On interpolation of compact operators

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To the memory of Professor José L. Rubio de Francia

0. Introduction

In 1964, J. L. Lions and J. Peetre [7] established the following compactness theorem

Theorem L-P. *Let* (A_0, A_1) *and* (B_0, B_1) *be compatible couples of Banach spaces, and let* T *be a linear operator such that* $T\colon A_0 \to B_0$ *is compact and* $T\colon A_1 \to B_1$ *is continuous.*

i) *If* $B_0 = B_1$ *and* E *is a Banach space of class* $\mathscr{C}_K(\theta; \bar{A})$, *then* $T\colon E \to B_0$ *is compact.*

ii) *If* $A_0 = A_1$ *and* E *is a Banach space of class* $\mathscr{C}_J(\theta; \bar{B})$, *then* $T\colon A_0 \to E$ *is compact.*

This originated the question of whether there are generalizations of Theorem L-P to the case $A_0 \neq A_1$ and $B_0 \neq B_1$.

Assuming a certain approximation condition on the couple (B_0, B_1), A. Persson [8] was able to give a positive answer (see also the papers by M. A. Krasnosel'skiĭ [5] and S. G. Kreĭn—Ju. I. Petunin [6]).

For the general case without an approximation property, some positive results are also known. They refer to the real interpolation method $(\cdot\,,\cdot)_{\theta,q}$ that, as is well-known, produces spaces of class $\mathscr{C}_K(\theta)$ and $\mathscr{C}_J(\theta)$.

In 1969, K. Hayakawa [4] stated that if $T\colon A_j \to B_j$ is compact for $j=0, 1$, $0<\theta<1$ and $1 \leqq q < \infty$, then real interpolation preserves compactness of the operator T. A transparent proof of this result (covering also the cases $0<q<1$ and $q=\infty$) has been given very recently by D. E. Edmunds, A. J. B. Potter and the

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first named author [2]. They also proved that the same conclusion holds when the assumption

$$T\colon A_1 \to B_1 \quad \text{compactly}$$

is replaced by

$$B_1 \text{ continuously embedded in } B_0.$$

Note that this last result is a natural extension of Theorem L-P/(i).

The aim of this paper is to show that the corresponding natural extension of part (ii) in Theorem L-P also holds (see Th. 2.1).

In contrast to [2], where the description of the real interpolation space through the K-functional is successfully used, our approach here will be based on the J-functional. The main ideas of these techniques were developed by K. Hayakawa [4], but in a rather involved way.

We shall also show how to derive by means of the J-functional the extended version of Hayakawa's result given in [2].

1. Preliminaries

Let (A_0, A_1) be a compatible couple of Banach spaces [that is A_0 and A_1 are continuously embedded in some Hausdorff topological vector space $\mathcal{A}$]. We equip $A_0 \cap A_1$ [respectively $A_0 + A_1$] with the norm $J(1, .)$ [respectively $K(1, .)$] where for $t > 0$

$$J(t, .) = J(t, .; A_0, A_1) \quad \text{and} \quad K(t, .) = K(t, .; A_0, A_1)$$

are the functionals of J. Peetre, defined by

$$J(t, a) = \max \{\|a\|_{A_0}, t\|a\|_{A_1}\}$$

and

$$K(t, a) = \inf \{\|a_0\|_{A_0} + t\|a_1\|_{A_1} \colon a = a_0 + a_1,\ a_0 \in A_0,\ a_1 \in A_1\}.$$

It is not hard to see that $A_0 \cap A_1$ and $A_0 + A_1$ are Banach spaces.

For $0 < \theta < 1$ and $0 < q \leqq \infty$, the real interpolation space $(A_0, A_1)_{\theta, q}$ consists of all $a \in A_0 + A_1$ which have a finite quasi-norm

$$\|a\|_{\theta, q} = \Big(\int_0^\infty (t^{-\theta} K(t, a))^q \, dt/t\Big)^{1/q} \quad (\text{if } 0 < q < \infty)$$

$$\|a\|_{\theta, \infty} = \sup_{t > 0} \{t^{-\theta} K(t, a)\}.$$

One can check that if $1 \leqq q \leqq \infty$ then $\big((A_0, A_1)_{\theta, q}, \|\ \|_{\theta, q}\big)$ is a Banach space, but if $0 < q < 1$ it is in general only a complete quasi-normed space (see [1] and [9]).

Let (B_0, B_1) be another compatible couple of Banach spaces, and let T be a linear operator which maps A_j continuously into B_j $(j=0, 1)$. The following interpolation property holds

$$\|T\|_{\theta,q} \leqq \|T\|_0^{1-\theta} \|T\|_1^{\theta}$$

where $\|T\|_0$, $\|T\|_1$ and $\|T\|_{\theta,q}$ are the norms of T as a mapping from A_0 to B_0, A_1 to B_1 and $(A_0, A_1)_{\theta,q}$ to $(B_0, B_1)_{\theta,q}$ respectively.

In order to establish the main results of [2] certain vector valued l_∞ spaces modelled on the sum B_0+B_1 are used. We shall require here vector-valued l_1 spaces modelled on the intersection $A_0 \cap A_1$.

Let $m=0, -1, -2, \ldots$ and denote by G_m the Banach space $A_0 \cap A_1$ endowed with the norm $J(e^m, .)$; for any θ with $0 \leqq \theta \leqq 1$, let $e^{-\theta m} G_m$ stand for the Banach space $(A_0 \cap A_1, e^{-\theta m} J(e^m, .))$. We designate by $l_q(e^{-\theta m} G_m)$ the collection of all sequences $(u_m)_{m=0}^{-\infty} \subset A_0 \cap A_1$ such that the quasi-norm

$$|||(u_m)|||_{\theta,q} = \left(\sum_{m=0}^{-\infty} (e^{-\theta m} J(e^m, u_m))^q\right)^{1/q} \quad (\text{if } 0 < q < \infty)$$

$$|||(u_m)|||_{\theta,\infty} = \sup \{e^{-\theta m} J(e^m, u_m)\}$$

is finite.

For later use we shall now state without proof an interpolation formula between these vector-valued sequence spaces.

Lemma 1.1. *Let $0<\theta<1$ and $0<q\leqq\infty$. Then we have with equivalent quasi-norms*

$$(l_1(G_m), l_1(e^{-m} G_m))_{\theta,q} = l_q(e^{-\theta m} G_m).$$

We end this section with a lemma that shows the relationship between $l_q(e^{-\theta m} G_m)$ and $(A_0, A_1)_{\theta,q}$. The lemma can be checked by adapting the proof of the Equivalence Theorem (see [1], 3.3 and 3.11).

Lemma 1.2. *Assume that A_0 and A_1 are Banach spaces with A_1 continuously embedded in A_0. Let $0<\theta<1$ and $0<q\leqq\infty$. Then $a\in(A_0, A_1)_{\theta,q}$ if and only if there exists a sequence $(u_m)_{m=0}^{-\infty} \subset A_0 \cap A_1$ with*

(1.1) $$a = \sum_{m=0}^{-\infty} u_m \quad (\text{convergence in } A_0+A_1)$$

and

(1.2) $$|||(u_m)|||_{\theta,q} < \infty.$$

Moreover

$$\|a\|_{\theta,q} \sim \inf \{|||(u_m)|||_{\theta,q}\}$$

where the infimum is extended over all sequences (u_m) satisfying (1.1) *and* (1.2).

2. Main results

Next we state the compactness theorem.

Theorem 2.1. *Let* (B_0, B_1) *be a compatible couple of Banach spaces and suppose that* A_0, A_1 *are Banach spaces such that* A_1 *is continuously embedded in* A_0. *Let* T *be a linear operator such that*

$$T\colon A_0 \to B_0 \quad \textit{is bounded}$$

and

$$T\colon A_1 \to B_1 \quad \textit{is compact.}$$

Then if $0<\theta<1$ *and* $0<q\leqq\infty$,

$$T\colon (A_0, A_1)_{\theta,q} \to (B_0, B_1)_{\theta,q}$$

is compact.

Proof. Given $(u_m)\in l_1(G_m)$, $n=1, 2, \ldots$ we put

$$P_n(u_m) = (u_0, u_{-1}, \ldots, u_{-n}, 0, 0, \ldots).$$

Each one of these operators is linear and bounded on $l_1(e^{-jm}G_m)$ where $j=0, 1$, and its norm is equal to 1. Consider also the bounded linear operator

$$Q\colon l_1(e^{-jm}G_m) \to A_j, \quad j=0, 1,$$

defined by

$$Q(u_m) = \sum_{m=0}^{-\infty} u_m$$

and write $\hat{T}=T\circ Q$. We shall first show that the bounded operator

$$\hat{T}\colon \big(l_1(G_m), l_1(e^{-m}G_m)\big)_{\theta,q} \to (B_0, B_1)_{\theta,q}$$

is compact.

The following diagram holds

$$\big(l_1(G_m), l_1(e^{-m}G_m)\big)_{\theta,q} \xrightarrow{P_n} l_1(e^{-m}G_m) \begin{cases} \nearrow l_1(G_m) \xrightarrow{\hat{T}} B_0 \\ \searrow l_1(e^{-m}G_m) \xrightarrow{\hat{T}} B_1. \end{cases}$$

In addition, compactness of T as an operator from A_1 into B_1 implies that

$$\hat{T}P_n = TQP_n\colon \big(l_1(G_m), l_1(e^{-m}G_m)\big)_{\theta,q} \to B_1$$

is compact. Whence, applying Theorem L-P/(ii), we have that

$$\hat{T}P_n\colon \big(l_1(G_m), l_1(e^{-m}G_m)\big)_{\theta,q} \to (B_0, B_1)_{\theta,q}$$

is also compact.

Hence, for the purpose of proving compactness of $\hat{T}$, it is enough to see that there exists a subsequence $(\hat{T}P_{n'})$ of $(\hat{T}P_n)$ such that

$$\|\hat{T}-\hat{T}P_{n'}\|_{\theta,q} \to 0 \quad \text{as} \quad n' \to \infty.$$

But

$$\|\hat{T}-\hat{T}P_{n'}\|_{\theta,q} \leqq \|\hat{T}-\hat{T}P_{n'}\|_0^{1-\theta}\|\hat{T}-\hat{T}P_{n'}\|_1^{\theta}.$$

Thus we only need to show that for some subsequence $(\hat{T}P_{n'})$

$$\|\hat{T}-\hat{T}P_{n'}\|_1 \to 0 \quad \text{as} \quad n' \to \infty.$$

With this aim, first note that

$$\|\hat{T}-\hat{T}P_n\|_1 \leqq \|\hat{T}\|_1$$

so, there is a subsequence $(\hat{T}P_{n'})$ of $(\hat{T}P_n)$ such that $(\|\hat{T}-\hat{T}P_{n'}\|_1)$ converges. Let λ be the limit. We can find $(x_{n'})\subset l_1(e^{-m}G_m)$ such that $|||x_{n'}|||_{1,1}\leqq 1$ and

$$\|\hat{T}(I-P_{n'})x_{n'}\|_{B_1} = \|(\hat{T}-\hat{T}P_{n'})x_{n'}\|_{B_1} \to \lambda \quad \text{as} \quad n' \to \infty.$$

Call $y_{n'}=(I-P_{n'})x_{n'}$. Then we obtain a sequence $(y_{n'})\subset l_1(e^{-m}G_m)$ satisfying

$$|||y_{n'}|||_{1,1} \leqq 1, \quad P_k y_{n'} = 0 \quad \text{if} \quad k \leqq n',$$

and

$$\|\hat{T}y_{n'}\|_{B_1} \to \lambda \quad \text{as} \quad n' \to \infty.$$

Now, since $\hat{T}\colon l_1(e^{-m}G_m)\to B_1$ is compact, there exists a subsequence $(y_{n''})$ of $(y_{n'})$ such that $(\hat{T}y_{n''})$ converges to some $b\in B_1$. In particular $\|b\|_{B_1}=\lambda$ and $(\hat{T}y_{n''})$ also converges to b in B_0+B_1.

On the other hand, if $k\geqq n''$, it follows from

$$|||(P_{k+1}-P_k)y_{n''}|||_{0,1} = e^{-(k+1)}|||(P_{k+1}-P_k)y_{n''}|||_{1,1} \leqq e^{-(k+1)}|||y_{n''}|||_{1,1} \leqq e^{-(k+1)}$$

that

$$\|\hat{T}(P_{k+1}-P_k)y_{n''}\|_{B_0} \leqq \|\hat{T}\|_0 e^{-(k+1)}.$$

Whence

$$\begin{aligned}\|\hat{T}y_{n''}\|_{B_0+B_1} &= \left\|\hat{T}\textstyle\sum_{k\geqq n''}(P_{k+1}-P_k)y_{n''}\right\|_{B_0+B_1} \\ &\leqq \textstyle\sum_{k\geqq n''}\|\hat{T}(P_{k+1}-P_k)y_{n''}\|_{B_0} \\ &\leqq \|\hat{T}\|_0\textstyle\sum_{k\geqq n''}e^{-(k+1)} \to 0 \quad \text{as} \quad n'' \to \infty.\end{aligned}$$

This yields $b=0$, and so $\lambda=0$. Therefore the operator

$$\hat{T}\colon \big(l_1(G_m), l_1(e^{-m}G_m)\big)_{\theta,q} \to (B_0, B_1)_{\theta,q} \quad \text{is compact.}$$

Next, in view of Lemma 1.1, we derive that the composition

$$l_q(e^{-\theta m}G_m)\xrightarrow{Q}(A_0, A_1)_{\theta,q}\xrightarrow{T}(B_0, B_1)_{\theta,q}$$

is compact. Finally, we complete the proof by using Lemma 1.2. ∎

Remark 2.2. The procedure used in Theorem 2.1 still works if we assume

$$T\colon A_0 \to B_0 \quad \text{compactly}$$

instead of

$$A_1 \hookrightarrow A_0.$$

In such a case the sequence spaces should be over $\mathbf{Z}$, operators P_n should be defined by

$$P_n(u_m) = (\ldots, 0, 0, u_{-n}, u_{-n+1}, \ldots, u_0, \ldots, u_{n-1}, u_n, 0, 0, \ldots)$$

and the projections

$$P_+(u_m) = (\ldots, 0, 0, u_0, \ldots, u_n, u_{n+1}, \ldots),$$

$$P_- = I - P_+$$

are also needed. Note that now

$$\|\hat{T}-\hat{T}P_n\|_{\theta,q} \leqq c[\|(\hat{T}-\hat{T}P_n)P_-\|_{\theta,q} + \|(\hat{T}-\hat{T}P_n)P_+\|_{\theta,q}]$$

where C is the constant in the quasi-triangle inequality for $\|\cdot\|_{\theta,q}$. Hence, in order to show that $(\hat{T}P_n)$ has a subsequence which converges to $\hat{T}$, we can proceed with $\|(\hat{T}-\hat{T}P_n)P_-\|_{\theta,q}$ as before, and then we can treat $\|(\hat{T}-\hat{T}P_n)P_+\|_{\theta,q}$ with a similar reasoning but using the fact that

$$T\colon A_0 \to B_0 \quad \text{is compact.}$$

In this way we derive Hayakawa's result (covering also the cases $0<q<1$ and $q=\infty$) by means of the J-functional. The resulting proof is, on the one hand, much more direct and simple than the original one [4], but on the other hand, it is slightly more involved than the proof given in [2] using the K-functional.

Remark 2.3. The techniques used in Theorem 2.1 also work for the (more general) method of interpolation with a function parameter. We refer to [3] for details on this method.

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