On the projective classification of smooth n-folds with n even

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Let $Y \subset \mathbf{P}_{\mathbf{C}}$ be an irreducible, n dimensional, projective variety with a smooth normalization $\alpha: M \to Y$ and let $\mathscr{L} = \alpha^* \mathcal{O}_{\mathbf{P}}(1)_Y$. Recent results of [5], [6], [16] imply that either $(M, \mathscr{L})$ is one of a list of specific, well understood polarized varieties or there is a projective manifold X and an ample line bundle L on X such that:

a) M is the blowup $\pi: M \to X$ of X at a finite set F,
b) $K_M \otimes \mathscr{L}^{n-1} = \pi^*(K_X \otimes L^{n-1})$ where $K_X \otimes L^{n-1}$ is ample and spanned by global sections,
c) $L = [\pi(S)]$ for a smooth $S \in |\mathscr{L}|$, or equivalently $\mathscr{L} = \pi^*(L) \otimes [\pi^{-1}(F)]^{-1}$,
d) $K_X \otimes L^{n-2}$ is semi-ample and big, i.e. some positive power $(K_X \otimes L^{n-2})^t$ is spanned by global sections and the map $\alpha: X \to \mathbf{P}_{\mathbf{C}}$ associated to $\Gamma((K_X \otimes L^{n-2})^t)$ has an n dimensional image.

The pair (X, L) is called the first reduction of $(M, \mathscr{L})$ and is very well behaved, see [12], [14] and [17]. It is easy to convert information between (X, L) and $(M, \mathscr{L})$.

Let $\Phi \circ s = \alpha$ be the Remmert—Stein factorization of the map α (in d) above) where $\Phi: X \to X'$ has connected fibres for a normal projective X', and $s: X' \to \mathbf{P}_{\mathbf{C}}$ is finite to one. There is an ample line bundle $\mathscr{K}$ on X' such that $\Phi^* \mathscr{K} = K_X \otimes L^{n-2}$. The pair $(X', \mathscr{K})$ is known as the 2^{nd} reduction and the map Φ is called the second adjunction map. Such pairs have been studied by the authors [4], [15]. X' has only isolated singularities, is 2-factorial and Gorenstein in even dimensions. Thus for n even, $n \geqq 4$, $\mathscr{K} = K_{X'} \otimes L'^{n-2}$ where for a smooth $A \in |L|$, $2\Phi(A)$ is Cartier, i.e. $[2\Phi(A)]$ is invertible and L' is 2-Cartier. This pleasant circumstance makes the 2^{nd} reduction almost as easy to use as the first reduction when n is even, and allows us to use the results of Fujita [5] in this case. Combining this with a recent result [2] we can push the known classification a good deal further. To state our main result it is useful to recall the notion of the spectral value of a pair.

Let $\mathscr{H}$ be a nef and big line bundle on a normal projective variety $\mathscr{X}$ of dimension $n\geqq 1$. In [16] the *spectral value*, $\sigma(\mathscr{X}, \mathscr{H})$, of the pair $(\mathscr{X}, \mathscr{H})$ is defined as the smallest real number τ such that given any fraction $p/q>\tau$, $\Gamma(K_{\mathscr{X}}^N\otimes\mathscr{H}^{(n+1-p/q)N})=0$ for all integers $N>0$ with $q|N$.

Note that $\sigma(M, \mathscr{L})=\sigma(X, L)=\sigma(X', L')\geqq 0$.

The normalization used in the above definition of spectral value is very useful in organizing the known results, e.g. $\sigma(M, \mathscr{L})=0$ if and only if $(M, \mathscr{L})=(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1))$, and $\sigma(M, \mathscr{L})=1$ if and only if $(M, \mathscr{L})=(\mathbf{Q}, \mathcal{O}_{\mathbf{Q}}(1))$ where $\mathbf{Q}\subset\mathbf{P}^{n+1}$ is a quadric or $(M, \mathscr{L})$ is a scroll over a curve. The known classification is for $\sigma(M, \mathscr{L})\leqq 3$. In [2] this is pushed to $\sigma(M, \mathscr{L})<4-3/(n+1)$. Our main result is

1.1. Theorem. *Assume* $\sigma(M, \mathscr{L})>3$ *(see* [16] *for the case* $\sigma(M, \mathscr{L})\leqq 3$*). Let* M *be of dimension* n *where* n *is even and* $n\geqq 4$. *Either* $h^0(K_M^{(n^2+1)}\otimes\mathscr{L}^{n(n-1)(n-2)})\neq 0$, *in which case* $\sigma(M, \mathscr{L})\geqq 4-(n+3)/(n^2+1)$ *and* $K_{X'}\otimes\mathscr{K}^{n-1}$ *is semi-ample and big or* (X', L') *is one of the following list:*

$$\sigma(X', L') = 3\tfrac{1}{3} \quad \textit{and} \quad (X', L') = (\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(3)).$$

$$\sigma(X', L') = 3\tfrac{1}{2} \quad \textit{and either a), b), c), or d), holds:}$$

a) $(X', L')=(\mathbf{P}^6, \mathcal{O}_{\mathbf{P}^6}(2))$,

b) $K_{X'}^{-4}=L'^6$, $\dim X'=4$, *and there is an ample line bundle* H *on* X' *with* $H^3=K_{X'}^{-1}$,

c) *there exists a holomorphic map* $\Psi\colon X'\to C$, *where* C *is a curve,* $K_{X'}^2\otimes L'^3=\Psi^*E$ *for an ample line bundle* E *on* C. *Further the general fibre of* Ψ *is* $(\mathbf{Q}^3, \mathcal{O}_{\mathbf{Q}^3}(2))$,

d) *there exists a holomorphic map* $\Psi\colon X'\to S$, *where* S *is a surface,* $K_{X'}^2\otimes L'^3=\Psi^*E$ *for an ample line bundle* E *on* S. *Further the general fibre of* Ψ *is* $(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(2))$.

$\sigma(X', L')=3\tfrac{2}{3}$ *and* $K_{X'}^{-3}=L'^{10}$, $\dim X'=6$, *and there is an ample line bundle* H *on* X' *with* $H^5=K_{X'}^{-1}$.

By passing to a general $H\in|\mathscr{L}|$ we get information about odd dimensional M.

1.1.2. Corollary. *Assume that* $\sigma(M, \mathscr{L})>3$. *Let* M *be of dimension* n *where* n *is odd and* $n\geqq 5$. *Either*

$$[((n-1)^2+1)K_M+(n^3-5n^2+9n-4)\mathscr{L}]\cdot\mathscr{L}$$

and

$$[((n-1)^2+1)K_X+(n^3-5n^2+9n-4)L]\cdot L$$

are effective, or one of the following is true:

a) (X', L') *is the cone on* $(\mathbf{P}^6, \mathcal{O}_{\mathbf{P}^6}(2))$,

b) X' *is* 5 *dimensional and* $K_{X'}^4\otimes L'^{10}=\mathcal{O}_{X'}$,

c) *there is a morphism* $\Psi: X \to C$ *where* C *is a curve,* $L_F = O_{P^4}(2)$ *for a general fibre* F *which is biholomorphic to* $\mathbf{P}^4$,

d) X' *is* 7 *dimensional and* $K_{X'}^6 \otimes L'^{26} = O_{X'}$.

To illustrate the use of these results which actually requires only that $\mathscr{L}$ be ample and spanned on M we give a single representative application in § 2. Let M be an n-fold with $n \geqq 4$ and assume that there is a family of lines on M with a line through most and hence all points of M. Let $t+n-1$ be the dimension of the family where $t \geqq 0$ by the hypothesis on the last line. Then $(M, \mathscr{L})$ has a 2nd reduction on the above lists if

$$n(n-1)(n-2) < (t+2)(n^2+1) \quad \text{and } n \text{ is even,}$$

or if

$$n^3-5n^2+9n-4 < (t+2)[(n-1)^2+1] \quad \text{and } n \text{ is odd.}$$

This should be contrasted with the work of Sato [10].

If $\mathscr{L}$ is very ample and the variety of singular hyperplane sections $\mathscr{H} \subset |\mathscr{L}|$ has codimension $k+1$ then using a theorem of Ein's ([3], see (0.5) for a statement and short proof) it follows that $(M, \mathscr{L})$ has a 2nd reduction on the list for $k>0$ if

$$n(n-1)(n-2) < \left(\frac{n+k+2}{2}\right)(n^2+1); \quad n \text{ is even}$$

$$n^3-5n^2+9n-4 < \left(\frac{n+k+2}{2}\right)((n-1)^2+1); \quad n \text{ is odd.}$$

Thus we are reduced to studying varieties on the above list and that of [16] if n is even and $k \geqq n-7$ (see also [8]).

It should be noted that the detailed classification of varieties on the lists with a special property, e.g. defect k discriminant locus, requires some further work.

We would like to thank the M. P. I. and the Italian Government for making this research possible. The second author's research was partially supported by the University of Notre Dame, the National Science Foundation (DMS 84—30315 and DMS 87—22330), and the Max Planck Institut für Mathematik in Bonn.

0. Background material

Throughout this paper $(M, \mathscr{L})$ will denote a pair, consisting of a smooth projective n-fold M, and an ample and spanned line bundle $\mathscr{L}$ on M such that the map $X \to P_C$ given by $\Gamma(\mathscr{L})$ is generically one to one.

0.1. Let L be a line bundle on M. We say that L is *nef* if $c_1(L) \cdot [C] \geqq 0$, for all effective curves C on M. We say that a nef line bundle L is *big* if $c_1(L)^n > 0$. We say that L is semi-ample if there exists an $m>0$ such that $Bs|mL|$, the base locus of $|mL|$, is empty.

0.2. A reduction (X, L) of a pair $(M, \mathscr{L})$ is a pair (X, L) consisting of an ample line bundle L on a projective manifold X such that:

a) M is the blowup $\pi: M \to X$ of X at a finite set F,

b) $\mathscr{L}=\pi^*(L)\otimes[\pi^{-1}(F)]^{-1}$ or equivalently $K_M\otimes\mathscr{L}^{n-1}=\pi^*(K_X\otimes L^{n-1})$.

The pair (X, L) is also called the **1st reduction** of $(M, \mathscr{L})$ if $K_X\otimes L^{n-1}$ is ample.

For the following theorems we refer to [5], [6], [12], [13], [16].

Theorem 0.3. *Let $(M, \mathscr{L})$ be as above. Then there exists a reduction (X, L) of $(M, \mathscr{L})$ such that $K_X\otimes L^{n-1}$ is ample and spanned by global sections unless one of the following holds:*

a) $(M, \mathscr{L})=(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1))$ *or* $(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(2))$,

b) $(M, \mathscr{L})=(\mathbf{Q}^n, \mathcal{O}_{\mathbf{Q}^n}(1))$, *where $\mathbf{Q}^n$ is a smooth hyperquadric in $\mathbf{P}^{n+1}$,*

c) *$(M, \mathscr{L})$ is a scroll over a smooth curve,*

d) *$(M, \mathscr{L})$ is a del Pezzo manifold, i.e. $K_M\otimes\mathscr{L}^{n-1}=\mathcal{O}_M$,*

e) *$(M, \mathscr{L})$ is a hyperquadric fibration over a smooth curve,*

f) *$(M, \mathscr{L})$ is a scroll over a surface.*

Theorem 0.4. *Let $(M, \mathscr{L})$ be as above. Assume that* $\dim M=n\geqq 3$. *If $(M, \mathscr{L})$ is not as listed in* 0.3 *there exists a reduction (X, L) of $(M, \mathscr{L})$ such that $K_X\otimes L^{n-2}$ is semi-ample and big unless one of the following holds:*

a) $(X, L)=(\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(2))$ *or* $(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(3))$,

b) $(X, L)=(\mathbf{Q}^3, \mathcal{O}_{\mathbf{Q}^3}(2))$,

c) *there is a holomorphic surjection $\varphi: X'\to C$ onto C, a smooth curve where $K_{X'}^2\otimes L^3\approx\varphi^*\zeta$ for an ample line bundle ζ on C; in particular the general fibre of φ is $(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(2))$,*

d) *$K_X\approx L^{-(n-2)}$, i.e. (X, L) is a Fano manifold of co-index* 3 (see [9]),

e) *(X, L) is a del Pezzo fibration over a curve,*

f) *(X, L) is a hyperquadric fibration over a surface,*

g) *$n\geqq 4$ and (X, L) is a scroll over a threefold.*

We need the following basic result (see [4], [15]).

Theorem 0.4.1. *Let $(M, \mathscr{L})$ be as above. Assume there is a reduction (X, L) with $K_X\otimes L^{n-1}$ ample. Assume that $K_X\otimes L^{n-2}$ is semiample and big and that $n\geqq 4$. Let $\Phi: X\to X'$ be the second adjunction map, i.e. there is a birational morphism, Φ, and an ample line bundle $\mathscr{K}$ on X', a normal projective variety, such that $\Phi^*\mathscr{K}=K_X\otimes L^{n-2}$. Then X' has isolated singularities. Precisely there exists an algebraic set $Z\subset X'$ such that* $\dim Z\leqq 1$ *and $\Phi_{X-\Phi^{-1}(Z)}: X-\Phi^{-1}(Z)\to X'-Z$ is a biholomorphism. If $C\subset Z$ is a pure one dimensional subvariety, then C is smooth, $C\subset X'_{\mathrm{reg}}$,* *and in a neighborhood U of C, Φ: $\Phi^{-1}(U)\to U$ is simply the blowup of C. If x is a zero dimensional irreducible component of Z then $\Phi^{-1}(x)$ is one of the following,*

a) $\mathbf{P}^{n-1}$ *with normal bundle* $\mathcal{O}_{\mathbf{P}^{n-1}}(-2)$, $L_{\mathbf{P}^{n-1}}=\mathcal{O}_{\mathbf{P}^{n-1}}(1)$,

b) $\mathbf{Q}$ *biholomorphic to an irreducible quadric in* $\mathbf{P}^n$ *with* $L_{\mathbf{Q}}=\mathcal{O}_{\mathbf{P}^n}(1)_{\mathbf{Q}}$, *and normal bundle* $\mathcal{O}_{\mathbf{P}^n}(-1)_{\mathbf{Q}}$.

Letting $Z=Z_1+Z_2$ where Z_1=set of points x with $\Phi^{-1}(x)$ as in a) and $Z_2=Z-Z_1$, and letting $L'=(\Phi*L)^{**}$ it can be seen that using $\mathbf{Q}$-Cartier divisors $\Phi^*L'-\Phi^{-1}(Z_2)-1/2\Phi^{-1}(Z_1)=L$.

0.4.2. *Let* $L'=(\Phi*L)^{**}$. This is a 2-Cartier divisor. Indeed except for points x with $\Phi^{-1}(x)$ of the form a) it is Cartier. Similarly $K_{X'}$ is 2-Cartier and Cartier if $n=\dim X'$ is even. We often write L'^{2a} for the line bundle $(2L')^a$.

0.4.3. Often we will have a surjective map f: $X'\to Y$ where $0<\dim Y<\dim X'$. If $\dim Y>1$ then by 0.4.1 we choose a general fibre $F\subset X'$ of f such that Φ gives a biholomorphism of $\Phi^{-1}(F)$ and F. Thus F can be identified with a general fibre of $f\circ\Phi$: $X\to Y$.

If $\dim Y=1$, then a general fibre F of f is smooth, $F\subset X'_{\text{reg}}$ and meets the set Z of 0.4.1 in a finite number of points $\mathcal{S}=\{x_1, \ldots, x_n\}\subset F$ obtained by intersecting F with a smooth curve $C\subset Z$. Note Φ: $\Phi^{-1}(F)\to F$ expresses $\Phi^{-1}(F)$ as F with $\mathcal{S}$ blown up and since $K_{X',F}=K_F$,

$$K_{\Phi^{-1}(F)}\otimes L^{n-2}_{\Phi^{-1}(F)}=\Phi^*(K_F\otimes L'^{n-2}_F).$$

Lemma 0.4.4. *Let $\mathcal{K}$ be a nef and big line bundle on a normal projective Gorenstein variety Y. Assume* Irr(Y) *is finite and $(K_Y^a\otimes\mathcal{K}^b)^t=\mathcal{O}_Y$ for some $a>0, b>0, t>0$. Then $K_Y^a\otimes\mathcal{K}^b=\mathcal{O}_Y$. Further $b/a\leqq n+1$.*

Proof. Choose the smallest integer $t>0$ such that $(K_Y^a\otimes\mathcal{K}^b)^t=\mathcal{O}_Y$. Let q: $Y'\to Y$ be the unramified cover associated to the t-th root of the constant function. By choice of t, Y' is irreducible and $K_{Y'}^a\otimes\mathcal{K}'^b=\mathcal{O}_{Y'}$ where $\mathcal{K}'=q^*\mathcal{K}$. Since $K_Y^{at}=\mathcal{K}^{-bt}$, $K_{Y'}^a=\mathcal{K}'^{-b}$ we see that K_Y^{-1}, $K_{Y'}^{-1}$ are nef and big. Thus by the Kawamata—Viehweg vanishing theorem (see [16], (0.2.1)), $h^i(\mathcal{O}_Y)=h^i(\mathcal{O}_{Y'})=0$, $i>0$. Thus $\chi(\mathcal{O}_Y)=\chi(\mathcal{O}_{Y'})=1$. But since q is an unramified cover $\chi(\mathcal{O}_{Y'})=t\chi(\mathcal{O}_Y)$. This implies $t=1$.

To see that $b/a\leqq n+1$ is a simple modification of an old Hirzebruch—Kodaira argument. Note that since $\mathcal{K}$ is nef and big, the polynomial $p(t)=\chi(K_Y\otimes\mathcal{K}^t)$ is an nth degree polynomial with nth degree term nonvanishing. By the Kawamata—Viehweg vanishing theorem used above, $p(t)=h^0(K_Y\otimes\mathcal{K}^t)$ for all $t>0$. If $b/a>n+1$, then $(K_Y\otimes\mathcal{K}^t)^a=(K_Y^a\otimes\mathcal{K}^b)\otimes\mathcal{K}^{ta-b}=\mathcal{K}^{ta-b}$ has a nef and big inverse for t between 1 and $n+1$. Thus we have the absurdity that $p(t)=h^0(K_Y\otimes\mathcal{K}^t)=0$ for $n+1$ integer values. □

0.5. Let $\mathscr{L}$ be a very ample line bundle on M. Let $\Delta \subset |\mathscr{L}|$ be the variety of singular hyperplane sections. If Δ has codimension $k+1$ then for a general point $x \in \Delta$, the set of singular points of the hyperplane section A corresponding to x is a linear $\wp = \mathbf{P}^k$, of non degenerate quadratic singularities. Thus the two jet τ of a section $s \in \Gamma(\mathscr{L})$ gives on $\wp$ a section of $N\wp^*(2) \otimes \mathscr{L}$ which is non degenerate as a symmetric form at all points of $\wp$. Thus

Theorem 0.5.1. (Ein.) *$N^*_{\wp} \otimes \mathscr{L} \cong N_{\wp}$. In particular given a line $\lambda \subset \wp$, $N_\lambda \cong N_{\lambda/\wp} \oplus N_{\wp,\lambda} \cong \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus(k-1)} \oplus \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus(n-k)/2} \oplus \mathcal{O}_{\mathbf{P}^1}^{\oplus(n-k)/2}$,*
Thus

$$\deg K_{M,\lambda} = -(n+k+2)/2 \quad \textit{and} \quad 0 = (n+k) \bmod 2.$$

Remark. The parity result had earlier been observed by A. Landman. The number k is also called the defect of M, $\operatorname{def}(M)$ (see [3] for details).

1. The main Theorem

Let $\mathscr{L}$ be an ample and spanned line bundle on a smooth projective manifold M. Assume the map $M \to \mathbf{P}_{\mathbf{C}}$ associated to $\Gamma(\mathscr{L})$ is generically one to one. Assume $\dim X = n$ is $\geqq 4$ and even. Assume $\sigma(M, \mathscr{L}) > 3$ and let X, L be as in 0.4. This section is devoted to proving the main theorem stated in the introduction.

Proof of Theorem 1.1. Recent results of [5], [6], [16] imply that the pair $(M, \mathscr{L})$ has a 2nd reduction $(X', \mathscr{K})$ unless $(M, \mathscr{L})$ is as listed in 0.3 and 0.4. It is easy to see from [4], [15] that X' has only isolated rational singularities and in fact X' is 2-factorial and Gorenstein in even dimensions. Thus for n even, $n \geqq 4$, the ample line bundle $\mathscr{K}$ is $K_{X'} \otimes L'^{n-2}$, where L' is as in the introduction.

Hence we can apply the results of Fujita [5], to the pair $(X', \mathscr{K})$. From ([5], Thm. 1, 2) we see that $K_{X'} \otimes \mathscr{K}^{n-1}$ is nef unless

a) $(X', \mathscr{K}) = (\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1))$,

b) X' is a hyperquadric $\mathbf{Q}^n$ in $\mathbf{P}^{n+1}$ and $\mathscr{K} = \mathcal{O}_{\mathbf{Q}^n}(1)$,

c) $(X', \mathscr{K})$ is a scroll over a smooth curve.

Noting that $\mathscr{K} = K_{X'} \otimes L'^{n-2}$, in a) we have $K_{\mathbf{P}^n} \otimes L'^{n-2} = \mathcal{O}_{\mathbf{P}^n}(1)$. Hence $-(n+1)+(n-2)d = 1$, where $d \in \mathbf{Z}$ and d is such that $L' = \mathcal{O}_{\mathbf{P}^n}(d)$. Using the ampleness of L', d is seen to be an integer >0. It follows that $3 \leqq n \leqq 6$. By assumption n is even, thus we have either

$$\text{a}_1) \quad (X', L') = (\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(3)),$$

or

$$\text{a}_2) \quad (X', L') = (\mathbf{P}^6, \mathcal{O}_{\mathbf{P}^6}(2)).$$

Note that L' is Cartier in case b) since the only singularity of 0.4.1 for which L' would not be Cartier doesn't occur on hyperquadrics.

Identical reasoning can be carried out for b) and c) and we obtain in b) either

b$_1$) $(X', L') = (\mathbf{Q}^3, \mathcal{O}_{\mathbf{Q}^3}(4))$,

or

b$_2$) $(X', L') = (\mathbf{Q}^5, \mathcal{O}_{\mathbf{Q}^5}(2))$

and in c) we see that $n = \dim X' = 3, 5$. Note that both b) and c) cannot occur since n is even. Thus $K_{X'} \otimes \mathscr{K}^{n-1}$ is nef unless (X', L') is as in a$_1$) and a$_2$). We will denote, for simplicity, $K_{X'} \otimes \mathscr{K}^{n-1}$ by $\mathscr{M}$. From (2.6) of [7] it follows that the linear system $|m\mathscr{M}|$ is base-point free for all $m \gg 0$. Let Ψ be the morphism associated to $|m\mathscr{M}|$ for $m \gg 0$.

Let $W = \Psi(X')$. Note that $\dim W \leqq 2$ or $\dim W = n$ (see [16]). To see that $\dim W \leqq 2$ note that the restriction of $\mathscr{M} = K_{X'} \otimes \mathscr{K}^{n-1}$ to a generic fibre, F, of Ψ is $K_F \otimes \mathscr{K}_F^{n-1}$ and $(K_F \otimes \mathscr{K}_F^{n-1})^m$ if $\mathcal{O}_F$ for some positive m. Now use lemma 0.4.4.

d) If $\dim W = 0$ then $(K_{X'} \otimes \mathscr{K}^{n-1})^t = \mathcal{O}_{X'}$. Thus by 0.4.4 $K_{X'} \otimes \mathscr{K}^{n-1} = \mathcal{O}_{X'}$. Thus $K_{X'}^n \otimes (2L')^{(n-1)(n-2)/2} = \mathcal{O}_{X'}$. If n is relatively prime to $(n-1)(n-2)/2$ then there exists an ample H such that $H^{(n-1)(n-2)/2} K_{X'}^{-1} =$. Since $(n-1)(n-1)/2 \leqq n+1$ and $n \geqq 4$ and even we conclude that $n = 4$. In this case $K_{X'}^4 \otimes L'^6 = \mathcal{O}_{X'}$. It is easy to see that if n and $(n-1)(n-2)/2$ have a common factor it is 2 and then $n/2$, $(n-1)(n-2)/4$ are relatively prime. Thus there exists an ample line bundle H such that $H^{(n-1)(n-2)/4} = K_{X'}^{-1}$. Since $n \geqq 4$ and even and $(n-1)(n-2)/4 \leqq n+1$ by Lemma 0.4.4 we conclude that $n = 6$. In this case we have $H^5 = K_{X'}^{-1}$, and $H^3 = 2L'$.

e) If $\dim W = 1$ and if we let F be a general fibre of Ψ we have $(K_F \otimes \mathscr{K}_F^{n-1})^t = \mathcal{O}_F$ for some $t > 0$. By (0.4.4, $K_F \otimes \mathscr{K}_F^{n-1} = \mathcal{O}_F$. Since F is smooth as noted in 0.4.3, F is a smooth quadric $\mathbf{Q} \subset \mathbf{P}^n$ and $K_F = \mathcal{O}_{\mathbf{Q}}(1)$. Since $L'_F = \mathcal{O}_{\mathbf{Q}}(d)$ for d a positive integer, this gives $-(n-1) + (n-2)\,d = 1$ or $(n-2)(d-1) = 2$. Since n is $\geqq 4$ and even, we conclude that $n = 4$, $d = 2$.

f) If $\dim W = 2$ and F denotes a general fibre of Ψ then by 0.4.3 W is smooth. We have $K_F \otimes \mathscr{K}_F^{n-1} = \mathcal{O}_F$, i.e. $(F, \mathscr{K}_F) = (\mathbf{P}^{n-2}, \mathcal{O}_{\mathbf{P}^{n-2}}(1))$. As before we can see that $n = 4$ and $d = 2$.

It is easy to see that $\sigma(X', L') = 3\frac{1}{3}$ if (X', L') is as in a$_1$), and $\sigma(X', L') = 3\frac{1}{2}$ if (X', L') is as in a$_2$), e), or f). In the first example of d) $\sigma(X', L') = 3\frac{1}{2}$; in the second $3\frac{2}{3}$. Hence (X', L') is as in Theorem 1.1 above.

If $\dim W = n$ then $\mathscr{M} = K_{X'} \otimes \mathscr{K}^{n-1}$ is nef and big. Consider the line bundle $K_{X'} \otimes \mathscr{M}^n$.

Either $h^0(K_{X'} \otimes \mathscr{M}^n) \neq 0$ or $h^0(K_{X'} \otimes \mathscr{M}^n) = 0$.

In ([2], Theorem 2.2) it is shown that if $\mathscr{K}=K_X\otimes L^{n-2}$ is nef and big and $h^0(K_X\otimes\mathscr{K}^n)=0$ then there is a birational morphism Φ: $X\to\mathbf{P}^n$ with $\mathscr{K}=\Phi^*\mathcal{O}_{\mathbf{P}^n}(1)$. The argument used there works for any line bundle $\mathscr{K}$ on a normal Y such that:

a) $\mathscr{K}^t$ is spanned by global sections for all sufficiently large t,

b) $\mathscr{K}$ is big,

c) $h^i(\mathscr{K}^j)=0$ for $i>0, j>0$.

Since X' is Gorenstein with rational singularities Kawamata's base point free theorem and the fact that $\mathscr{K}$ and $\mathscr{M}$ are nef and big imply a) and b). Since $\mathscr{M}^j=K_{X'}\otimes(\mathscr{K}^{n-1}\otimes\mathscr{M}^{j-1})$ for $j\geqq 1$, $\dim\mathrm{Sing}(X')=0$, and $\mathscr{K}$, $\mathscr{M}$ are nef and big, the Kawamata—Viehweg vanishing theorem implies c).

Thus if $\mathscr{M}=\Phi^*\mathcal{O}_{\mathbf{P}^n}(1)$, then $\Phi*(L')^{**}=\mathcal{O}_{\mathbf{P}^n}(d)$ where

$$-(n+1)+(n-1)((n-2)d-n-1)=1$$

or $(n-1)((n-2)d-n-2)=3$. Since $n\geqq 4$, this implies $n=4$ and $2d=7$. This is clearly not possible.

Proof of Corollary 1.1.2. Let $A\in|L|$ be a general element. Corollary 1.1.2 will follow from (1.1) if we show that $(A', L'_{A'})$ can be one of the exceptions of (1.1) only if

a) (X', L') is the cone on $(\mathbf{P}^6, \mathcal{O}_{\mathbf{P}^6}(2))$,

b) X' is 5 dimensional and $K_{X'}^4\otimes L'^{10}=\mathcal{O}_{X'}$,

c) there is a morphism Ψ: $X\to C$ where C is a smooth curve, $L_F=\mathcal{O}_{\mathbf{P}^4}(2)$ for a general fibre F which is biholomorphic to $\mathbf{P}^4$,

d) X' is 7 dimensional and $K_{X'}^6\otimes L'^{26}=\mathcal{O}_{X'}$.

Note if $A'=\mathbf{P}^4$, then X' is smooth in a neighborhood of A'. This follows by looking over the possible singularities of 0.4.1. Since A' is therefore Cartier and ample it follows from Scorza's theorem (see [1]) that X' is a cone over $\mathbf{P}^4$. The only singularity on X' is the vertex. Checking the list in 0.4.1 it doesn't occur.

If $K_{A'}^{-4}=L'_{A'^6}$, with $\dim A'=4$, $A\in|L|$, $A'=\Phi(A)$, then $(K_X^4\otimes L^{10})_A$ has a section zero only on the inverse image of the positive dimensional fibre of $A\to A'$ and $h^0((K_X^4\otimes L^{10})_A)=1$. Consider $0\to K_X\otimes(K_X^3\otimes L^9)\to K_X^4\otimes L^{10}\to(K_X^4\otimes L^{10})_A\to 0$. Since $K_X\otimes L^3$ is nef and big by assumption, we conclude $h^1(K_X^4\otimes L^9)=0$ by the Kodaira vanishing theorem. Also since A is a general element of $|L|$ and $h^0((K_X^4\otimes L^{10})_A)=1$ we conclude $h^0(K_X^4\otimes L^{10})\geqq 1$. Thus $4K_X+10L=D$ where D is an effective divisor supported on the set of positive dimensional fibre of Φ: $X\to X'$. From this we conclude the Cartier divisor $4K_{X'}+10L'$ is trivial.

Assume now that for $A\in|L|$, $A'=\Phi(A)$ there exists a Ψ: $A'\to C$, C a curve, $K_{A'}^2\otimes L'^3=\Psi^*E$ for an ample line bundle E on C with general fibre F of Ψ equal $(\mathbf{Q}^3, \mathcal{O}_{\mathbf{Q}^3}(2))$. By [11], the map $\Psi\circ\Phi_A$: $A\to S$ extends to a map f: $X\to S$. By 0.4.3 we can assume that for a general fibre f of $X\to C$, (f, L_f) has a first reduction

$(f', L'_{f'})$ with $F \in |L'_{f'}|$. Since $K_F^2 \otimes L_F'^2 = \mathcal{O}_F$ we conclude by the first Lefschetz theorem, $(K_{f'} \otimes L'_{f'})^2 \otimes L_{f'}'^3 = \mathcal{O}_{f'}$. Thus there is an ample line bundle H with $H^5 = K_{f'}^{-1}$. Thus $f' = \mathbf{P}^4$. Since $H^2 = L'_{f'}$, $L'_{f'} = \mathcal{O}_{\mathbf{P}^4}(2)$.

Similarly the 4[th] case leads to a map $X' \to S$ with $K_F{}^2 \otimes L'_F{}^5 = \mathcal{O}_F$ for a general fibre F with $\dim F = 3$. This implies $K_F^{-1} = H^5$ for an ample line bundle H which is easily seen to be impossible. In the last case we conclude as in the 3[rd] case that $6K_X = 26L = D$ where D is an effective divisor supported on the set of positive dimensional fibre of $X \to X'$. Thus $K_{X'}^6 \otimes L'^{26} = \mathcal{O}_{X'}$. □

Theorem 1.2. *Let $Y \subset \mathbf{P}_{\mathbf{C}}$ be an n dimensional irreducible projective variety whose normalization M is smooth of dimension $n \geqq 4$. Assume that $(M, \mathscr{L})$ is not as listed in* 0.3 *and* 0.4. *Let $S = \bigcap_{1 \leqq i \leqq n-2} H_i$ for the general $H_i \in |\mathscr{L}|$. Then if n is even either*

$$\deg M \leqq (g-1)\left(1 + \frac{n+3}{2n^2-n-1}\right)$$

and

$$K_S \cdot L \leqq \left(1 + \frac{n+3}{n^2-n-2}\right) K_S^2$$

or $(M, \mathscr{L})$ has a 2nd reduction $(X', \mathscr{K})$ such that (X', L') is as in Theorem 1.1.

If n is odd then either

$$\deg M \leqq (g-1)\left(1 + \frac{n+2}{2n^2-5n+2}\right)$$

and

$$K_S \cdot L \leqq \left(1 + \frac{n+2}{n^2-3n}\right) K_S^2$$

or $(M, \mathscr{L})$ has a 2nd reduction $(X', \mathscr{K})$ such that (X', L') is as in Corollary 1.1.2.

Proof. From 1.1 it follows that either $h^0(K_M^{(n^2+1)} \otimes \mathscr{L}^{n(n-1)(n-2)}) \neq 0$ or $(M, \mathscr{L})$ has a 2[nd] reduction $(X', \mathscr{K})$ such that (X', L') is as listed in the Theorem 1.1 or in 1.1.2.

If $h^0(K_M^{(n^2+1)} \otimes \mathscr{L}^{n(n-1)(n-2)}) \neq 0$ then since $(K_M \otimes \mathscr{L}^{(n-2)})_S = K_S$ and $\mathscr{L}$ is ample we have

$$K_S \cdot \mathscr{L} \geqq \frac{(n+1)(n-2)}{n^2+1} \mathscr{L} \cdot \mathscr{L}.$$

By the adjunction formula and the above inequality we see that

$$2g-2 = (K_S + \mathscr{L}) \cdot \mathscr{L} = K_S \cdot \mathscr{L} + \mathscr{L} \cdot \mathscr{L} \geqq \frac{2n^2-n-1}{n^2+1} \mathscr{L} \cdot \mathscr{L}.$$

Hence

$$\deg M = \mathscr{L}\cdot\mathscr{L} \leqq (g-1)\left(1+\frac{n+3}{2n^2-n-1}\right).$$

Similar reasoning with Corollary 1.1.2 yields the given result. □

Remark 1.2.1. Assume $n\geqq 4$ and $h^0(\mathscr{L})\geqq n+3$. Using Castelnuovo's bound for the genus of a curve in terms of its degree we get $g\geqq 8$ and further

a) if n is even then $\deg M \leqq (g-1)\left(1+\frac{n+3}{2n^2-n-1}\right)$,

b) if n is odd then $\deg M \leqq (g-1)\left(1+\frac{n+2}{2n^2-5n+2}\right)$.

2. An application

Proposition 2.1. *Let M be an n dimensional manifold. Assume that there is a family of lines on M with at least a $t\geqq 0$ dimensional subfamily of lines through most points of M. Then $(M, \mathscr{L})$ is as in 0.3 or 0.4 or has a 2^{nd} reduction as in Theorem 1.1 or Corollary 1.1.2 if*

$$n(n-1)(n-2) < (t+2)(n^2+1) \quad \textit{and } n \textit{ is even } \geqq 4$$

$$n^3-5n^2+9n-4 < (t+2)[(n-1)^2+1] \quad \textit{and } n \textit{ is odd } \geqq 5.$$

Proof. Let λ be a line through a general point p of M. Let N_λ be the normal bundle of λ in M. By hypothesis, N_λ is generically spanned by global sections. Hence

$$N_\lambda = \oplus_{i=1}^{n-1} O_\lambda((a_i)) \quad \text{with} \quad a_i \geqq 0. \tag{2.1.1}$$

Let $I_{p/\lambda}$ denote the ideal sheaf on λ of germs of holomorphic functions vanishing at p. Since $h^1(N_\lambda\otimes I_{p/\lambda})=0$, where the Hilbert scheme Λ of lines in X through p is smooth at the point t_0 corresponding to λ. Hence there is a unique irreducible component Λ_0 of the Hilbert scheme containing t_0. Also

$$\dim \Lambda_0 = h^0(N_\lambda\otimes I_{p/\lambda}) = \textstyle\sum_{i=1}^{n-1} a_i.$$

For simplicity we denote this dimension by t.

Unless $(M, \mathscr{L})$ is as in 0.3 or 0.4 or has a second reduction $(X', \mathscr{K})$ as in 1.1 or 1.1.2 it follows that

a) $(n^2+1)K_M+n(n-1)(n-2)\mathscr{L}$ is effective if n is even and $\geqq 4$,

b) $[((n-1)^2+1)K_M+(n^3-5n^2+9n-4)\mathscr{L}]\cdot\mathscr{L}$ is effective if n is odd and $\geqq 5$.

By the adjunction formula $K_M \cdot \lambda = -2 - \deg(\det N_\lambda) = -2 - t$. Since $\mathscr{L} \cdot \lambda = 1$ it follows from a) that

$$\alpha) \quad -(n^2+1)(2+t)+n(n-1)(n-2) \geqq 0 \quad \text{if } n \text{ is even and } \geqq 4,$$

and from b) that

$$\beta) \quad -[(n-1)^2+1](2+t)+(n^3-5n^2+9n-4) \geqq 0 \quad \text{if } n \text{ is odd and } \geqq 5. \quad \square$$

Proposition 2.2. *Let $\mathscr{L}$ be a very ample line bundle on an n-fold M with $n \geqq 4$. Assume that* $\operatorname{def}(M)=k>0$. *Then $(M, \mathscr{L})$ has a* 2^{nd} *reduction as in Theorem* 1.1 *or Corollary* 1.1.2 *if*

$$n \text{ is even and } n(n-1)(n-2) < (n+k+2)[n^2+1]/2,$$

or

$$n \text{ is odd and } n^3-5n^2+9n-4 < (n+k+2)[(n-1)^2+1]/2.$$

Proof. From 0.5.1 and the adjunction formula it follows that $\deg K_{X', \lambda} = -(n+k+2)/2$.

Hence as in the proof of 2.1 we conclude that $(M, \mathscr{L})$ has a 2^{nd} reduction as in Theorem 1.1 or Corollary 1.1.2 unless the above inequalities occur. $\square$

Conjecture 2.3. *Let L be a very ample line bundle on a smooth connected projective n-fold, X. Assume that the spectral value, $\sigma(X, L)$, of the pair (X, L)* is $\leqq n$. *Then the only possible values of $\sigma(X, L)$ are $n+1-\frac{p}{q}$ where p, q are integers satisfying* $0<q \leqq p \leqq n+1$.

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Received May 31, 1987

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