac{m}{p}=\delta>0$ and as a consequence $\delta=\frac{2a}{p'}-\frac{m}{p}=2a-\frac{m+2a}{p}$, if $1+\frac{m}{2a}<p\leqq 2$, and $\delta=\frac{2a}{p}-\frac{m}{p}=\frac{2a-m}{p}$, if $2\leqq p\leqq\infty$.

We continue with the necessary condition in the remark.

Take a fixed $\varphi\in C_0^{\infty}(\mathbf{R})$ such that $\emptyset\neq\operatorname{supp}\varphi\subset(0,b)$, $\psi(t,y)=ty$ and a fixed k in $\{2,3,\ldots,n-1\}$. Let a prime on a variable denote an element in $\mathbf{R}^{k+1}$ and double-prime one in $\mathbf{R}^{n-k-1}$, and decompose a variable in $\mathbf{R}^n$ into these, for example $x=(x',x'')$, where $x'=(x_1,\ldots,x_{k+1})$ and $x''=(x_{k+2},\ldots,x_n)$. Let $\omega\in C_0^{\infty}(\mathbf{R})$ be supported in $[-1,1]$ with $\int_{\mathbf{R}}\omega(s)\,ds=1$ and put

$$\bar{\omega}(x'') = \Pi^n_{i=k+2}\,\omega(x_i).$$

Denote by $d\theta$ the induced Lebesgue measure on the unit sphere S^k in $\mathbf{R}^{k+1}$ and define a measure on $S^k\times[-1,1]^{n-k-1}$ in $\mathbf{R}$ by:

$$\mu(x)\,d\sigma(x) = d\theta(x')\otimes\bar{\omega}(x'')\,dx''.$$

$S^k\times[-1,1]^{n-k-1}$ has k principal curvatures different from zero.

Take a $g\in C_0^{\infty}(\mathbf{R}^{n-k-1})$ such that $g\equiv 1$ on the cube $[-b-1,b+1]^{n-k-1}$, and as a consequence we have that

$$(16)\qquad \int_{\mathbf{R}^{n-k-1}} g(x''-ty'')\bar{\omega}(y'')\,dy'' = \Pi_{i=k+2}^{n}\int_{-1}^{1} \omega(x_i)\,dx_i = 1,$$

for $t\in\operatorname{supp}\varphi$ and $x''\in Q=[-1,1]^{n-k-1}$.

Define, for $h\in L^p(\mathbf{R}^{k+1})$, the spherical mean in $\mathbf{R}^{k+1}$

$$H_{x'}(t) = \int_{S^k} h(x'-ty')\,d\theta(y'),$$

and put $f(x)=h(x')g(x'')$, so $f\in L^p(\mathbf{R}^n)$. Then for the mean F of f we have by (16)

$$\begin{aligned}\varphi(t)F_x(t) &= \int_S f(x-ty)\,\mu(y)\,d\sigma(y)\\ &= \varphi(t)\int_{\mathbf{R}^{n-k-1}}\int_{S^k} h(x'-ty')g(x''-ty'')\,d\theta(y')\bar{\omega}(y'')\,dy''\\ &= \varphi(t)\int_{\mathbf{R}^{n-k-1}} g(x''-ty'')\bar{\omega}(y'')\,dy''\int_{S^k} h(x'-ty')\,d\theta(y')\\ &= \varphi(t)H_{x'}(t),\end{aligned}$$

if $x''\in Q$.

Estimating $\|\varphi F_x\|_{\Lambda_\delta}$ gives then

$$\begin{aligned}\Big(\int_{\mathbf{R}^n}\|\varphi F_x\|_{\Lambda_\delta}^r\,dx\Big)^{1/r} &\geqq \Big(\int_{\mathbf{R}^{k+1}}\int_Q \|\varphi F_{(x',x'')}\|_{\Lambda_\delta}^r\,dx''\,dx'\Big)^{1/r}\\ &= \Big(\int_{\mathbf{R}^{k+1}}\int_Q \|\varphi H_{x'}\|_{\Lambda\delta}^r\,dx''\,dx'\Big)^{1/r}\\ &= C\Big(\int_{\mathbf{R}^{k+1}}\|\varphi H_{x'}\|_{\Lambda_\delta}^r\,dx'\Big)^{1/r},\end{aligned}$$

and under the assumption of (6) and that $1+\frac{1}{k}<p\leqq 2$ we get

$$\begin{aligned}\Big(\int_{\mathbf{R}^{k+1}}\|\varphi H_{x'}\|_{\Lambda_\delta}^r\,dx'\Big)^{1/r} &\leqq C\Big(\int_{\mathbf{R}^n}\|\varphi F_x\|_{\Lambda_\delta}^r\,dx\Big)^{1/r}\\ &\leqq C\|f\|_p = C\|h\|_p\|g\|_p\\ &= C\|h\|_p.\end{aligned}$$

But by a counter-example in [S2] this implies that $\delta\leqq k-\dfrac{k+1}{p}$.

Take a fixed k in $\{0,1,\dots,n-1\}$ and assume that $1\leqq p<1+\frac{1}{k}$, if $k\geqq 1$, and $1\leqq p\leqq 2$, if $k=0$. Replacing Λ_δ by L^q in the norms above together with (4) we also obtain (from [S2]) that $\dfrac{1}{q}\geqq\dfrac{k+1}{p}-k$. (The counter-example in [S2] is restricted to $k\geqq 1$, but can easily be modified to cover the case $k=0$.)

Thus the theorem gives the best possible values of δ and p.

Assume that $k\geqq 1$ and $p=1+\frac{1}{k}$ and set

$$h(x') = \begin{cases} |x'|^{-k}\left(\log\frac{1}{|x'|}\right)^{-1}, & \text{if } 0<|x'|\leqq\frac{1}{2} \\ 0, & \text{otherwise.}\end{cases}$$

Then the mean F of $f=hg$ (g as before), defined by the measure $\mu\, d\sigma$ above, gives

$$\sup_{t>0}|\varphi(t)F_x(t)| = \infty,$$

for x in a set of positive measure. But f belongs to $L^p(\mathbf{R}^n)$. Cf. [St2]. This shows that BMO in (3) cannot be replaced by L^∞.

Using the above construction of S with measure $\mu\, d\sigma$ one also obtains, from counterexamples for the spherical mean in [S5], that $p\leqq r\leqq p'$ is necessary if $1\leqq p\leqq 2$ in (1)—(4).

Proof of Theorem 2. The proof is a small extension of the proof of Theorem 2.2 of [CM], from which we only give the main lines. For a more thorough treatment the reader should consult [CM].

Assume that $f\in C_0^\infty(\mathbf{R}^n)$. Let u be the distribution defined by the measure $\mu\, d\sigma$ and, for $\Re z>0$, set

$$R_z f(x) = 2\Gamma(z)^{-1}\int_0^1 (1-t^2)^{z-1} f\left(\frac{x}{t}\right)\frac{dt}{t}.$$

$z\to R_z f$ continues analytically into C, and $R_z u$ is defined by duality, i.e.

$$\langle R_z u, f\rangle = \langle u, R_z f\rangle.$$

Note that $R_0 f=f$.

For $\beta>\frac{1}{2}-a$ and $\beta\leqq\Re z\leqq 1$, let

$$F_{x,z}(t) = t^{(n(z-\beta))/(1-\beta)}\langle R_z u, f(x-t\cdot)\rangle,\quad t>0.$$

Let $T(x)=|h(x)|$, where h is the function defined in the proof of Theorem 1 and μ the weight on S extended to Ω. Put $E(t)=\{x\in\mathbf{R}^n;\ T(x)\leqq t\}\cup\{0\}$, $t>0$. Then, if $\Re z>0$, $R_z u$ "is" the function

$$R_z u(x) = \begin{cases} 2\Gamma(z)^{-1}(1-T(x)^2)^{z-1}|\nabla T(x)|\,\mu(z), & \text{if } x\in E(1) \\ 0, & \text{otherwise.}\end{cases}$$

See [CM, Prop. 2.1]. ∇T is an outward normal vector to S. Hence, if $\Re z=1$, we have the following estimate of $F_{x,z}(t)$.

$$\begin{aligned}|F_{x,z}(t)| &= \left|t^{(n(z-\beta))/(1-\beta)}\langle R_z u, f(x-t\cdot)\rangle\right| \\ &= \left|t^n 2\Gamma(z)^{-1}\int_{E(1)}|\nabla T(y)\,\mu(y)\,f(x-ty)|\,dy\right| \\ &\leqq Ce^{\pi|\Im z|}t^n\int_{\mathbf{R}^n}|f(x-ty)|\,dy \\ &= Ce^{\pi|\Im z|}\|f\|_1,\end{aligned}$$

since $|\nabla T(x)|$ and $\mu(y)$ are bounded on $E(1)$. Therefore

$$\Big\|\sup_{t>0}|F_{x,z}(t)|\Big\|_\infty \leqq Ce^{\pi|\Im z|}\|f\|_1,$$

if $\Re z=1$. But by Theorem 1.4 of [CM] we also have that

$$\Big\|\sup_{t>0}|F_{x,z}(t)|\Big\|_2 \leqq Ce^{A|\Im z|}\|f\|_2,$$

if $\Re z=\beta>\frac{1}{2}-a$. We apply Stein's complex interpolation method and deduce that

$$\Big\|\sup_{t>0}|F_{x,\varrho}(t)|\Big\|_{p'} \leqq C\|f\|_p, \tag{17}$$

where $1\leqq p\leqq 2$, $\beta>\frac{1}{2}-a$ and $\varrho=\frac{2}{p'}(\beta-1)+1$. For $\varrho=0$, this becomes

$$\Big\|\sup_{t>0}|t^{(-n\beta)/(1-\beta)}F_x(t)|\Big\|_{p'} \leqq C\|f\|_p,$$

if $\beta>\dfrac{1}{2}-a$ and $p=\dfrac{2(1-\beta)}{1-2\beta}$, or equivalently

$$\Big\|\sup_{t>0}|t^{n((2/p)-1)}F_x(t)|\Big\|_{p'} \leqq C\|f\|_p,$$

for $1+\dfrac{1}{2a}<p\leqq 2$. This can be interpolated with

$$\Big\|\sup_{t>0}|F_x(t)|\Big\|_p \leqq C\|f\|_p,$$

if $1+\dfrac{1}{2a}<p\leqq 2$. This is Theorem 2.2 of [CM]. We obtain the following result.

$$\Big\|\sup_{t>0}|t^{n((1/p)-(1/r))}F_x(t)|\Big\|_r \leqq C\|f\|_p.$$

Here $1+\dfrac{1}{2a}<p\leqq 2$ and $p\leqq r\leqq p'$.

The extension from $C_0^\infty(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$ follows as in [SWa, p. 1285—1287].

Proof of Theorem 3. The proof is partly contained in the proof of Theorem 2. But if we in this case use that $a=\dfrac{n-1}{2}$, and set

$$F_{x,z}(t) = t^{(n(z-\beta))/(1-\beta)}M_t^z f(x), \quad t>0,$$

then (17) can be rewritten as

$$\Big\|\sup_{t>0}|t^{n((2/p)-1)}M_t^\varrho f|\Big\|_{p'} \leqq C\|f\|_p,$$

with $1\leqq p\leqq 2$ and $\varrho>1-\frac{n}{p'}$. Interpolating this with Stein's estimate [St2, Th. 2]

$$\Big\|\sup_{t>0}|M_t^\varrho f|\Big\|_p \leqq C\|f\|_p,$$

where $1<p\leqq 2$ and $\varrho>1-\frac{n}{p'}$, gives the full result.

5. Further results

(1) can be compared to the following estimate.

$$\Big(\int_{\mathbf{R}^n} \|\varphi F_x\|_{H_2^\beta}\, dx\Big)^{1/2} \leqq C\,\|f\|_p, \tag{18}$$

for $\frac{2n}{n+2a} \leqq p \leqq 2$ and $\beta = n\Big(\frac{1}{2}-\frac{1}{p}\Big)+a(m\geqq 1)$. (18) is the result of interpolation between the end-point $\beta=\frac{2a}{p'}$, $p=2$, which is also an end-point for α in (1), and the end-point $\beta=0$, $p=\frac{2n}{n+2a}$. The latter can be shown by an application of the continuity of the Riesz potential I_a to (8) (see [St1, Ch. V, 1]), viz.

$$\begin{aligned}\Big(\int_{\mathbf{R}^n} \|\varphi F_x\|^2_{H_2^0}\, dx\Big)^{1/2} &\leqq C\Big(\int_{\mathbf{R}^n} \big|(1+|\xi|)^{-a}\hat f(\xi)\big|^2\, d\xi\Big)^{1/2}\\ &\leqq C\,\|\widehat{I_a f}\|_2 = C\,\|I_a f\|_2\\ &\leqq C\,\|f\|_p,\end{aligned}$$

if $\frac{1}{2}=\frac{1}{p}-\frac{a}{n}$ or (equivalently) $p=\frac{2n}{n+2a}$. For $\frac{n}{n-\frac{1}{2}} \leqq p \leqq 2$, $a=\frac{n-1}{2}$, $m=1$ and $r=2$ we obtain the corollary from (18) by the same type of embeddings which proved the corollary. Cf. [S2, pp. 282—283]. Note that this doesn't require any extra assumption on $\psi(t,y)$ or on the orientation of S as was the case in (3). Trying this method for $a<\frac{n-1}{2}$ we get weaker results than the corollary gives for $r=2$. Consider for example $S=S^k\times[-1,1]^{n-k-1}$, $1\leqq k\leqq n-1$, with measure $\mu(x)\,d\sigma(x)= d\theta(x')\otimes\varpi(x'')\,dx''$ defined as in the proof of the corollary. Estimating the Fourier transform $m=\widehat{\mu\, d\sigma}$ of this measure gives, for an arbitrary number M, that

$$\begin{aligned}|m(\xi)| &= \Big|\int_S e^{-i\xi\cdot x}\mu(x)\,d\sigma(x)\Big|\\ &= \Big|\int_{S^k} e^{-i\xi'\cdot x'}\,d\theta(x')\int_{\mathbf{R}^{n-k-1}} e^{-i\xi''\cdot x''}\varpi(x'')\,dx''\Big|\\ &= |\mathscr{F}'\,d\theta(\xi')\cdot\mathscr{F}''\varpi(\xi'')|\\ &= \big||\xi'|^{-((k-1)/2)}J_{(k-1)/2}(|\xi'|)\cdot\mathscr{F}''\varpi(\xi'')\big|\\ &\leqq C(1+|\xi'|)^{-(k/2)}(1+|\xi''|)^{-M}.\end{aligned}$$

Here $\mathscr{F}'$ and $\mathscr{F}''$ denote the Fourier transforms in $\mathbf{R}^{k+1}$ and $\mathbf{R}^{n-k-1}$ respectively. The computation of $\mathscr{F}'\,d\theta$ and the estimate of the Bessel function can be found in [SWe, pp. 154 and 158]. Thus, the decay of $m(\xi)$ is better in some directions.

The Riesz potential $I'_{k/2}$ in $\mathbf{R}^{k+1}$ is bounded as an operator from $L^p(\mathbf{R}^{k+1})$ to $L^2(\mathbf{R}^{k+1})$, if $\frac{1}{2}=\frac{1}{p}-\frac{k/2}{k+1}$, i.e. $1<p=\frac{k+1}{k+\frac{1}{2}}<2$. For this p choose M such that the Riesz potential in $\mathbf{R}^{n-k-1}$ satisfies $I''_M\colon L^p(\mathbf{R}^{n-k-1})\to L^2(\mathbf{R}^{k-n-1})$. By Minkowski's inequality for integrals we get an improved estimate of the $L^2(H_2^0)$-norm.

$$
\begin{aligned}
&\Big(\int_{\mathbf{R}^n}\|\varphi F_x\|^2_{H_2^0}\,dx\Big)^{1/2}\\
&= C\Big(\int_{\mathbf{R}^n}\int_{\mathbf{R}^m}|\varphi(t)\,m(\psi(t,\xi'),\psi(t,\xi''))\hat{f}(\xi)|^2\,dt\,d\xi\Big)^{1/2}\\
&\leqq C\Big(\int_{\mathbf{R}^n}\int_{\mathbf{R}^m}|\varphi(t)(1+|\psi(t,\xi')|)^{-k/2}(1+|\psi(t,\xi'')|)^{-M}|^2\,dt\,|\hat{f}(\xi)|^2\,d\xi\Big)^{1/2}\\
&\leqq C\Big(\int_{\mathbf{R}^n}|(1+|\xi'|)^{-k/2}(1+|\xi''|)^{-M}\hat{f}(\xi)|^2\,d\xi\Big)^{1/2}\\
&\leqq C\Big(\int_{\mathbf{R}^{n-k-1}}|\xi''|^{-2M}\int_{\mathbf{R}^{k+1}}\big||\xi'|^{-k/2}\mathscr{F}'\mathscr{F}''f(\xi',\xi'')\big|^2\,d\xi'\,d\xi''\Big)^{1/2}\\
&= C\Big(\int_{\mathbf{R}^{n-k-1}}|\xi''|^{-2M}\int_{\mathbf{R}^{k+1}}|\mathscr{F}'I'_{k/2}\mathscr{F}''f(\xi',\xi'')|^2\,d\xi'\,d\xi''\Big)^{1/2}\\
&\leqq C\Big(\int_{\mathbf{R}^{n-k-1}}|\xi''|^{-2M}\Big(\int_{\mathbf{R}^{k+1}}|\mathscr{F}''f(x',\xi'')|^p\,dx'\Big)^{2/p}d\xi''\Big)^{1/2}\\
&= C\Big(\int_{\mathbf{R}^{n-k-1}}\Big(\int_{\mathbf{R}^{k+1}}\big||\xi''|^{-M}\mathscr{F}''f(x',\xi'')\big|^p\,dx'\Big)^{2/p}d\xi''\Big)^{(p/2)\cdot(1/p)}\\
&\leqq C\Big(\int_{\mathbf{R}^{k+1}}\Big(\int_{\mathbf{R}^{n-k-1}}\big||\xi''|^{-M}\mathscr{F}''f(x',\xi'')\big|^2\,d\xi''\Big)^{p/2}dx'\Big)^{1/p}\\
&= C\Big(\int_{\mathbf{R}^{k+1}}\Big(\int_{\mathbf{R}^{n-k-1}}|\mathscr{F}''I''_M f(x',\xi'')|^2\,d\xi''\Big)^{p/2}dx'\Big)^{1/p}\\
&\leqq C\Big(\int_{\mathbf{R}^{k+1}}\Big(\int_{\mathbf{R}^{n-k-1}}|f(x',x'')|^p\,dx''\Big)^{(1/p)\cdot p}dx'\Big)^{1/p}\\
&= \|f\|_p,
\end{aligned}
$$

where we have set

$$\psi(t,\xi')=(\psi_1(t)\,\xi_1,\ldots,\psi_{k+1}(t)\,\xi_{k+1})$$

and

$$\psi(t,\xi'')=(\psi_{k+2}(t)\xi_{k+2},\ldots,\psi_n(t)\xi_n.)$$

Taking $k<n-1$, $a=\frac{k}{2}$, and $p=\frac{2n}{n+2a}=\frac{2n}{n+k}$ in (18) ($\beta=0$) we obtain a p which is larger than $\frac{k+1}{k+\frac{1}{2}}$. We note also that $\frac{k+1}{k+\frac{1}{2}}$ is the value of p in (4) of the corollary if $q=r=2$ and $a=\frac{k}{2}\,(m=1)$.

That the decay of $\widehat{\mu\, d\sigma}(\xi)=m(\xi)$ is better, in certain directions, than $|\xi|^{-a}$, if $a<\frac{n-1}{2}$, is a universal property of smooth measures. This is the content of the following theorem of Erik Svensson [Sv].

Theorem. *Assume that S has k principal curvatures different from zero. Let $\varkappa(\xi)$ be the least acute angle that a vector ξ forms with any normal vector of S. Then, for any positive number R,*

$$|m(\xi)| \leqq C_R(1+|\xi|)^{-k/2}(1+|\xi| \sin \varkappa(\xi))^{-R}.$$

This improves W. Littman's [L] estimates of the Fourier transform of this kind of measures.

If $\psi(t, y)=ty$, $\varphi\in C_0^\infty(\mathbf{R})$ and $\eta>\eta_0=\frac{n}{r'}-1-\frac{n-1-2a}{p'}$, then (4)—(6) holds with φF_x replaced by $|t|^\eta \varphi F_x$. A sketch of the proof goes as follows. $|t|^\eta\varphi F_x$ is split up in a sum $\sum_{k=1}^\infty |t|^\eta \varphi_k F_x$ by a dyadic partition of unity. Each $|t|^\eta\varphi_k F_x$ is estimated by (4)—(6) followed by a dilation argument which collects the dependence on k in the constant $2^{k\eta_0}C$ replacing the constant C in (4)—(6). Then by summing the geometric series $\sum_{k=1}^\infty 2^{-k(\eta-\eta_0)}$, which converges if $\eta>\eta_0$, gives the desired result. Note that the bound η_0 becomes independent of p if $a=\frac{n-1}{2}$. In the case $S=S^{n-1}\left(a=\frac{n-1}{2}\right)$; η_0 is best possible. For details see [S4].

If we also admit the smooth mass μ to depend smoothly on $t\in\mathbf{R}^m$ and substitute the growth condition on the Fourier transform of $\mu\, d\sigma$ by

$$\left|\sum_{k=1}^m D_k^N(\mu_t\, d\sigma)^\wedge(\xi)\right| \leqq C_N(1+|\xi|)^{-a},$$

for $t\in\operatorname{supp}\varphi$ and $N=0, 1, \ldots, [a]+1$, then (1)—(6) are still valid for the average defined by the measure $\mu_t d\sigma$. Here $D_k^N=\frac{\partial^N}{\partial t_k^N}$ and $[a]$ is the integer part of a. This can be seen as follows.

We compute the derivatives of

$$m_t(\psi(t, \xi)) = (\mu_t\, d\sigma)^\wedge(\psi(t, \xi))$$

and get

$$\begin{aligned} D_k^j(m_t(\psi(t, \xi))) &= \int_S D_k^j(e^{-ix\cdot\psi(t,\xi)}\mu_t(x))\, d\sigma(x) \\ &= \int_S \sum_{l=0}^j D_k^{j-l}(e^{-ix\cdot\psi(t,\xi)}) D_k^l\mu_t(x)\, d\sigma(x) \\ &= \sum_{l=0}^j \sum_{|\gamma|=j-l} P_k^\gamma(t)\xi^\gamma \int_S e^{-ix\cdot\psi(t,\xi)} x^\gamma D_k^l\mu_t(x)\, d\sigma(x), \end{aligned}$$

for $j=1, \ldots, N$. But, since

$$\left|((D_k^l \mu_t)\, d\sigma)\hat{}\,(\xi)\right| = |D_k^l(\mu_t\, d\sigma)\hat{}\,(\xi)| \leqq C(1+|\xi|)^{-a},$$

if $t \in \operatorname{supp} \varphi$ and $l=1, \ldots, N$, we get by the same reasons as before that

$$\left|\int_S e^{-ix\cdot\xi} x^\gamma D_k^l \mu_t(x)\, d\sigma(x)\right| \leqq C(1+|\xi|)^{-a},$$

and consequently

$$\left|D_k^j(m_t(\psi(t, \xi)))\right| \leqq \sum_{|\gamma| \leqq j} |P_k^\gamma(t)\xi^\gamma| \cdot C(1+|\psi(t, \xi)|)^{-a} \leqq C(1+|\xi|)^{j-a},$$

for $t \in \operatorname{supp} \varphi$ and $l=1, \ldots, N$. From this, the desired L^2 estimate follows as before and also the L^1 and L^∞ estimates, since μ_t is bounded on $\operatorname{supp} \varphi$. The interpolation and the extension of f to $L^p(\mathbf{R}^n)$ is as before.

Let u be a compactly supported distribution in $\mathbf{R}^n$. Define u_t by

$$\langle u_t, f\rangle = \langle u, f(\psi(t, y))\rangle_{[y]},$$

where f is a test function and $\langle\, ,\rangle_{[y]}$ indicates that we apply the distribution on test functions of y, and an average by

$$F_x(t) = u_t * f(x) = \langle u, f(x-\psi(t, y))\rangle_{[y]},$$

for f in $C_0^\infty(\mathbf{R}^n)$. Then

$$(u_t * f)\hat{} = \hat{f}\hat{u}_t$$

as distributions. But since u_t has compact support for t in $\operatorname{supp} \varphi$ the Fourier transform of u_t is given by the function

$$\hat{u}_t(\xi) = \langle u_t, e^{-i\xi\cdot y}\rangle_{[y]} = \langle u, e^{-i\xi\cdot\psi(t, y)}\rangle_{[y]} = \langle u, e^{-i\psi(t, \xi)\cdot y}\rangle_{[y]} = \hat{u}(\psi(t, \xi)),$$

for $t \in \operatorname{supp} \varphi$ (see [GS, pp. 196—197]), and we get that

$$\hat{F}_\xi(t) = \hat{u}(\psi(t, \xi))\hat{f}(\xi).$$

Hence, if $|\hat{u}(\xi)| \leqq C(1+|\xi|)^{-a}$, $a \geqq 0$, Theorem 1 and its corollary are true also in the case $p=2$, for $f \in C_0^\infty(\mathbf{R}^n)$, because we consider only the Fourier transform of $F_x(t)$, and $\langle u, e^{-ix\cdot\xi} x^\gamma\rangle_{[x]}$ decreases like $\hat{u}(\xi)$. Cf. [CM, Th. 1.4].

We get an example by taking u equal to the function $(\max(1-|x|^2, 0))^\delta$, $\delta > -1$, where $a = \frac{n+1}{2} + \delta$ will suffice.

References

BP. Benedek A. and Panzone R., The spaces L^p, with mixed norm, *Duke Math. J.* **28** (1961), 301—324.

BL. Bergh, J. and Löfström, J., *Interpolation spaces*, Springer-Verlag, Berlin—Heidelberg—New York, 1976.

Bö. Börjeson, L., Mixed norm estimates for certain means, to appear in *Trans. Amer. Math. Soc.*

B1. Bourgain, J., Estimations de certaines fonctions maximales, *C.R. Acad. Sc. Paris* **301** (1985), 499—502.

B2. Bourgain, J., On the spherical maximal function in the plane, *Preprint IHES*, June 1985.

B3. Bourgain, J., Averages in the plane over convex curves and maximal operators, *J. Analyse Math.* **47** (1986), 69—85.

CS. Carlsson, H. and Sjögren, P., Estimates for maximal functions along hypersurfaces, *Ark. Mat.* **25** (1987), 1—14.

CSS. Carlsson, H., Sjögren, P. and Strömberg, J.-O., Multiparameter maximal functions along dilation-invariant hypersurfaces, *Trans. Amer. Math. Soc.* **292** (1985), 335—343.

C. Colzani, L., Regularity of spherical means and localization of spherical harmonic expansions, *J. Austral. Math. Soc. Ser. A* **41** (1986), 287—297.

CM. Cowling, M. and Mauceri, G., Inequalities for some maximal functions. II, *Trans. Amer. Math. Soc.* **296** (1986), 341—365.

GS. Gel'fand I. M. and Shilov, G. E., *Generalized functions, Vol. 1.* Academic Press, New York—London, 1966.

G. Greenleaf, A., Principal curvature and harmonic analysis, *Indiana Math. J.* **30** (1981), 519—537.

HP. Hille, E. and Phillips, R. S., *Functional analysis and semi-groups*, Amer. Math. Soc. Coll., Providence, 1957.

L. Littman, W., Fourier transforms of surface-carried measures and differentiability of surface averages, *Bull. Amer. Math. Soc.* **69** (1963), 766—770.

OS. Oberlin, D. M. and Stein, E. M., Mapping properties of the Radon transform, *Indiana Univ. Math. J.* **31** (1982), 641—650.

PS. Peyrière J. and Sjölin, P., Regularity of spherical means, *Ark. Mat.* **16** (1978), 117—126.

R. Rubio de Francia, J. L., Maximal functions and Fourier transforms, *Duke Math. J.* **53** (1986), 395—404.

S1. Sjölin, P., Lipschitz continuity of spherical means, in *Linear spaces and approximation, Proc. Conf. Oberwolfach, 1977*, Ed. by P. L. Butzer and B. Sz.-Nagy, Birkhäuser, 1978, 229—234.

S2. Sjölin, P., Regularity and integrability of spherical means, *Monatshefte f. Math.* **96** (1983), 277—291.

S3. Sjölin, P., Regularity and integrability properties of spherical means, *Proc. of the Nineteenth Nordic Congress of Mathematicians*, Reykjavík, 1984, 215—218.

S4. Sjölin, P., Regularity properties of spherical means, in *The Proceedings of Alfred Haar Memorial Conference, Budapest, 1985*, Colloq. Math. Soc. János Bolyai 49, 823—846.

S5. Sjölin, P., Norm inequalities for spherical means, *Monatshefte f. Math.* **100** (1985), 153—161.

So. SOGGE, C. D., On maximal functions associated to hypersurfaces and the Cauchy problem for strictly hyperbolic operators, *Trans. Amer. Math. Soc.* **304** (1987), 733—749.

SS1. SOGGE, C. D. and STEIN, E. M., Averages of functions over hypersurfaces in R^n, *Invent. Math.* **82** (1985), 543—556.

SS2. SOGGE, C. D. and STEIN, E. M., Averages over hypersurfaces: II, *Invent. Math.* **86** (1986), 233—242.

St1. STEIN, E. M., *Singular integrals and differentiability properties of functions,* Princeton University Press, Princeton, 1970.

St2. STEIN, E. M., Maximal functions: Spherical means, *Proc. Nat. Acad. Sci. U.S.A.* **73** (1976), 2174—2175.

SWa. STEIN, E. M. and WAINGER, S., Problems in harmonic analysis related to curvature, *Bull. Amer. Math. Soc.* **84** (1978), 1239—1295.

SWe. STEIN, E. M. and WEISS G., *Introduction to Fourier analysis in Euclidean spaces,* Princeton University Press, Princeton, 1971.

Sv. SVENSSON, E., Fourier—Stieltjestransformer av mått på developpabla ytor, Mimeographed manuscript, Uppsala, May 1972.

T. TRIEBEL, H., *Interpolation theory, function spaces, differential operators,* North-Holland, Amsterdam, New York, Oxford, 1978.

Wa. WAINGER, S., Averages and singular integrals over lower dimensional sets, in *Beijing Lectures in Harmonic Analysis* (ed. E. M. Stein), Annals Math. Studies, No. 112, Princeton Univ. Press, 1986, pp. 357—421.

W. WATSON, G. N., *Theory of Bessel functions,* Cambridge University Press, Cambridge, 1944.

Received Aug. 1, 1988

Lennart Börjeson
University of Stockholm
Department of Mathematics
Box 6701
S-113 85 Stockholm
Sweden