On unbounded hyponormal operators

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0. Introduction

The theory of hyponormal operators in Hilbert spaces is now well developed. We refer to [1] for a good presentation of the theory. In this work we generalize the notion of hyponormality to unbounded operators. It turns out that unbounded hyponormal operators share some properties of bounded ones. However it is unknown whether other properties of bounded hyponormal operators also hold in the unbounded case.

The paper is divided into three parts. Part I contains a few simple results concerning mostly spectral properties of unbounded hyponormal operators. Part II introduces a class of hyponormal operators which have their spectra contained in an angle $\{z \in \mathbf{C}, |\arg z| \leqq \theta < \pi/2\}$. We also prove that hyponormal operators with spectra contained in the half plane $\{z \in \mathbf{C}, \operatorname{Re} z \geqq 0\}$ are maximal accretive operators. Part III gives certain examples of unbounded hyponormal operators (differential operators of the order one or two and a class of composition operators in $L^2(\mu)$).

In what follows the following notation will be used. For an operator T in a complex Hilbert space H we denote by $D(T)$, T^*, $\sigma(T)$, the domain of T, the adjoint to T, the spectrum of T; respectively. If $D \subset D(T)$, then $\bar{T}|_D$ stands for the closure of the restriction of T do D. Other symbols are standard or will be defined in the text.

I.

Let T be a densely defined operator in H.

Definition. We say that T is hyponormal in H if

i) $D(T) \subset D(T^*)$

ii) $\|Tx\| \geqq \|T^*x\|, \quad x \in D(T)$.

Remark 1. If, moreover, T is closed then $\alpha T+\beta$ is hyponormal for any $\alpha, \beta \in \mathbf{C}$, and the operator $\bar{T}|_D$ is hyponormal for any dense, linear subspace $D \subset D(T)$.

Here are simple examples of unbounded hyponormal operators.

Example 1. Let $\alpha_k \in \mathbf{C}$ be a sequence such that $|\alpha_k| \leqq |\alpha_{k+1}|$ for every k. Then the weighted shift with the above weights is hyponormal.

Example 2. If T is hyponormal and V is an isometry, then $\tilde{T}=VTV^*$ is also hyponormal.

Example 3. Let A and B be hyponormal operators. Then $T=A\otimes B$ is hyponormal in $H\otimes H$. Here $A\otimes B$ denotes the closure of the algebraic tensor product $A\odot B$, defined on the algebraic tensor product $D(A)\odot D(B)$.

Example 4. If S is an unbounded subnormal operator (see [9] for the definition) then S is hyponormal.

Later we shall give more interesting examples of unbounded hyponormal operators.

The following proposition states a few simple properties of unbounded hyponormal operators.

Proposition 1. *Let T be a densely defined operator in H. We have*

i) *T is hyponormal iff $T^*|_{D(T)}=KT$, where K is a contraction.*

ii) *If T is hyponormal,* Ker $T=\{0\}$, $\overline{R(T)}=H$, *then T^{-1} is hyponormal.*

iii) *If T_1 and T_2 are closed hyponormal operators and there exist injective, bounded operators X and Y with dense ranges such that $XT_1 \subseteqq T_2X$, $T_1Y \supseteqq YT_2$, then $\sigma(T_1) = \sigma(T_2)$.*

iv) *If T_k are hyponormal, $D(T_k)\subset D(T_{k+1})\subset D(T_{k+1}^*)\subset D(T_k^*)$ and for any $\varepsilon>0$ there exists $n_0=n_0(\varepsilon)$ such that for every $m>n>n_0$*

$$\|T_m x - T_n x\| \leqq \varepsilon(\|x\|+\|T_n x\|+\|T_m x\|)$$

for all $x\in D(T_n)$ and

$$\|T_m^* y - T_n^* y\| \leqq \varepsilon(\|y\|+\|T_n^* y\|+\|T_m^* y\|)$$

for all $y\in D(T_m^*)$, then there exists $\lim_k T_k x = Tx$, $x\in \bigcup_n D(T_n)$, *T is also hyponormal and*

$$\sigma(T) = \bigcap_{n=1} \overline{\bigcup_{k=n}^{\infty} \sigma(T_k)}.$$

Proof. i) The proof is similar to the one given for bounded hyponormal operators, see [1, p. 3].

ii) Let $y, w \in D(T^{-1}) = R(T)$, so $y = Tx$, $w = Tv$. Using i) we can write

$$T^*|_{D(T)} = KT, \quad \|K\| \leqq 1.$$

We have

$$(T^{-1}w, y) = (v, Tx) = (T^*v, x) = (KTv, x) = (Kw, x) = (w, K^*x),$$

and so $y \in D((T^{-1})^*)$.

A similar reasoning gives

$$\|(T^{-1})^* y\| \leqq \|T^{-1}y\|.$$

iii) By symmetry it is enough to prove that $XT_1 \subseteqq T_2 X$ implies that $\sigma(T_2) \subset \sigma(T_1)$. Suppose that $\lambda \notin \sigma(T_1)$ (if $\sigma(T_1) = \mathbf{C}$ then there is nothing to prove). Let $c = \|(\lambda - T_1)^{-1}\|$. Following ideas of [2] we define the sequence

$$g_n = c^{-n} X(\lambda - T_1)^{-n} f, \quad n = 0, 1, \dots .$$

Note that $g_n \in D(T_2)$. Moreover, by repeating the proof of Lemma A of [2] in our case, one can check that the sequence $\|g_n\|$ is monotone decreasing. In particular, we have

$$\|g_1\| \leqq \|g_0\|$$

i.e.

$$\|(\lambda - T_2) X(\lambda - T_1)^{-1} f\| = \|Xf\| \geqq c^{-1} \|X(\lambda - T_1)^{-1} f\|.$$

Thus $(\lambda - T_2)$ is bounded from below on $XD(\lambda - T_1)$. Let $S = (\lambda - T_2)|_{XD(\lambda - T_1)}$. One can easily check that $R(S) = R(X)$. Hence there exists $S^{-1} \in L(H)$.

Thus $(\lambda - T_2)^* \subseteqq S^*$ has a bounded inverse on $R((\lambda - T_2)^*)$. Now $\overline{R((\lambda - T_2)^*)} \supseteqq \overline{R(\lambda - T_2)} = H$, so $(\lambda - T_2)^{*-1}$ exists on $R((\lambda - T_2)^*) = H$ and $\lambda \notin \sigma(T_2)$.

iv) This result is implicitly contained in [4] and its proof is omitted.

The proof of Proposition 1 is complete. The next result concerns generators of hyponormal semi-groups.

Proposition 2. *If* $\mathbf{R}_+ \ni t \to T_t$ *is a continuous semi-group of hyponormal operators, then its generator*

$$Af = \lim_{t \to 0_+} t^{-1}(T_t f - f)$$

is hyponormal.

Proof. Let $A_t = t^{-1}(T_t - I)$. For any $f, g \in D(A)$ we have

$$|(Af, g)| = \lim_{t \to 0_+} |(A_t f, g)| = \lim_{t \to 0_+} |(f, A_t^* g)|$$

$$\leqq \|f\| \lim_{t \to 0_+} \|A_t^* g\| \leqq \|f\| \lim_{t \to 0_+} \|A_t g\| = \|f\| \, \|Ag\|.$$

Hence $g \in D(A^*)$ and $\|A^* g\| \leqq \|Ag\|$.

II.

Now we shall restrict ourselves to hyponormal operators with spectra contained in angles. Namely, for $0<\theta\leqq\pi/2$ define $S_\theta=\{z\in\mathbf{C},\ |\arg z|<\theta\}$. It turns out that hyponormal operators with spectra contained in S_θ have nice properties. In what follows, by hyponormal operator we mean closed operator.

Proposition 3. *Let T be a hyponormal operator with $\sigma(T)\subseteqq S_{\pi/2-\theta}$, where $0<\theta\leqq\pi/2$. Then $-T$ generates a bounded holomorphic semi-group in S_θ.*

Proof. Applying Th. X. 52 of [7] it is enough to prove that for $0<\theta_1<\theta$

$$\|(\lambda+T)^{-1}\|\leqq M_{\theta_1}[\operatorname{dist}(-\lambda,S_{\pi/2-\theta_1})]^{-1},\quad \lambda\in\mathbf{C}\setminus S_{\pi/2-\theta_1}.$$

Since $(\lambda+T)^{-1}$ is also hyponormal, this is immediate by the following inequalities

$$\|(\lambda+T)^{-1}\|\leqq[\operatorname{dist}(\lambda,\sigma(-T))]^{-1}\leqq[\operatorname{dist}(-\lambda,S_{\pi/2-\theta_1})]^{-1}.$$

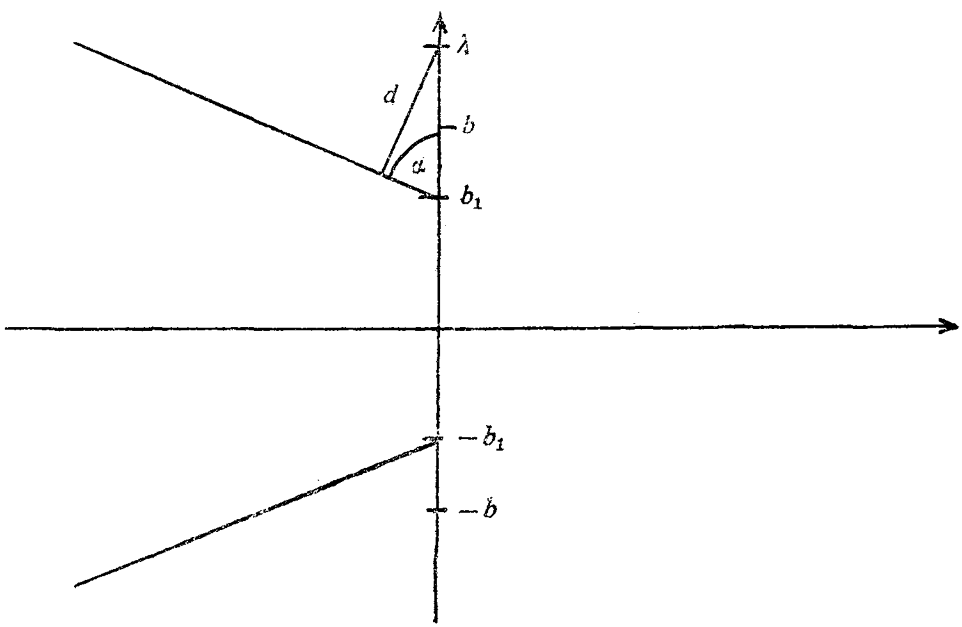
Remark 2. Note that $-T$ also generates a holomorphic semi-group (of exponential growth) if its spectrum is contained in set

$$S_{\theta,a}=\{z\in\mathbf{C},\ \operatorname{Re}z\geqq 0\}\cap\{z,\ |\arg(z+a)|<\theta,\ a\geqq 0\}.$$

In fact, applying Th. 5.1 of [3] we have to show that there exists $b>0$ such that

$$\|(\lambda+T)^{-1}\|\leqq C|\lambda|^{-1}$$

for all λ with $\operatorname{Re}\lambda=0$ and $|\operatorname{Im}\lambda|\geqq b$. Let $b>b_1=a\cdot\tan\theta$. Now this is clear from the following picture.



For $\alpha=\frac{\pi}{2}-\theta$ we have

$$\|(\lambda+T)^{-1}\| \leqq \frac{1}{d} = \frac{1}{|\lambda-b_1|\sin\alpha} \leqq \frac{b}{(b-b_1)|\lambda|\sin\alpha}.$$

Corollary 1. *If T is a hyponormal operator and $\sigma(T)\subset S_{\theta,a}$, then $\sigma(e^{-T})=\{e^{-z}, z\in\sigma(T)\}\cup\{0\}$.*

It turns out that hyponormal operators with spectrum contained in the half plane $\mathbf{C}_+=\{z\in\mathbf{C}, \operatorname{Re} z\geqq 0\}$ are maximal accretive.

Lemma 4. *If T is a hyponormal operator in H with $\sigma(T)\subseteqq\mathbf{C}_+$ then T is a maximal accretive operator in H.*

Proof. For $\lambda\in\operatorname{Int}\mathbf{C}_+$ the resolvent $(\lambda+T)^{-1}$ is hyponormal, hence

$$\|(\lambda+T)^{-1}\| \leqq [\operatorname{dist}(-\lambda, \sigma(T))^{-1}] \leqq (\operatorname{Re}\lambda)^{-1}.$$

Hence T is m-accretive and so it must be maximal accretive [6, p. 729].

Corollary 2. *Let T be a hyponormal operator with* $\operatorname{Ker} T=\{0\}$, $\sigma(T)\subset\bar{S}_\theta$, $0<\theta\leqq\pi/2$. *Then for any bounded and holomorphic function f in $S_{\theta+\varepsilon}$ we have $f(T)\in\mathscr{L}(H)$ and* $\|f(T)\|\leqq C_\theta\|f\|_\infty$, $\varepsilon>0$.

Proof. By Lemma 4 we know that T is a maximal accretive operator in H. Hence T must satisfy the so called quadratic estimates, see [5, p. 227]. Applying the theorem in [5, Sect. 7] we obtain the desired inequality.

The following application of Corollary 2 seems to deserve mentioning. For this we need to recall the notion of analytic vectors.

Definition. Let A be a closed linear operator in H. We say that $f\in C^\infty(A)=\bigcap_{k=1}^\infty D(A^k)$ is analytic for A if there exist $r>0$ and $a>0$ such that

$$\|A^k f\| \leqq a k!\, r^k, \quad k = 1, 2, \ldots.$$

Let us denote by $D^\omega(A)$ the set of all analytic vectors for A. In the case when A is a positive, selfadjoint operator in H, Nelson proved that

$$D^\omega(A) = \bigcup_{t>0} e^{-tA}(H).$$

It turns out that the same equality holds for hyponormal operators which have their spectra contained in S_θ.

Proposition 5. *Let T be a hyponormal operator in H with $\sigma(T)\subset\bar{S}_\theta$, $0<\theta<\pi/2$. Then*

$$D^\omega(T) = \bigcup_{t>0} e^{-tT}(H).$$

Proof. First note that e^{-tT} is well defined (by Proposition 3). Let $f\in D^{\omega}(T)$ then by the definition we can write

$$\|T^k f\| \leqq ak!r^k, \quad k = 1, 2, \dots .$$

Take $0<s<\frac{1}{r}$. We have

$$\sum_{k=0}^{\infty} \frac{\|T^k f\|}{k!} s^k \leqq a \sum_{k=0}^{\infty} (rs)^k < \infty,$$

and so $f\in\bigcup_{t>0} e^{-tT}H$. To prove the opposite inclusion suppose that $g=e^{-sT}h$ for a certain $s>0$. Then for any $k\in\mathbf{N}$

$$\|T^k g\| \leqq \|T^k \exp(-sT)\| \, \|h\|.$$

By Corollary 2 we know that

$$\|T^k \exp(-sT)\| \leqq C_\theta \, \|z^k e^{-sz}\|_\infty .$$

Therefore it suffices to estimate $\sup_{z\in S_\theta} |z^k e^{-sz}|$. Denoting $\tilde{C}_\theta = \sqrt{1+\tan^2\theta}$, $z=x+iy$ we have

$$\sup_{z\in S_\theta} |(x+iy)^k| e^{-sx} \leqq \tilde{C}_\theta^k \sup_{x>0} x^k e^{-sx} = \tilde{C}_\theta^k k!(1/s)^k$$

and so g must belong to $D^{\omega}(T)$. This completes the proof.

We end this section with a lemma which will be useful in the next one, where some specific examples of hyponormal operators will be given.

Lemma. *Let A be a densely defined operator in H possessing a formal adjoint A^+ such that*

$$(*) \qquad (Au, v) = (u, A^+ v)$$

for all $u, v\in D$, where $D\subseteqq D(A)\cap D(A^+)$ is a dense subspace of H. Assume that $\|A^+ u\| \leqq \|Au\|$ for $u\in D$. Then A is closable and $T=\bar{A}|_D$ is hyponormal.

Proof. It is obvious that A and A^+ are closable. Moreover, by $(*)$ we have $A^+\subset T^*$ and so $\bar{A}^+\subset T^*$. One can check easily that $D(T)\subseteqq D(\bar{A}^+)$. Hence $D(T)\subseteqq D(T^*)$. Now for $x\in D(T)$ there exists a sequence $x_k\in D$ such that $x_k\to x$ and $Ax_k\to Tx$. Thus

$$\|Tx\| = \lim_k \|Ax_k\| \geqq \lim_k \|A^+ x_k\| = \|A^+ x\| = \|T^* x\|.$$

III.

Despite the simplicity of the definition of hyponormality it is far from trivial to check when a given operator is hyponormal. In what follows we shall give some nonobvious examples of unbounded hyponormal operators. We start with a differential operator of the first order.

Example 1. Let $T=\left(a_0-\frac{i}{2}a_1'-ia_1\frac{d}{dx}\right)\Big|_{C_0^\infty(\mathbf{R})}$, where a_0 and a_1 are functions of C^1 class and satisfy the following conditions

1) $\operatorname{Re} a_0(x)\neq 0$
2) $a_1=ib_1$, where $b_1(x)=cx/\operatorname{Re} a_0(x)$, $c\in\mathbf{R}$.

Proposition 7. *Under the above conditions* 1) *and* 2) *the closure of T defines a hyponormal operator in $L^2(\mathbf{R})$ if*

α) $c<0$ and $2-x(\operatorname{Re} a_0'/\operatorname{Re} a_0)\geqq 0$

or

β) $c>0$ and $2-x(\operatorname{Re} a_0'/\operatorname{Re} a_0)\leqq 0$.

The proof can obtained by direct but tedious computations (integration by parts and the last Lemma). Note that nontrivial a_0 satisfy the above requirements. For example: $a_0(x)=x^\alpha$, $\alpha\leqq 2$, $\alpha_0(x)=e^{-x^2/2}$ or $a_0(x)=e^{-x}$ for $x>0$ (in the last case T is hyponormal in $L^2(\mathbf{R}_+)$).

Example 2. Let $D_j=i^{-1}\partial/\partial x_j$, $j=1,\dots,n$. For $a_j, b_j\in\mathbf{C}$, define $L=L(X,D)=\sum_{j=1}^n(a_jx+b_jD_j)|_{\mathcal{S}}$, where $\mathcal{S}$ stands for the Schwartz space in $\mathbf{R}^n$. Denote by $L^+=\sum_j(\bar a_jx+\bar b_jD_j)$. By direct computation we find that

$$(Lu,v)=(u,L^+v)$$

and

$$(L^+L-LL^+)w=4\operatorname{Im}(a,b)w,$$

where $w,v,u\in\mathcal{S}$ and $(a,b)=\sum_j a_j\bar b_j$. Hence $\bar L$ (the closure) defines a hyponormal operator in $L^2(\mathbf{R}^n)$ whenever $\operatorname{Im}(a,b)\geqq 0$. Moreover by a recent result of [8] we know that $\bar L$ is even subnormal.

Example 3. Let $\sigma(x,\xi)=a_0(x)+a_1(x)\xi+\xi^2$, $x\in\mathbf{R}$, $\xi\in\mathbf{R}$, where $a_0(x)=\beta x+\gamma$ and $a_1(x)=\varepsilon x+\varrho$ with $\beta,\gamma,\varepsilon,\varrho\in\mathbf{C}$. We associate with $\sigma(x,\xi)$ the differential operator $A=\sigma_w(X,D)$ of order 2 restricted to $\mathcal{S}$, and given by the Weyl prescription [9]. For numbers β, ε, ϱ we define the following conditions

i) $\operatorname{Im}\varepsilon\geqq 0$
ii) $\operatorname{Im}\bar\varepsilon\beta=0$
iii) $[\operatorname{Im}(\varepsilon\bar\varrho+2\beta)]^2\leqq 8\operatorname{Im}\varepsilon(\operatorname{Im}\bar\varrho\beta-\operatorname{Im}\varepsilon)$.

Let A^+ be a formal adjoint to A i.e.

$$(Au,v)=(u,A^+v)\quad\text{for all}\quad u,v\in\mathcal{S}.$$

From the general theory of the Weyl correspondence we know how to find the Weyl symbol of $A^+\cdot A-A\cdot A^+$. This enables us to prove the following result (whose proof will be given elsewhere).

Proposition 7. *The closure $\bar{A}$ gives a hyponormal operator in $L^2(\mathbf{R})$ if β, ε, ϱ satisfy the above conditions* i), ii) *and* iii).

Example 4. The last example describes a class of composition operators C_τ in $L^2(X, \mu)$ which are cohyponormal (i.e. C_τ^* are hyponormal).

Let (X, B, μ) be a measure space with a σ-finite measure μ i.e. $X=\bigcup_{k=1}^{\infty} X_k$, where $\mu(X_k)<+\infty$. Suppose we are given a measurable bijection $\tau\colon X\to X$ such that τ^{-1} is also measurable and $\mu\circ\tau^{-1}\ll\mu$.

Let $C_\tau f=f\circ\tau$, $f\in L^2(\mu)$. We shall find a condition for C_τ to be a cohyponormal operator in $L^2(\mu)$. Let $p=d(\mu\circ\tau^{-1})/d\mu$ be the Radon—Nikodym derivative. Note that characteristic functions of sets of finite measure belong to the domain of C_τ^* provided that $p\chi_A\in L^2(\mu)$ for all A such that $\mu(A)<\infty$, and

$$C_\tau^* f(z) = p(z)f(\tau^{-1}(z)).$$

Hence

$$\|C_\tau^* f\|^2 = \int p^2(y)|f(\tau^{-1}(y))|^2\,d\mu \tag{1}$$
$$= \int |f(w)|^2 p^2(\tau(w))\,d(\mu\circ\tau) = \int |f(w)|^2 p(\tau(w))\,d\mu.$$

Take $f=\chi_A$, where $\mu(A)<\infty$. Then by (1)

$$\|C_\tau^* f\|^2 = \int_A p(\tau(x))\,d\mu < \infty.$$

Since

$$\int |f\circ\tau|^2\,d\mu = \int_A p(z)\,d\mu < \infty \tag{2}$$

it is evident that $f\in D(C_\tau)$.

Under the above assumptions we have

Lemma 8. *The operator C_τ^* is hyponormal in $L^2(\mu)$ if and only if $p\circ\tau\geqq p$ a.e.*

Proof. Necessity.

If $\|C_\tau^* h\|^2\geqq\|C_\tau h\|^2$ for every $h\in D(C_\tau)$ then put $h=\chi_A$, $\mu(A)<\infty$ into this inequality.

We have

$$\|C_\tau^* h\|^2 = \int_A p^2(\tau(w))\,d(\mu\circ\tau) = \int_A p(\tau(w))\,d\mu \geqq \int_A p(w)\,d\mu = \|C_\tau h\|^2.$$

Since A is arbitrary it follows that $p\circ\tau\geqq p$ a.e. Sufficiency.

If $p\circ\tau\geqq p$ a.e., then for $f\in D(C_\tau^*)$ we have by (1) and (2)

$$\|C^* f\|^2 = \int |f(w)|^2 p(\tau(w))\,d\mu \geqq \int |f(w)|^2 p(w)\,d\mu = \|C_\tau f\|^2.$$

The proof is complete.

We conclude by considering special C_τ in $L^2(\mu)$ over the Heisenberg group $X=H_3=\mathbf{C}\times\mathbf{R}$. Namely let $d\mu=\exp(-|u|^4/2)\,dV$, where dV is Lebesgue measure in $\mathbf{C}\times\mathbf{R}$ and $|u|^4=|(z,t)|^4=|z|^4+t^2$. The group H_3 has natural dilations $\delta_r(z,t)=(rz,r^2t)$, $r>0$ (note that $|\delta_r(z,t)|=r|(z,t)|$). If $r\leqq 1$ they induce a bounded operator C_r in $L^2(\mu)$. By direct computation we find that

$$p(z,t)=r^{-4}\exp[(1-r^{-4})|(z,t)|^4/2].$$

Hence $p\circ\delta_r\geqq p$ for all $r\leqq 1$ and so, by Lemma 8, the C_r are cohyponormal. Moreover the C_r form a semi-group

$$(0,1)\ni r\to C_r.$$

It turns out that its generator can be found explicitly and we have:

Corollary 3. *The infinitesimal generator C of the semi-group C_r given by*

$$Cf=z\frac{\partial f}{\partial z}+2t\frac{\partial f}{\partial t}$$

is cohyponormal in $L^2(H_3,d\mu)$.

Appendix. When this work was completed we learned from J. Stochel that S. Ôta and K. Schmüdgen also defined unbounded hyponormal operators. They called them formally hyponormal, probably by analogy to formally normal operators. If in addition $D(T)=D(T^*)$, then T was called hyponormal. Moreover they also proved an analog of our Proposition 1 iii).

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