Poles of $|f(z, w)|^{2s}$ and roots of the B-function

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Introduction

Let f: $(\mathbf{C}^2, \bar{0}) \to (\mathbf{C}, 0)$ be an analytically irreducible germ. To f there is associated its local b-function at $\bar{0}$, denoted $b_f(s)$. Properties of $b_f(s)$ have been found by many authors [1, 2, 7, 11, 12, 14, 15, 16, 17]. In this paper a geometric construction is used to give precise formulae for roots (not *all* roots however) of $b_f(s)$ in terms of the geometry of the branch (i.e. germ of an analytically irreducible plane curve) defined by f at $\bar{0}$.

Brauner showed that the topological properties of a branch are determined by the finite set of integers $(n, \beta_1, \ldots, \beta_g)$ comprising its "characteristic sequence". That is, the topology of the link $\{f=0\} \cap S_\varepsilon$, S_ε a small 3-sphere centered at $\bar{0}$, is completely determined by this sequence. This is discussed in [19, pgs. 5—13]. Moreover, the canonical embedded resolution of f, an important component of the work described here, is also determined by this sequence. This is discussed in [12, Sec. 1].

To state the main result, let $e^{(0)}=n$ and $e^{(i)}=gcd(e^{(i-1)}, \beta_i)$, $i=1, 2, \ldots, g$. For each $i=1, 2, \ldots, g$, define

$$r_i = \frac{\beta_i + n}{e^{(i)}}, \tag{0.1}$$

$$R_i = \frac{\beta_i e^{(i-1)} + \beta_{i-1}(e^{(i-2)} - e^{(i-1)}) + \ldots + \beta_1(e^{(0)} - e^{(1)})}{e^{(i)}}$$

and $\varrho_i = -r_i/R_i$.

In [12, pg. 151], it was shown that if f is the complexification of a real analytic germ at $\bar{0}$ and if $gcd(r_i, R_i)=1$, then ϱ_i was a root of $b_f(s)$. Here, for any germ f as above, and independent of the gcd condition, one shows

Theorem 1. *The ratios* $\varrho_1, \ldots, \varrho_g$ *are roots of* $b_f(s)$.

Remark. For the precise relation to polar invariants of f see [11, pg. 153].

Closely allied to $b_f(s)$ is a generalized function, denoted $|f|^{2s}$, and defined as follows [1]. Let U resp. T be open neighborhoods of $\bar{0}$ resp. 0. Set $U^* = U - f^{-1}(0)$ and $T^* = T - \{0\}$. One chooses U and T so that $f^{-1}(t)$ is transverse to ∂U for each $t \in T$ and $f\colon U^* \to T^*$ is a C^∞ locally trivial fibration (Ehresman—Milnor fibration theorem). This is a "good representative" of the germ f. Set

$$\Omega_U^{(2,2)} = \{C^\infty \text{ forms of type } (2,2) \text{ with support in } U\}.$$

Define $|f|^{2s}$, for $\operatorname{Re}(s) \gg 1$, on $\Omega_U^{(2,2)}$ by the rule

$$\langle |f|^{2s}, \omega \rangle = \int_U |f|^{2s} \omega. \tag{0.2}$$

It is more convenient to think of this as a function of s. So, (0.2) is denoted by $I_f(s, \omega)$ in the following.

Using Hodge-theoretic techniques, Loeser [14, Th. 1.9] has shown that a consequence of Theorem 1 is the

Corollary 1. *$\varrho_1, \ldots, \varrho_g$ are poles of the meromorphic continuation of $I_f(s, -)$.*

Section 2 gives a direct "classical-style" proof of the corollary. In so doing, a precise expression for the value of $\operatorname{Res}_{s=\varrho_i} I_f(s, \omega)$ is derived if $gcd(r_i, R_i) = 1$. The residue is zero if $\omega(\bar{0}) = 0$.

In Section 3, formulae for certain poles of $I_f(s, \omega)$ are determined even if $\omega(\bar{0}) = 0$. This is accomplished by extending the analysis in [12, Sec. 1] of the ordering properties of candidates for poles of $I_f(s, \omega)$ arising from the canonical resolution of f at $\bar{0}$. Section 4 proves a general result about the roots of any *local* b-function $b_f(s)$, for a function f on $\mathbf{C}^n$ at any point $x \in \mathbf{C}^n$. It shows that one can estimate their numerators by the multiplicities of a jacobian of an embedded resolution π (for the germ of f at x) along the divisors of the exceptional locus. Theorem 1 is then a specific (and more precise) example of this phenomenon.

Section 1

Because the discussion below uses the canonical resolution, before proceeding to the proof of Theorem 1, it is useful to recall certain aspects of the resolution algorithm (cf. [12, Sec. 1]).

For $f\colon U \to T$ a good representative of the germ f, as defined in the introduction, let $\pi\colon X_{\text{res}} \to U$ denote the canonical resolution of f. π is determined by the characteristic sequence $(n, \beta_1, \ldots, \beta_g)$ at $\bar{0}$ as follows.

It is helpful to think of π as being segmented into blocks $\mathscr{B}_0, \mathscr{B}_1, \ldots, \mathscr{B}_{g-1}$ consisting of compositions of quadratic transformations

$$\mathscr{B}_i\colon X_0(i) \leftarrow \ldots \leftarrow X_{N(i)}(i).$$

When $i=0$, set $X_0(i)=U$. In general, $X_{N(i-1)}(i-1)=X_0(i)$.

To determine the length $N(i)$ of $\mathscr{B}_i$, one writes the continued fraction expansion for the first Puiseux ratio $\beta_1^{(i)}/n^{(i)}$ of the strict transform of f in $X_0(i)$

$$(1.1)_i \qquad \frac{\beta_1^{(i)}}{n^{(i)}} = [k_1(i), \ldots, k_q(i)]$$

and then

$$(1.2) \qquad N(i) = \sum_{j=1}^{q} k_j(i).$$

Each quadratic transformation creates one divisor in the exceptional locus of π. There are g divisors $\mathscr{D}_1, \ldots, \mathscr{D}_g$ of special interest here. Each $\mathscr{D}_i$ is created by the final quadratic transformation of the block $\mathscr{B}_{i-1}$. Moreover, each $\mathscr{D}_i$ for $i<g$, is characterized by the property that it intersects three components of $\pi^{-1}(\{\bar{0}\})$, while $\mathscr{D}_g$ intersects two components and the desingularized strict transform of $\{f=0\}\cap U$. Any other divisor intersects two other divisors in the exceptional locus.

Because the proof of Theorem 1 has a geometric and analytic component, it is also important to have expressions for $f\circ\pi$ and $\pi^*(dz\,dw)$ $\big((z, w)$ holomorphic coordinates in U, cf. (3.6) also$\big)$ in a neighborhood of each $\mathscr{D}_i$. To this end, one finds [12, Props. (2.2), (2.5)] that there are coordinate charts $\mathscr{U}_{1i}(x_i, y_i)$, $\mathscr{U}_{2i}(x_i', y_i')$ such that

(1.3)

i) $\mathscr{D}_i$ is contained in $\mathscr{U}_{1i}(x_i, y_i)\cup\mathscr{U}_{2i}(x_i', y_i')$.

ii) The overlap $\mathscr{U}_{1i}\cap\mathscr{U}_{2i}$ is determined by

$$x_i' = x_i^2 y_i$$
$$y_i' = 1/x_i.$$

iii) $f\circ\pi(x_i, y_i) = x_i^{A_1} y_i^{R_i}(1+\alpha_i x_i+x_i\psi_i(x_i y_i))^{S_i} T_{1,i}(x_i, y_i)$

$\det d\pi|_{\mathscr{U}_{1i}}(x_i, y_i) = x_i^{a_1} y_i^{r_i-1}(1+\alpha_i x_i+x_i\psi_i(x_i y_i))^{s_i-1} J_{2,i}(x_i, y_i)$.

iv) $f\circ\pi(x_i', y_i') = (x_i')^{R_i}(y_i')^{A_2}(y_i'+\alpha_i+\psi_i(x_i' y_i'))^{S_i} T_{2,i}(x_i', y_i')$

$\det d\pi|_{\mathscr{U}_{2i}}(x_i', y_i') = (x_i')^{r_i-1}(y_i')^{a_2}(y_i'+\alpha_i+\psi_i(x_i' y_i'))^{s_i-1} J_{2,i}(x_i', y_i')$.

For purposes here, one only needs to know that a_1, a_2, A_1, A_2 are positive integers.

v) $R_i = \operatorname{ord}_{\mathscr{D}_i}(f\circ\pi)$, $\quad r_i = 1+\operatorname{ord}_{\mathscr{D}_i}(\det d\pi)$.

The important properties concerning the expressions in iii), iv) are the following.

(1.4)

i) $C_i = T_{1,i}(0, y_i) = T_{1,i}(x_i, 0) = T_{1,i}(0, 0) = T_{2,i}(0, y_i') = T_{2,i}(x_i', 0) = T_{2,i}(0, 0) \neq 0.$

$c_i = J_{1,i}(0, y_i) = J_{1,i}(x_i, 0) = J_{1,i}(0, 0) = J_{2,i}(0, y_i') = J_{2,i}(x_i', 0) = J_{2,i}(0, 0) \neq 0.$

ii) The section of $\mathcal{O}_{X_{\mathrm{Res}}}$ which is $x_i^{A_1}y_i^{R_i}(1+\alpha_i x_i+x_i\psi_i(x_iy_i))^{S_i}$ in $\mathscr{U}_{1i}$ and is $(x_i')^{R_i}(y_i')^{A_2}(y_i'+\alpha_i+\psi_i(x_i'y_i'))^{S_i}$ in $\mathscr{U}_{2i}$ is a global section of $\mathcal{O}_{X_{\mathrm{Res}}}|_{\mathscr{U}_{1i}\cup\mathscr{U}_{2i}}$. A similar conclusion holds for the corresponding product of factors in $\det d\pi$.

iii) Let

$$\varepsilon_1(i) = \varrho_i A_1 + a_1,$$
$$\varepsilon_2(i) = \varrho_i S_i + s_i - 1,$$
$$\varepsilon_3(i) = \varrho_i A_2 + a_2.$$

Then $\varepsilon_2(i), \varepsilon_3(i) \in (-1, 0)$, $\varepsilon_1(i) \in (-2, -1)$ and the "cocycle relation"

$$\varepsilon_1(i)+\varepsilon_2(i)+2 = -\varepsilon_3(i)$$

holds. (For precise formulae of the $\varepsilon_j(i)$ cf. [12, Prop. 2.12]).

In order to prove Theorem 1, it suffices to proceed as follows. First, we may assume $i \geqq 2$ because it is already known that ϱ_1 is a root of $b_f(s)$ [9, pg. 88]. From now on an arbitrary $i \geqq 2$ is chosen and fixed. The basic criterion to find a root of $b_f(s)$ is this [15, 17]. If one can find a continuous family of 1-cycles $\zeta(t)$ lying in $U_t = f^{-1}(t) \cap U$ and if $\psi\, dz\, dw$ is a holomorphic 2-form defined in U satisfying $\psi(0) \neq 0$, then ϱ_i is a root of $b_f(s)$ if

(1.5) $$\lim_{t\to 0} t^{\varrho_i+1} \int_{\zeta(t)} \psi/df \neq 0.$$

Proof of Theorem 1. The proof has two parts. Part 1 constructs the 1-cycles $\zeta(t)$. Part 2 shows (1.5).

Part 1. This is an adaptation of the construction of Steenbrink—Varcenko [17, Sec. 4]. In the notation (1.3) let $M_i = lcm\{R_i, A_1, A_2\}$. Set $\sigma\colon \tilde{T} \to T$ to be an M_i fold cover branched only at $t=0$. Let τ be the coordinate in $\tilde{T}$. Define $\mathscr{X} = (\mathscr{U}_{1i} \cup \mathscr{U}_{2i})/\sim$, $\sim$ the equivalence relation determined by (1.3) (ii). Let $\mathscr{N}\colon \tilde{\mathscr{X}} \to \mathscr{X} \times_T \tilde{T}$ be the normalization. This gives the commutative diagram

$$\begin{array}{ccc} \tilde{\mathscr{X}} & \xrightarrow{\nu} & \mathscr{X} \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f\circ\pi} \\ \tilde{T} & \xrightarrow[\sigma]{} & T \end{array}$$

If D is any divisor in $\mathscr{X}$, set $\tilde{D} = \nu^{-1}(D)$. Let $m_1 = gcd\{R_i, A_1\}$, $m_2 = gcd\{R_i, A_2\}$.

In $\mathscr{U}_{1i}\times_T \tilde{T}$ resp. $\mathscr{U}_{2i}\times_T \tilde{T}$ one has the relations

$$\tau^{M_i} = f\circ\pi(x_i, y_i) \quad \text{resp.} \quad \tau^{M_i} = f\circ\pi(x_i', y_i').$$

Set D_0 to be the divisor of $\mathscr{X}$ defined by the conditions

$$D_0\cap\mathscr{U}_{1i} = \{(1+\alpha_i x_i+x_i\psi(x_i y_i))T_{1i} = 0\}, \quad D_0\cap\mathscr{U}_{2i} = \{(y_i'+\alpha_i+\psi(x_i' y_i'))T_{2i} = 0\}.$$

Define $\mathscr{X}_0=\mathscr{X}-D_0$ and $\tilde{\mathscr{X}}_0=\mathscr{N}^{-1}(\mathscr{X}_0\times_T\tilde{T})$. Then, above $(\mathscr{X}_0\cap\mathscr{U}_{1i})\times_T\tilde{T}$ resp. $(\mathscr{X}_0\cap\mathscr{U}_{2i})\times_T\tilde{T}$ in $\tilde{\mathscr{X}}_0$ there exist m_1 resp. m_2 disjoint open sets $W_1^{(1)}, \ldots, W_{m_1}^{(1)}$ resp. $W_1^{(2)}, \ldots, W_{m_2}^{(2)}$. Outside the singular locus of $\tilde{\mathscr{X}}_0$ are defined coordinates for the $W_j^{(k)}$. In the following, we shall refer to "coordinates" for any $W_j^{(k)}$ to mean coordinates defined only on the nonsingular part of $W_j^{(k)}$.

One can find coordinates $(z_1^{(j)}, z_2^{(j)}, u^{(j)})$, $j=1, \ldots, m_1$ resp. $(w_1^{(k)}, w_2^{(k)}, v^{(k)})$, $k=1, \ldots, m_2$ on $W_j^{(1)}$ $j=1, \ldots, m_1$ resp. $W_k^{(2)}$ $k=1, \ldots, m_2$ so that the relations (1.6) below hold. This is done as follows.

Let $(z_1^{(j)}, z_2^{(j)})$, $j=1, \ldots, m_1$ resp. $(w_1^{(k)}, w_2^{(k)})$, $k=1, \ldots, m_2$, be coordinates on m_1 resp. m_2 distinct copies of $\mathbf{C}^2$. Define the maps

$$\theta_j\colon \mathbf{C}^2 \to \mathscr{U}_{1i}\colon (z_1^{(j)}, z_2^{(j)}) \to (x_i, y_i) = ((z_1^{(j)})^{M_i/A_1}, (z_2^{(j)})^{M_i/R_i})$$

$$\eta_k\colon \mathbf{C}^2 \to \mathscr{U}_{2i}\colon (w_1^{(k)}, w_2^{(k)}) \to (x_i', y_i') = ((w_1^{(k)})^{M_i/R_i}, (w_2^{(k)})^{M_i/A_2}).$$

Next define

$$\varphi_1(x_i, y_i) = (1+\alpha_i x_i+x_i\psi(x_i y_i))^{S_i}T_{1i}(x_i, y_i)$$

$$\varphi_2(x_i', y_i') = (y_i'+\alpha_i+\psi(x_i' y_i'))^{S_i}T_{2i}(x_i', y_i')$$

and set for each j and k

$$\Phi_{1,j} = \varphi_1\circ\theta_j \quad \Phi_{2,k} = \varphi_2\circ\eta_k.$$

On the charts $W_j^{(1)}$ resp. $W_k^{(2)}$ of $\tilde{\mathscr{X}}_0$ one defines the $(z_1^{(j)}, z_2^{(j)}, u^{(j)})$, $j=1, \ldots, m_1$ resp. $(w_1^{(k)}, w_2^{(k)}, v^{(k)})$, $k=1, \ldots, m_2$ by the maps $\boldsymbol{\theta}_j$, $\boldsymbol{\eta}_k$, given by

$$(1.6) \qquad \boldsymbol{\theta}_j\colon \tau = u^{(j)}\Phi_{1j}^{1/M_i}e^{2\pi ij/M_i}, \quad x_i = (z_1^{(j)})^{M_i/A_i}, \quad y_i = (z_2^{(j)})^{M_i/R_i}$$

resp.

$$\boldsymbol{\eta}_k\colon \tau = v^{(k)}\Phi_{2k}^{1/M_i}e^{2\pi ik/M_i}, \quad x_i' = (w_1^{(k)})^{M_i/R_i}, \quad y_i' = (w_2^{(k)})^{M_i/A_2}.$$

Then $\tilde{\mathscr{X}}_0\cap W_j^{(1)}$ resp. $\tilde{\mathscr{X}}_0\cap W_k^{(2)}$ is defined by the equation

$$u^{(j)} = z_1^{(j)}z_2^{(j)} \quad \text{resp.} \quad v^{(k)} = w_1^{(k)}w_2^{(k)}.$$

One now constructs a 1-cycle in an open subset of $\tilde{\mathscr{D}}_i\cap\tilde{\mathscr{X}}_0$.

Set $\tilde{\mathscr{D}}_i' = \tilde{\mathscr{D}}_i - \bigcup_{\{D\cap\mathscr{D}_i\neq\emptyset\}}\tilde{D}$. Here, D is any component of the divisor $(f\circ\pi)^{-1}(0)$. Choose one chart $W_1^{(1)}$ resp. $W_1^{(2)}$ each from $\{W_j^{(1)}\}$ resp. $\{W_k^{(2)}\}$, denote it by $\mathscr{A}_1$ resp. $\mathscr{A}_2$. Let (z_1, z_2, u) resp. (w_1, w_2, v) denote the coordinates on $\mathscr{A}_1$ resp. $\mathscr{A}_2$. Denote by $\boldsymbol{\theta}$ resp. $\boldsymbol{\eta}$ the maps determined by (1.6)

$$\boldsymbol{\theta}_1\colon \mathscr{A}_1 \to \mathscr{X}_0\times_T\tilde{T} \quad \text{resp.} \quad \boldsymbol{\eta}_1\colon \mathscr{A}_2 \to \mathscr{X}_0\times_T\tilde{T}.$$

Then one notes that $\tilde{\mathscr{D}}_i' \cap \mathscr{A}_1$ resp. $\tilde{\mathscr{D}}_i' \cap \mathscr{A}_2$ is defined by

$$z_2 = u = 0 \quad \text{and} \quad 1+\alpha_i z_1^{\gamma_1} \neq 0 \quad \text{resp.} \quad w_1 = v = 0 \quad \text{and} \quad w_2^{\gamma_2}+\alpha_i \neq 0,$$

where $\gamma_1 = M_i/A_1$ resp. $\gamma_2 = M_i/A_2$. Define $\mathscr{R} = \{\lambda_1(1) < \ldots < \lambda_1(\gamma_1)\}$ to be the γ_1 roots of the equations

$$1+\alpha_i z_1^{\gamma_1} = 0$$

ordered by increasing argument. Denote $1/\gamma_1 = \beta$, $1/\gamma_2 = \delta$. Set $\omega_1 = e^{2\pi i\beta}$ resp. $\omega_2 = e^{2\pi i\delta}$. Define the following 1-cycle $\Lambda(\varepsilon, B)$ in $\tilde{\mathscr{D}}_i'$.

Let ε be a small positive number, B a large positive number, $\xi \notin \mathscr{R}$ the point in $\tilde{\mathscr{D}}_i' \cap \mathscr{A}_1$ equal to $(1/\alpha_i)^\beta$. This implies that $t\xi \notin \mathscr{R}$ for any $t > 0$.

Set $\Lambda(\varepsilon, B) = \sum_{i=1}^4 \Lambda_i$ where Λ_i are the following oriented paths in $\tilde{\mathscr{D}}_i' \cap \mathscr{A}_1$.

i) Λ_1 is the segment from $B^\beta \xi$ to $\varepsilon^\beta \xi$.
ii) Λ_2 is the sector of the circle from $\varepsilon^\beta \xi$ to $\varepsilon^\beta \omega_1 \xi$.
iii) Λ_3 is the segment from $\varepsilon^\beta \omega_1 \xi$ to $B^\beta \omega_1 \xi$.
iv) Λ_4 is the sector of the circle from $B^\beta \omega_1 \xi$ to $B^\beta \xi$.

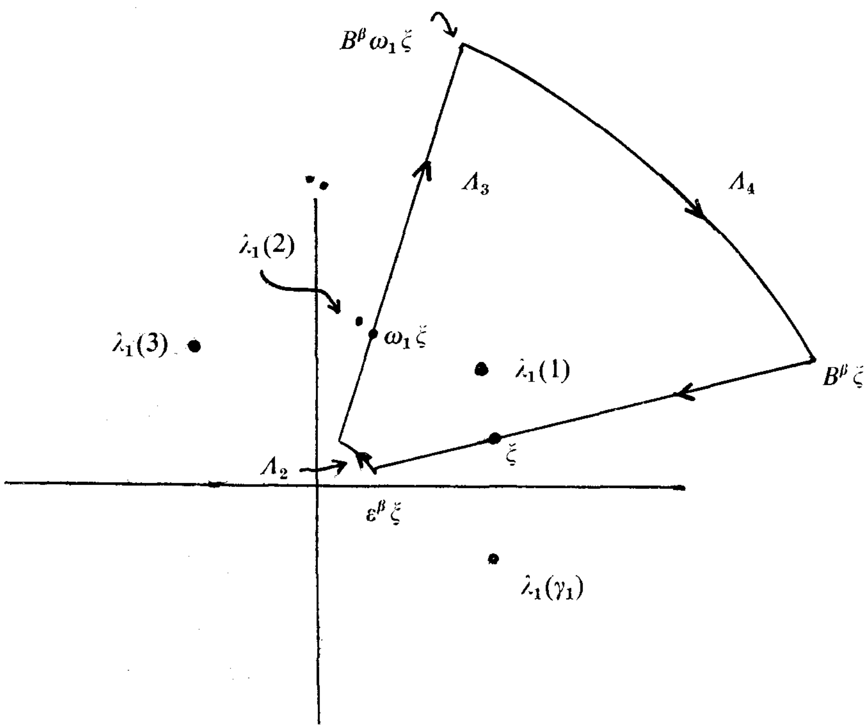

Fig. 1

Because $\Lambda(\varepsilon, B)$ must enclose a root in $\mathscr{R}$, it follows that the cycle determined by $\Lambda(\varepsilon, B)$ is non-homologous to zero in $\tilde{\mathscr{D}}_i'$. Moreover, it is clear that for any $\varepsilon_1 \in (0, \varepsilon)$ and $B_1 \in (B, \infty)$, $\Lambda(\varepsilon_1, B_1)$ is homologous in $\tilde{\mathscr{D}}_i'$ to $\Lambda(\varepsilon, B)$. It is sometimes convenient to note that one may view $\Lambda(\varepsilon, B)$ equivalently by describing Λ_4 as

iv)' the sector of the circle from $\omega_2^{-1}B^{-\delta}\xi^{-\delta/\beta}$ to $B^{-\delta}\xi^{-\delta/\beta}$ in $\tilde{\mathscr{D}}_i' \cap \mathscr{A}_2$.

It is clear by the construction that there exists an open tubular neighborhood $\mathscr{T}$ of $\Lambda(\varepsilon, B)$ in $\tilde{\mathscr{X}}$ which is disjoint from any divisor $\tilde{D}$ intersecting $\tilde{\mathscr{D}}_i$. Moreover, $\mathscr{T}$ may be constructed to lie in $\tilde{\mathscr{X}}_{sp}$ on which $\tilde{f}$ is smooth. As observed by Varcenko [17, pg. 489], one can then use the flow of the vector field grad $(\tilde{f}|_{\mathscr{T}})$ to push $\Lambda(\varepsilon, B)$ out to the fibers $\tilde{f}^{-1}(\tau)$ for $\tau = \tau(\theta|_{\mathscr{T}})$ sufficiently small, say in the set $\mathscr{W}$. This determines a continuous family of 1-cycles $\tilde{\zeta}(\tau)$ with $\tilde{\zeta}(0) = \Lambda(\varepsilon, B)$. Let $\mathscr{Z}_1(M)$ denote the group of 1-cycles on any space M. The map υ induces a homomorphism for each $\tau = \tau(\theta|_{\mathscr{T}})$

$$(\upsilon_\tau)_*\colon \mathscr{Z}_1(\tilde{f}^{-1}(\tau)) \to \mathscr{Z}_1((f\circ\pi)^{-1}\sigma(\tau)).$$

Thus, one defines for these values of τ

$$\zeta(\sigma(\tau)) = (\upsilon_\tau)_*(\tilde{\zeta}(\tau)). \tag{1.7}$$

Because the flow out used to construct the $\tilde{\zeta}(\tau)$ is a continuous mapping, it follows that the homology class of each $\tilde{\zeta}(\tau)$ is independent of the choice of ε and B. Thus, the same holds for the induced classes of the 1-cycles $\zeta(\sigma(\tau))$. In this way a continuous section of the 1-homology bundle for the fibration $f\circ\pi\colon \mathscr{X} - \{f\circ\pi = 0\} \to T^*$ has also been constructed. This completes Part 1. ∎

For Part 2, introduce the notation

$$I_i(t, \psi) = t^{\varrho_i+1} \int_{\zeta(t)} \pi^*(\psi/df)$$

for ψ a holomorphic 2-form defined on U. In addition, define the following 1-cycle in $\mathscr{D}_i$.

Given the loop $\Lambda(\varepsilon, B)$ in $\tilde{\mathscr{D}}_i'$ the map $z_1 \to z_1^{\gamma_1} = x_i$ transforms this loop to another loop in $\mathscr{D}_i$ which is classically known as a "Hankel contour" [13. pg. 50]. Denote this by $\lambda(\varepsilon, B)$. Set $\xi^{\gamma_1} = 1/\alpha_i$ by ξ_0 below. In fig. 2, $\lambda(\varepsilon, B)$ is sketched.

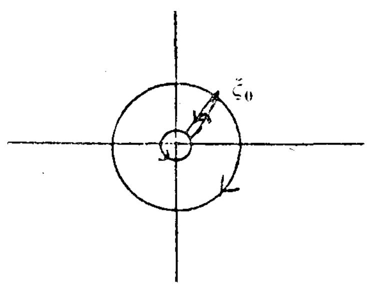

Fig. 2

Part 2. Analysis of $\lim_{t\to 0} I_i(t, \psi)$.

The main calculation is (cf. (1.4) (iii))

Proposition 1. *There is a constant $b(\psi)$ such that $b(\psi)\neq 0$ iff $\psi(0)\neq 0$ and so that*

$$\lim_{t\to 0} I_i(t, \psi) = b(\psi)\int_{\lambda(\varepsilon, B)} x_i^{\varepsilon_1(i)}(1+\alpha_i x_i)^{\varepsilon_2(i)}\, dx_i.$$

Proof. Define the quantity $\mu_i = M_i \varrho_i + 1$. Then Varcenko [17, Lemma 4.4] observed that the 2-form

$$\tilde{f}^{\mu_i}(\pi \circ \upsilon)^* \psi$$

extends to a holomorphic form over $\tilde{\mathscr{D}}_i'$ in $\tilde{\mathscr{X}}$, so that

$$R(\psi, i) = \tilde{f}^{\mu_i}(\pi \circ \upsilon)^*(\psi)/d\tilde{f}$$

is a non-zero holomorphic form over $\tilde{\mathscr{D}}_i'$. By the construction in part 1, one has

$$\lim_{\tau\to 0}\int_{\tilde{\zeta}(\tau)} R(\psi, i) = \int_{\Lambda(\varepsilon, B)} R(\psi, i). \tag{1.8}$$

Moreover, one also has [17, (9) pg. 489]

$$\tau^{-\mu_i}\int_{\tilde{\zeta}(\tau)} R(\psi, i) = M_i \tau^{M_i - 1}\int_{\zeta(\tau^{M_i})} \psi/df. \tag{1.9}$$

Since $M_i + \mu_i - 1 = M_i(\varrho_i + 1)$ and $\tau^{M_i} = t$, one concludes

$$\int_{\tilde{\zeta}(\tau)} R(\psi, i) = M_i I_i(t, \psi).$$

The proposition will therefore follow from an evaluation of the right hand side of (1.8). This is now easy.

A straightforward calculation shows the following.

$$R(\psi, i)|_{\mathscr{T}\cap\tilde{\mathscr{D}}_i} = \frac{M_i}{R_i}\begin{Bmatrix} \theta^*(\omega) & \text{on} & \mathscr{T}\cap\tilde{\mathscr{D}}_i'\cap\mathscr{A}_1 \\ \eta^*(\omega) & \text{on} & \mathscr{T}\cap\tilde{\mathscr{D}}_i'\cap\mathscr{A}_2 \end{Bmatrix},$$

where ω is the following 1-form on $\mathscr{D}_i' =_{\text{def.}} \mathscr{D}_i - \bigcup_{\{D\neq\mathscr{D}_i\}} D$ (cf. (1.3), (1.4)).

$$\omega = \begin{Bmatrix} \psi(\bar{0}) C_i^{\varrho_i} c_i (x_i)^{\varepsilon_1(i)}(1+\alpha_i x_i)^{\varepsilon_2(i)}\, dx_i & \text{on} & \mathscr{D}_i'\cap\mathscr{U}_{1i} \\ -\psi(\bar{0}) C_i^{\varrho_i} c_i (y_i')^{\varepsilon_3(i)}(y_i' + \alpha_i)^{\varepsilon_2(i)}\, dy_i' & \text{on} & \mathscr{D}_i'\cap\mathscr{U}_{2i} \end{Bmatrix},$$

The proposition follows immediately with the constant

$$b(\psi) = \frac{\psi(\bar{0}) C_i^{\varrho_i} c_i}{R_i}.$$

The is finishes the proof of the proposition. ∎

To conclude the proof of Theorem 1 one proceeds in a classical manner. For $\lambda \in \mathbf{C}$, define

$$\mathscr{I}(\lambda) = \int_{\lambda(\varepsilon, B)} x_i^\lambda (1+\alpha_i x_i)^{\varepsilon_2(i)}\, dx_i.$$

This is an analytic function of λ. To evaluate at $\lambda = \varepsilon_1(i)$, one may interpret this value as that obtained by an analytic continuation of $\mathscr{I}(\lambda)$ constrained to a subset of the halfplane $\operatorname{Re}(\lambda) > -1$, defined as follows. Let $\mathscr{L}$ be the region $\mathbf{C}^2$ defined by

$$\mathscr{L} = \{(p, q)\colon p+q = -\varepsilon_2(i),\ \operatorname{Re}(p),\ \operatorname{Re}(q) > 0\}.$$

Since $-\varepsilon_2(i) \in (0, 1)$ by (1.4) (iii), $\mathscr{L} \neq \emptyset$. Now set

$$\mathscr{Y} = \{\lambda\colon\ (\lambda+1,\ -\lambda-\varepsilon_2(i)-1) \in \mathscr{L}\}.$$

$\mathscr{Y}$ is a nonempty open vertical strip in the λ plane. Define $l(\xi_0)$ to be the line connecting the points $x_i = 0$, ξ_0, and $y_i' = 0$ (i.e. $x_i = \infty$). A classical computation [13, pg. 50] shows

Proposition 2. *For $\lambda \in \mathscr{Y}$, one has*

$$\mathscr{I}(\lambda) = e^{2\pi i \lambda}(1-e^{-2\pi i(\lambda+\varepsilon_2(i))}) \int_{l(\xi_0)} x_i^\lambda (1+\alpha_i x_i)^{\varepsilon_2(i)}\, dx_i.$$

Proposition 2 allows the evaluation of $\mathscr{I}(\varepsilon_1(i))$ to be made. A simple coordinate change gives for $\lambda \in \mathscr{Y}$

$$\mathscr{I}(\lambda) = v(\lambda) \int_0^\infty t^\lambda (1+t)^{\varepsilon_2(i)}\, dt,$$

with

$$v(\lambda) = e^{2\pi i \lambda}(1/\alpha_i)^{\lambda+1}(1-e^{-2\pi i(\lambda+\varepsilon_2(i))}).$$

One converts now to an integral over $(0, 1)$ by setting $t = v/(1-v)$, yielding

$$\mathscr{I}(\lambda) = v(\lambda) \int_0^1 v^\lambda (1-v)^{-\lambda-\varepsilon_2(i)-2}\, dv.$$

When $\lambda \in \mathscr{Y}$ it follows that the integral equals $B(\lambda+1,\ -\lambda-\varepsilon_2(i)-1)$, where $B(p, q)$ is the Eulerian beta function. One now uses the identity

$$B(\lambda+1,\ -\lambda-\varepsilon_2(i)-1) = \frac{\Gamma(\lambda+1)\Gamma(-\lambda-\varepsilon_2(i)-1)}{\Gamma(-\varepsilon_2(i))}$$

to analytically continue the left-hand side of this identity, and therefore $\mathscr{I}(\lambda)$, into $\mathbf{C}$. In particular, for $\lambda = \varepsilon_1(i)$, one obtains

$$\mathscr{I}(\varepsilon_1(i)) = v(\varepsilon_1(i)) \frac{\varepsilon_1(i)\varepsilon_3(i)\Gamma(\varepsilon_1(i))\Gamma(\varepsilon_3(i))}{\Gamma(-\varepsilon_2(i))}.$$

This is well-defined and nonzero by (1.4) (iii), completing the proof of Theorem 1.

Section 2

This section proves the corollary stated in the Introduction. For each $i\geqq 2$, one finds that the value of $\mathrm{Res}_{s=\varrho_i} I_f(s, \omega)$ is a negative real number multiple of a simple arithmetic expression involving certain coefficients of f.

For simplicity, an $i\geqq 2$ is fixed and each $\varepsilon_j(i)$ is denoted ε_j.

Theorem 2. *If $gcd(r_i, R_i)=1$, then ϱ_i is a simple pole of the meromorphic continuation of $I_f(s, \omega)$ when $\omega\in\Omega_U^{(2,2)}$ satisfies the condition $\omega(\bar{0})\neq 0$.*

Proof. Use is made of a calculation by H. Cohen, appearing in an article of Barlet [3].

As in [12, Sec. 2], the value of $\mathrm{Res}_{s=\varrho_i} I_f(s, \omega)$ is obtained as follows [6].

Let B be a positive number satisfying the condition

$$B > \max\{|\alpha_i|, 1/|\alpha_i|\}+1,$$

α_i the coefficient appearing explicitly in (1.3).

As specified in [12, (2.8)], there is a simple arithmetic expression involving coefficients of $f(z, w)$ which determine a non-zero constant c such that

$$\begin{aligned}&\mathrm{Res}_{s=\varrho i} I_f(s, \omega)\\ &= c\left\{\int_{|x_i|<B} |x_i|^{2\varepsilon_1}|1+\alpha_i x_i|^{2\varepsilon_2}\, dx\, d\bar{x} + \int_{|y_i'|<1/B} |y_i'|^{2\varepsilon_3}\, |y_i'+\alpha_i|^{2\varepsilon_2}\, dy_i'\, d\bar{y}_i'\right\}.\end{aligned}$$

Observe now that because $\varepsilon_3\in(-1, 0)$, the second integral in y_i' is an integral of an integrable function in a neighborhood of $y_i'=0$. Thus, as $B\to\infty$ this integral converges to 0. Hence,

$$\mathrm{Res}_{s=\varrho_i} I_f(s, \omega) = c\int_{\mathbf{C}} |x_i|^{2\varepsilon_1}|1+\alpha_i x_i|^{2\varepsilon_2}\, dx_i'\, dx_i'. \tag{2.1}$$

Remark also that because $\varepsilon_2\in(-1, 0)$ the singularity at $x_i=-1/\alpha_i$ will not contribute effectively to the value of the residue. This follows from the argument just given for the point $y_i'=0$.

If therefore suffices to show that the right hand side of (2.1) is not zero. Let $-\alpha_i x_i=re^{i\theta}$. (2.1) becomes

$$e\int_0^\infty \int_0^{2\pi} r^{2\varepsilon_1+1}(1+r^2-2r\cos\theta)^{\varepsilon_2}\, d\theta\, dr, \tag{2.2}$$

with a positive constant e.

(2.2) is interpreted as a value of the analytic continuation of

$$I(a, b) = e\int_0^\infty \int_0^{2\pi} r^a(1+r^2-2r\cos\theta)^b\, d\theta\, dr$$

where a, b are in the region $\mathrm{Re}\,(a)>-1$, $\mathrm{Re}\,(b)>-1$, $\mathrm{Re}\,(a-b)<1$. Indeed, the theorem is equivalent to determining the value of the analytic continuation at $a=2\varepsilon_1+1$, $b=\varepsilon_2$ and showing it is non-zero.

To do this, the calculation of Cohen found in [3, Sec. 1] is followed. One arrives at the value (consulting [18, Chapter 12] when needed)

$$I(a, b) = e\pi\frac{\Gamma(\varepsilon_1+1)\Gamma(\varepsilon_3+1)\Gamma(-1-\varepsilon_1-\varepsilon_3)}{\Gamma(-\varepsilon_1)\Gamma(-\varepsilon_2)\Gamma(-\varepsilon_3)} = e\pi\frac{\Gamma(\varepsilon_1+1)\Gamma(\varepsilon_2+1)\Gamma(\varepsilon_3+1)}{\Gamma(-\varepsilon_1)\Gamma(-\varepsilon_2)\Gamma(-\varepsilon_3)}$$

$$= -e\pi^4\frac{csc(\pi\varepsilon_1)\,csc(\pi\varepsilon_2)\,csc(\pi\varepsilon_3)}{[\Gamma(-\varepsilon_1)\Gamma(-\varepsilon_2)\Gamma(-\varepsilon_3)]^2}.$$

Indeed, by (1.4) (iii), one sees that $I(a, b)$ is negative. This completes the proof of Theorem 2.

Remark. Igusa has derived Theorem 2 in a different manner [10].

Section 3

The intent of this section is to extend Theorems 1, 2 to certain holomorphic differentials ψ such that $\psi(\bar{0})\neq 0$. In a fixed pair of coordinates (z, w) in the neighborhood U (of Section 1) (chosen to satisfy condition (3.7) below), we assume

$$\psi(z, w) = z^{i_1}w^{i_2}\mathscr{V}(z, w)\,dz\,dw, \tag{3.1}$$

where $\mathscr{V}(z, w)$ is an analytic unit in U. Set $I=(i_1, i_2)$.

Let ϱ be a C^∞ function, identically 1 on a smaller open neighborhood U' of $\bar{0}$, U' relatively compact in U, and so that ϱ has support in U. Then set

$$\omega = \varrho\psi\wedge\bar{\psi}. \tag{3.2}$$

ω belongs to $\Omega_U^{(2,2)}$.

It is first necessary to identify the ratios analogous to the ϱ_i, $i=1, \ldots, g$ for such ψ, ω. Secondly, one must determine if the analogue of (1.4) (iii) holds. Theorem 3, its corollary, and Theorem 4 describe the success at this determination. Because the arguments are quite similar to those in [12, Sec. 1] and the prior sections, complete proofs will not be given.

Let $i\in\{1, \ldots, g\}$ be chosen and fixed. Let $D_i=\{E_1, E_2, E_3\}$ be the divisors in X_{res} intersecting the divisor $\mathscr{D}_i$ (cf. Sec. 1). One assigns a label to each exceptional divisor by indexing the divisor E with subscript u if and only if E is created during the u^{th} quadratic transformation of the canonical resolution π. Recalling the quantities $N(v)$ defined in (1.2), one has $v\in\{1, \ldots, M_{tot}\}$ where

$$M_{tot} = \sum_{v=0}^{q-1} N(v).$$

Let (cf. $(1.1)_{i-1}$)

$$\frac{\beta_1^{(i-1)}}{n^{(i-1)}} = [k_1(i-1), \ldots, k_q(i-1)].$$

Set

$$M_{t,i-1} = \sum_{v=0}^{i-2} N(v) + \sum_{d=1}^{t} k_d(i-1)$$

$t=0, 1, \ldots, q-1$, and

$$M(i) = \sum_{v=0}^{i-1} N(v).$$

From [12, Props. (1.18, 1.19)], one sees that if the block $\mathscr{B}_i$ of π begins in "case A" resp. "case B" [12, pg. 143], then

$$D_i = \{E_{M_{q-1,i-1}}, E_{M_{q-1,i-1}+k_q(i-1)-1}, E_{M(i)+1}\}$$

resp.

$$D_i = \{E_{M_{q-1,i-1}}, E_{M_{q-1,i-1}+k_q(i-1)-1}, E_{M(i)+k_1(i)+1}\} \quad [12, \text{pg. } 152].$$

In the following, the "case A" possibility is assumed. This is a robust assumption in the sense that for fixed n, $\beta_1, \ldots, \beta_i$ there are only finitely many possible values for β_{i+1} which would force $\mathscr{B}_i$ to begin in "case B". In any case, the minor modifications, needed to adjust the arguments below if case B occurs, are left to the reader.

To each divisor E_u, associate the ratio

$$\lambda(u) = \frac{-[1+\mathrm{ord}_{E_u}(\pi^*\psi)]}{\mathrm{ord}_{E_u}(f\circ\pi)}. \tag{3.3}$$

For $i=1, \ldots, g$, set

$$\varrho_i(I) = \lambda(M(i)).$$

The analogues of the $\varepsilon_j(i)$ are defined as particular values of the function

$$\varepsilon(u) = \mathrm{ord}_{E_u}(f\circ\pi)\,\varrho_i(I) + \mathrm{ord}_{E_u}(\pi^*\psi).$$

Thus, set

$$\begin{aligned} \varepsilon_1(i, I) &= \varepsilon(M_{q-1,i-1}), \\ \varepsilon_2(i, I) &= \varepsilon(M_{q-1,i-1}+k_q(i-1)-1), \\ \varepsilon_3(i, I) &= \varepsilon(M(i)+1). \end{aligned} \tag{3.4}$$

The analogue of the cocycle condition (1.4) (iii) is then

Theorem 3. *For each i, I one has*

$$\varepsilon_1(i, I)+\varepsilon_2(i, I)+2 = -\varepsilon_3(i, I). \tag{3.5}$$

Proof. With π the canonical resolution map, let $\pi^{(i)}\colon X_0(i)\to U$ be the composition of the blow-ups comprising blocks $\mathscr{B}_0, \ldots, \mathscr{B}_{i-1}$. In $X_0(i)$, for $i \leqq g-1$, there is a unique non-normal crossing point $\xi(i)$ between the strict transform $f^{(i)}$

of f and the divisor $\mathcal{D}_i$. Define

$$N_1(i) = \operatorname{mult}_{\xi(i)}(z \circ \pi^{(i)})$$
$$N_2(i) = \operatorname{mult}_{\xi(i)}(w \circ \pi^{(i)})$$
$$E_I(i) = i_1 N_1(i) + i_2 N_2(i).$$

One checks that

$$N_1(1) = \frac{n}{e^{(1)}}, \quad N_2(1) = \frac{\beta_1}{e^{(1)}},$$

if the coordinates (z, w) in U are such that

$$f(z, w) = w^n + a_2(z) w^{n-2} + \ldots + a_n(z) \tag{3.6}$$

with $\operatorname{ord}_z a_j(z) > j$, $j=2, \ldots, n$. One remarks that up to an inessential unit factor local coordinates can always be found so that (3.6) holds. Geometrically, this condition means that the curve defined by f in U has maximal contact with the z-axis [8].

Set

$$n^{(d)} = \operatorname{mult}_{\xi(d)} f^{(d)}, \quad d = i-1, i.$$

Using the set of recurrence relations within $\mathcal{B}_{i-1}$ [12, (1.7.1, 1.7.2, 1.8)], one deduces that

$$N_j(i) = N_j(i-1) \frac{n^{(i-1)}}{e^{(i)}}, \quad j = 1, 2.$$

Thus,

$$E_I(i) = E_I(i-1) \frac{n^{(i-1)}}{e^{(i)}}. \tag{3.7}$$

For fixed i, $t \in \{0, 1, \ldots, q-1\}$, and $u \in [M_{t,i-1}, M_{t+1,i-1} - 1]$, define

$$\delta_t(\psi) = \operatorname{ord}_{E_u}(f \circ \pi) \operatorname{ord}_{E_{u+1}}(f \circ \pi) \{\lambda(u+1) - \lambda(u)\}.$$

One shows by induction (exactly as in [12]) that $\delta_t(\psi)$ depends only upon t and satisfies $\delta_t(\psi) = \delta_{t-2}(\psi)$.

Moreover,

$$\delta_0(\psi) = R_{i-1} - n^{(i-1)}(r_{i-1} + E_I(i-1)) \tag{3.8}$$
$$\delta_1(\psi) = -(R_{i-1} + \beta_1^{(i-1)}).$$

This suffices to determine $\varepsilon_2(i, I)$, $\varepsilon_3(i, I)$. To evaluate $\varepsilon_1(i, I)$ one proceeds in a manner similar to that in [12, Prop. (1.18)]. Finally, one obtains

$$\varepsilon_1(i, I) + 1 = \varepsilon_1(i) + 1 + \frac{n^{(i-1)} E_I(i-1)}{R_i}, \tag{3.9}$$
$$\varepsilon_2(i, I) + 1 = \varepsilon_2(i) + 1,$$
$$\varepsilon_3(i, I) + 1 = \varepsilon_3(i) + 1 - \frac{n^{(i)} E_I(i)}{R_i}.$$

By (1.4) (iii) and (3.7), Theorem 3 follows. ∎

Imposing the condition

$$(3.10) \qquad \varepsilon_1(i, I),\ \varepsilon_2(i, I),\ \varepsilon_3(i, I) \notin \mathbf{Z}$$

one observes that the proof of Theorem 1 applies immediately to ψ and the family of 1-cycles $\{\zeta(t)\}$ constructed in Section 1. Thus, as a corollary one obtains

Corollary 2. i) *If* $\varrho_i(I) \in (-1, 0)$ *and* (3.10) *is satisfied, then* $\varrho_i(I)$ *is a root of* $b_f(s)$.

ii) *More generally, if* (3.10) *is satisfied and no ratio of the form* $\varrho_i(I)+k$, $k=1, 2, \ldots$, *is a root of* $b_f(s)$, *then* $\varrho_i(I)$ *is a root of* $b_f(s)$.

A further extension of Section 2 is also possible. Using (3.8) one observes at once that if

$$(3.11)_i \qquad R_i - n^{(i)}\big(r_i + E_I(i)\big) > 0$$

is true for each i, then one finds the same situation as in [12, Sec. 1] with respect to the ordering properties of the ratios $\lambda(u)$ (3.3). In particular, the graph of the function $u \to \lambda(u)$, $u \in \{1, \ldots, M_{tot}\}$, would exhibit the same alternating increase-decrease behavior as the graphs on pg. 143—144 of [12]. Moreover, the same arguments from Section 1 of [12] would then show that if $(3.11)_i$ holds for each i then

$$(3.12) \qquad \varrho_1(I) > \varrho_2(I) > \ldots > \varrho_g(I).$$

From these observations one obtains

Theorem 4. a) *Assume* $(3.11)_i$ *is satisfied for each* $i=1, 2, \ldots, g$. *If* $\varrho_1(I) \in (-1, 0)$ *then* $\varrho_1(I)$ *is the largest pole of* $I_f(s, \omega)$, ω *defined by* (3.2).

b) *Let* $r_i(I) = 1 + \mathrm{ord}_{\mathcal{D}_i}(\pi^* \psi)$. *Assume* $gcd(r_i(I), R_i) = 1$ *and* $(3.11)_i$ *is true for each* i. *Then if* $\varrho_i(I) \in (-1, 0)$, *it is a pole of* $I_f(s, \omega)$, ω *defined by* (3.2).

(3.13) *Remark.* From (3.7), (3.9), and [12, (2.12)], one sees that the conditions (3.10), (3.11), "$\varrho_i(I) \in (-1, 0)$", and "$gcd(r_i(I), R_i) = 1$" can all be expressed in terms of and thus checked by simple arithmetic upon the characteristic numbers n, $\beta_1, \ldots, \beta_i$, $i=1, \ldots, g$.

Section 4

In [7, Th. (5.1)], Kashiwara proved the rationality of the roots of the local b-function at any point x of a complex manifold X for a complex analytic function f. In the following this polynomial is denoted $b_x(s)$, the function f being fixed throughout. In the proof he also derived a general estimate for the roots in terms of the local resolution data near x.

Let π: $X' \to X$ be a resolution of a representative of the germ of the function f at x, defined in a small neighborhood X of x. That is, $I=(f\circ\pi)\mathcal{O}_{X'}$ is locally in normal crossing form at each point of $\pi^{-1}(f^{-1}(x))$. To the divisors $\{D_i\}_{i=1}^{N}$ in the support of I there are associated the multiplicities

$$M_i = \mathrm{ord}_{D_i}(f\circ\pi), \quad m_i = \mathrm{ord}_{D_i}(\det d\pi). \tag{4.1}$$

Kashiwara showed that the set of roots of $b_x(s)$ had the form

$$\theta = -\frac{u}{M_i} - d, \tag{4.2}$$

where $u\in\{1, 2, \ldots, M_i\}$, $d\in\{0, 1, 2, \ldots\}$, and $i\in\{1, 2, \ldots, N\}$. This general estimate can be improved, slightly, as follows.

Theorem 5. *Each root α of $b_x(s)$ has the form*

$$\alpha = -\frac{(m_i+e)}{M_i} - d,$$

for some $e\in\{1, 2, \ldots, M_i\}$, $d\in\{0, 1, 2, \ldots\}$, *and* $i\in\{1, 2, \ldots, N\}$. ∎

The proof will be given in (4.10). Beforehand, it is useful to make some preliminary remarks and introduce notation and basic constructions.

(4.3) *Remark.* Observe that each element of the set $\mathscr{P}_f$ of poles of the generalized function $I_f(s, -)$ (cf. (0.2)) must have the form described in Theorem 5. Thus, this theorem indicates that the roots of the local b-function behave in a manner similar to the poles in $\mathscr{P}_f$ under pullback of f by a morphism of resolution of singularities. It would be interesting to know, in this regard, if the result of Loeser [14, Th. (1.9)] extends to the $n>1$ case.

More generally, Bernstein has asked whether each root of $b_x(s)$ is the pole of some "generalized $(n+1, 0)$ current" $I_\Gamma(s, -)$, defined (perhaps) as follows. Let Γ be a relative cycle class in $H_{n+1}(U, U\cap\{f=0\})$, where U is some neighborhood of x. Assume Γ' is a representative of Γ so that a single-valued branch of $\log(f)$ can be defined on $\Gamma'-\partial\Gamma'$. Let φ be a holomorphic $n+1$-differential defined in U. Then, for $\mathrm{Re}(s)\gg 1$ define

$$I_\Gamma(s, \varphi) = \int_{\Gamma'} f^s\varphi.$$

Evidently, if this conjecture was true then each root of $b_x(s)$ would also have to have the form given by Theorem 5. In this way, one sees that this theorem lends a small amount of credence to this conjecture, whose philosophy is that each root of $b_x(s)$, for any $x\in\{f=0\}$, should have an analytical significance. ∎

Notations and constructions needed in the proof of Theorem 5 are these.

In the following a point $x \in \{f=0\}$ will be fixed. X will denote a sufficiently small open neighborhood of x. It will therefore be assumed to be equipped with a fixed set of holomorphic coordinates $(z_1, \ldots, z_n)$. Denote by $\pi: X' \to X$ the proper bimeromorphic map placing the ideal sheaf $(f \circ \pi)\mathcal{O}_{X'}$ in locally normal crossing form over $\mathcal{X} = \pi^{-1}(f^{-1}(0))$. Let $\mathcal{M} = \mathcal{D}_X[s]f^s$ be the standard $\mathcal{D}_X[s]$ module of interest in the study of the local b-function [7]. Set $F = f \circ \pi$ and $\mathcal{M}' = \mathcal{D}_{X'}[s]F^s$. Each module admits an action by $\mathbf{C}[t]$, where $[t, s] = t$ and so that $tf^s = f^{s+1}$, $tF^s = F^{s+1}$. If p is a point of $\mathcal{X}$ in a neighborhood of which there are holomorphic coordinates $(x_1, \ldots, x_n)$ so that

$$F(x_1, \ldots, x_n) = x_1^{M_1} \ldots x_r^{M_r} \cdot (\text{local unit at } p)$$

then it is easy to see that the local b-function for F at p equals [4, pg. 245]

$$b_p(s) = \Pi_{i=1}^{r} \left[\left(s + \frac{1}{M_i}\right)\left(s + \frac{2}{M_i}\right) \ldots \left(s + \frac{M_i - 1}{M_i}\right)(s+1)\right]. \tag{4.4}$$

The following properties about $\mathcal{M}$, $\mathcal{M}'$ will be needed.

(4.5)

A) Let $\int_\pi \mathcal{M}'$ be the integration (direct image) of $\mathcal{M}'$ along the fibers of π. This is the complex $\mathbf{R}\pi_*(\mathcal{D}_{X \leftarrow X'} \otimes^L_{\mathcal{D}_{X'}} \mathcal{M}')$ [4, pg. 234]. Let $\mathcal{R} = \int_\pi^0 \mathcal{M}'$. Then $\mathcal{R}/t\mathcal{R} = \int_\pi^0 \mathcal{M}'/t\mathcal{M}'$ is a holonomic $\mathcal{D}_X$ module, admitting s-action, with support on $f^{-1}(0)$.

B) There is a global section u of $\mathcal{R}$ on X such that

i) $\mathcal{R}/\mathcal{D}'_X u$ is either zero or a holonomic $\mathcal{D}_X$ module with support on $f^{-1}(0)$.

ii) There is a $\mathcal{D}_X[s]$ sheaf surjection [4, pg. 245]

$$\mathcal{D}_X[s]\, u \twoheadrightarrow \mathcal{D}_X[s] f^s$$

C) Let $\mathcal{P} = \{p_i\}$ be a finite set of points in $\mathcal{X}$ for which coordinate charts $\mathcal{U}(x_1, \ldots, x_n)$ can be found such that $\bigcup \mathcal{U}_i(x)$ covers $\pi^{-1}(x)$ and for which $b_{p_i}(s)$ is given by an expression like (4.4). Set $B_F(s) = \Pi_{p \in \mathcal{P}} b_p(s)$. Then $B_F(s) F^s$ is a section of $\mathcal{D}_{X'}[s]F^{s+1}$ over $\mathcal{X}$. Now let $b_F(s)$ be the minimal polynomial of s action on $\mathcal{D}_{X'}[s]F^s/t\mathcal{D}_{X'}[s]F^s$ in a neighborhood of $\pi^{-1}(x)$. Clearly, $b_F(s)|B_F(s)$. Moreover, one has

$$b_f(s) | b_F(s) b_F(s+1) \ldots b_F(s+D) \tag{4.6}$$

for some non-negative integer D.

For the convenience of the reader, this crucial point is now explained. For a detailed discussion one should obviously consult [7] or [4, pg. 244 ff]. The t action on $\mathcal{M}'$ is a $\mathcal{D}_{X'}$ linear action. Thus, the property

$$b_F(s) F^s \text{ is a section of } \mathcal{D}_{X'} F^{s+1} \text{ over } \mathcal{X}$$

implies

$$b_F(s)F^s \text{ is a section of } t\mathscr{D}_{X'}F^s \text{ over } \mathscr{X}.$$

Since t action has no torsion on $\mathscr{M}'$, as operators on $\mathscr{M}'$ one has

$$b_F(s) = t\circ\theta$$

where θ is a $\mathscr{D}_{X'}$ homomorphism on $\mathscr{M}$.

One obtains corresponding operators on $\mathscr{R}$ via the functor $\int_\pi^0 \square$. Thus, $b_F(s)\mathscr{R}=t\theta(\mathscr{R})$ is a subsheaf of $t\mathscr{R}$.

(4.5) (Bi) implies that for some k, $t^k\mathscr{R}$ is a subsheaf of $\mathscr{D}_X u$. By the relation $[t, s]=t$ on $\mathscr{R}$, one finds

$$b_F(s)\, b_F(s+1)\dots b_F(s+k-1)\mathscr{R} \hookrightarrow t^k\mathscr{R} \hookrightarrow \mathscr{D}_X\, u.$$

This implies

$$b_F(s)\, b_F(s+1)\dots b_F(s+k-1)\, b_F(s+k)\,\mathscr{R} \hookrightarrow t^{k+1}\mathscr{R} \hookrightarrow t\mathscr{D}_X\, u.$$

(4.5) (Bii) now implies

$$b_F(s)\, b_F(s+1)\dots b_F(s+k-1)\, b_F(s+k) f^s \hookrightarrow \mathscr{D}_X[s] f^{s+1}.$$

This proves (4.6).

In the following, for a complex manifold W of dimension m, Ω_W is the sheaf of germs of maximal degree holomorphic differentials on W.

Because the m_i in the (4.1) appear as multiplicities via the jacobian of π, it is reasonable to belive that the sheaf $\Omega_{X'}$, should be explicitly incorporated into a $\mathscr{D}_{X'}$ module along with $\mathscr{M}'$, before the direct image functor is used to define $\mathscr{R}$. The fact that in Kashiwara's proof this is not done is the reason for the estimates (4.2). It also appears to be the case that (4.2) can not be obtained purely within the category of left modules used in [7]. Now $\mathscr{M}'$ is a left $\mathscr{D}_{X'}$ module and $\Omega_{X'}$ is a right $\mathscr{D}_{X'}$ module. Thus, to deal with global objects it is necessary to transfer considerations into the category of right $\mathscr{D}_{X'}$ modules. So, it is useful to recall a few preliminary remarks on the left-right $\mathscr{D}$-module pairings.

On the complex manifold W, let $\mathscr{M}^L(\mathscr{D}_W)$ resp. $\mathscr{M}^R(\mathscr{D}_W)$ be the category of left resp. right $\mathscr{D}_W$ modules. There is an equivalence between the two categories as follows.

(4.7)
$$\varrho: K\in ob(\mathscr{M}^L(\mathscr{D}_W)) \to K^{(r)} = \Omega_W \otimes_{\mathcal{O}_W} K \in ob(\mathscr{M}^R(\mathscr{D}_W))$$
$$\lambda: M\in ob(\mathscr{M}^R(\mathscr{D}_W)) \to M^{(l)} = \mathscr{H}om_{\mathcal{O}_W}(\Omega_W, M) \in ob(\mathscr{M}^L(\mathscr{D}_W)).$$

In the derived category of $\mathscr{M}^R(\mathscr{D}_W)$, $\mathbf{D}^b(\mathscr{M}^R(\mathscr{D}_W))$, there is a functorial notion of direct image for a map $g: W\to V$ [5, pg. 240]. It is denoted also by $\int_g$. The functors λ, ϱ extend to the derived category and one has the following diagram

$$\begin{array}{ccccc} \mathcal{M}^R(\mathcal{D}_W) & \xrightarrow{\lambda} & \mathcal{M}^L(\mathcal{D}_W) & \xrightarrow{\varrho} & \mathcal{M}^R(\mathcal{D}_W) \\ \int_g \downarrow & & \int_g \downarrow & & \downarrow \int_g \\ \mathbf{D}^b(\mathcal{M}^R(\mathcal{D}_W)) & \xrightarrow[\lambda]{} & \mathbf{D}^b(\mathcal{M}^L(\mathcal{D}_W)) & \xrightarrow[\varrho]{} & \mathbf{D}^b(\mathcal{M}^R(\mathcal{D}_W)) \end{array} \tag{4.8}$$

In particular, if one only considers the functor $\int_g^0 \square = \mathcal{H}^0(\mathbf{R}g_*(\square \otimes^L_{\mathcal{D}_W} \mathcal{D}_{W\to V}))$, then (4.8) has the lower row replaced by

$$\mathcal{M}^R(\mathcal{D}_W) \xrightarrow{\lambda} \mathcal{M}^L(\mathcal{D}_W) \xrightarrow{\varrho} \mathcal{M}^R(\mathcal{D}_W).$$

The other property needed concerns this observation. Assume that on the manifold W there is a global $\mathcal{O}_W$ isomorphism $\Omega_W \to \mathcal{O}_W$. Then to each section u of a right $\mathcal{D}_W$ module $\mathcal{R}$ there corresponds a unique section u^* of the left module $\mathcal{R}^{(l)}$ such that for any operator P in $\Gamma(W, \mathcal{D}_W)$, there is an operator P^* in $\Gamma(W, \mathcal{D}_W)$ satisfying

$$(u \cdot P)^* = P^* \cdot u^*. \tag{4.9}$$

Without the global isomorphism, one could only assert (4.9) locally near a given point of W.

Concretely, one sees this by first considering in coordinates $(x_1, \ldots, x_n)$, the vector field $\xi = \sum \xi_i \partial_{x_i}$ for P and the adjoint $P^* = -\sum \partial_{x_i} \circ \xi_i$. For the section u one has the section u^* of $\mathcal{R}^{(l)}$ determined by $u^*(dx_1 \wedge \ldots \wedge dx_n) = u$. The left $\mathcal{D}_W$ action on $\mathcal{R}^{(l)}$ is seen to be, setting $dx = dx_1 \wedge \ldots \wedge dx_n$,

$$\begin{aligned} (\xi^* \cdot u^*)(dx) &= u^*[(dx)\cdot \xi^*] - (u^*(dx))\cdot \xi^* = u^*[dx \cdot \xi^*] - u \cdot \xi^* \\ &= u^*\left[\left(\sum \frac{\partial \xi_i}{\partial x_i}\right) dx\right] + \sum \xi_i (u \cdot \partial_{x_i}) - \left(\sum \frac{\partial \xi_i}{\partial x_i}\right) u \\ &= \sum \xi_i \cdot (u \cdot \partial_{x_i}) \\ &= (u \cdot \xi)^*(dx). \end{aligned}$$

One iterates the formula to extend to $\mathcal{D}_W$.

(4.10) With these remarks the proof of Theorem 5 can begin. The left $\mathcal{D}_X[s]$ module $\mathcal{M}$ generated by f^s has associated to it a right $\mathcal{D}_X[s]$ module $\mathcal{M}^{(r)}$, defined above. The s-action and right $\mathcal{D}_X$ actions commute. Since coordinates $(z_1, \ldots, z_n)$ have been fixed on X, it follows that $\Omega_X \cong \mathcal{O}_X$ over X. Thus, there is a global section $(f^s)^*$ of $\mathcal{M}^{(r)}$ corresponding to f^s.

By converting the local functional equation

$$\mathcal{P} f^{s+1} = b_x(s) f^s$$

into the right module category, using (4.9), one notes that at each x there is a local functional equation in the stalk at any $x\in X$

$$(4.11)\qquad f(f^s)^*\cdot\tilde{\mathscr{P}} = b_x^*(s)(f^s)^*.$$

It follows that $b_x(s)=b_x^*(s)$. So, Theorem 5 can then be interpreted as a property of the local b-function for the section $(f^s)^*$ of $\mathscr{M}^{(r)}$.

Using the left-right equivalences (4.7), (4.8), one proves the theorem by adapting Kashiwara's proof of (4.2) to the right module $\mathscr{M}^{(r)}$. In the notation above, set

$$\mathscr{N} = \Omega_{X'}\otimes_{\mathscr{O}_{X'}}\mathscr{D}_{X'}F^s.$$

Then $\mathscr{N}$ is a subholonomic right $\mathscr{D}_{X'}$ module. There is an evident submodule to consider. Set $u=\pi^*(dx)\otimes F^s$. This is a global section of $\mathscr{N}$. Set

$$\mathscr{T} = u\cdot\mathscr{D}_{X'}.$$

It is clear that $\mathscr{T}$ resp. $\mathscr{M}^{(r)}$ admit a $\mathbf{C}[t]$ action which is multiplication by F resp. f. Set

$$\mathscr{S} = \int_\pi^0\mathscr{T} = \mathscr{H}^0(\mathbf{R}\pi_*(\mathscr{T}\otimes^L_{\mathscr{D}_{X'}}\mathscr{D}_{X'\to X})).$$

Then $\mathscr{S}$ is a right $\mathscr{D}_X$ module and a submodule of $\int^0\mathscr{N}$. The singular support of $\mathscr{S}$ agrees with that of the left module $\mathscr{R}$ of (4.5) (A). So, it is subholonomic. Thus, it suffices to prove the analogues of (4.5) (B—C).

(4.12)

(1) There is a global section v of $\mathscr{S}$ such that there is a right $\mathscr{D}_X$ surjection

$$v\cdot\mathscr{D}_X[s]\twoheadrightarrow(f^s)^*\cdot\mathscr{D}_X[s](=(\mathscr{D}_X[s]f^s)^{(r)}).$$

(2) $\mathscr{S}/v\cdot\mathscr{D}_X[s]$ is zero or holonomic with support on $f^{-1}(0)$.

(3) Let $b_F^*(s)$ be the minimal polynomial of s-action on $\mathscr{T}/t\mathscr{T}$. Let $\mathscr{P}=\{p_i\}$ be a finite set of points as in (4.5) (C). Then $b_F^*(s)|\prod_{p\in\mathscr{P}}b_{F,p}^*(s)$, where, if p is a point of $\pi^{-1}(x)$, in a neighborhood of which coordinates $(x_1, \dots, x_n)$ exist so that

$$F(x_1, \dots, x_n) = \prod_1^r x_i^{M_i}\quad\text{and}\quad\det d\pi(x_1, \dots, x_n) = \prod_1^r x_i^{m_i}\,(\text{local unit at } p),$$

then

$$b_{F,p}^*(s) = \prod_1^r\left[\left(s+\frac{m_i+1}{M_i}\right)\cdots\left(s+\frac{m_i+M_i}{M_i}\right)\right].$$

(4.13)

The proofs of (1—3) are, to be sure, simple modifications of those for (4.5). As such, sketches of the arguments will suffice.

(1) For any right $\mathscr{D}_{X'}$ module $\mathscr{Q}$, one determines $\int_\pi \mathscr{Q}$ by embedding π into a triangle

$$\begin{array}{ccc} X' & \overset{\iota}{\hookrightarrow} & Z \\ & \underset{\pi}{\searrow} & \downarrow{\scriptstyle\eta} \\ & & X \end{array}$$

where Z is a projective variety, ι is a closed embedding, and η is a submersion. Denote $\mathscr{H}om_{\mathcal{O}_Z}(\Omega_Z, \mathcal{O}_Z)$ by Ω_Z^{-1} below.

Then $\int_\pi \mathscr{Q} = \mathbf{R}\pi_*[\iota_*(\mathscr{Q} \otimes_{\mathscr{D}_{X'}} \mathscr{D}_{X'\to Z}) \otimes^L_{\mathscr{D}_Z} \mathscr{D}_{Z\to X}]$, where, because of the sheaf identification Ω_X with $\mathcal{O}_X$, one identifies

$$\mathscr{D}_{Z\to X} = \mathscr{D}_{X\leftarrow Z} \otimes_{\mathcal{O}_Z} \Omega_Z^{-1}.$$

Since Ω_Z^{-1} is a flat $\mathcal{O}_Z$ module, a resolution of $\mathscr{D}_{Z\to X}$ by left $\mathscr{D}_Z$ modules is given by the complex

$$\Omega^\bullet_{Z/X} \otimes_{\mathcal{O}_Z} \mathscr{D}_Z \otimes_{\mathcal{O}_Z} \Omega_Z^{-1} \to \mathscr{D}_{Z\to X} \to 0$$

where $\Omega^\bullet_{Z/X}$ is the relative de Rham complex for η [4, p. 235].

Since $\Omega^n_{Z/X} \cong \Omega_Z$ as an $\mathcal{O}_Z$ module, one sees that

$$\iota_*(\mathscr{D}_{X'\to Z}) \otimes_{\mathscr{D}_Z} (\Omega^n_{Z/X} \otimes_{\mathcal{O}_Z} \mathscr{D}_Z \otimes_{\mathcal{O}_Z} \Omega_Z^{-1}) \cong \iota_*(\mathscr{D}_{X'\to Z}),$$

and for $\mathscr{T} = u\mathscr{D}_{X'}$,

$$\begin{aligned} &\iota_*(\mathscr{T} \otimes_{\mathscr{D}_{X'}} \mathscr{D}_{X'\to Z}) \otimes_{\mathscr{D}_Z} (\Omega^n_{Z/X} \otimes_{\mathcal{O}_Z} \mathscr{D}_Z \otimes_{\mathcal{O}_Z} \Omega_Z^{-1}) \\ &\quad = \iota_*(u) \otimes_{\mathbf{C}_{X'}} (\mathcal{O}_{X'} \otimes_{\iota^{-1}\mathcal{O}_Z} \iota^{-1}\mathscr{D}_Z). \end{aligned}$$

There is a global section, determined by $\iota_*(u)$, of the group of n-cocycles of the complex $\Gamma(Z, \iota_*(\mathscr{T} \otimes_{\mathscr{D}_{X'}} \mathscr{D}_{X'\to Z}) \otimes^L_{\mathscr{D}_Z} \mathscr{D}_{Z\to X})$. It is obtained by taking the global section $1_{X'}$ of $\mathcal{O}_{X'}$ and 1_Z of $\mathscr{D}_Z$. Let $[u]$ denote the cohomology class in

$$H^n(\Gamma(Z, \iota_*(\mathscr{T} \otimes_{\mathscr{D}_{X'}} \mathscr{D}_{X'\to Z}) \otimes^L_{\mathscr{D}_Z} \mathscr{D}_{Z\to X})).$$

Let v denote the image of $[u]$ under the edge homomorphism [4, p. 283]. This defines a global section of $\mathscr{S}$ which agrees with $(f^s)^*$ off $f^{-1}(0)$.

Properties (1), (2) of (4.12) now follow by arguments easily translated (i.e. "left" becomes "right") from those proving (4.5) B (i, ii).

So, (3) is the only new calculation that needs verification.

Given u and point p as in the statement of (3), denote by $\mathscr{W}(x)$ the local unit factor of $\det d\pi$. One looks for an operator $\mathscr{P}(s, x, D_x)$ such that

$$\begin{aligned} &[(\Pi_1^r x_1^{m_i})\mathscr{W}(x)\, dx \otimes \Pi_1^r (x_1^{M_i})^{s+1}]\,\mathscr{P} \\ &\quad = b^*_{F,p}(s)\,[\Pi_1^r (x_i^{m_i})\mathscr{W}(x)\, dx \otimes \Pi_1^r (x_i^{M_i})^s] \end{aligned} \tag{4.14}$$

for all points x in the neighborhood $\mathscr{U}(x_1, \dots, x_n)$ of p.

Let

$$\mathscr{P}_1 = (-)\frac{\partial^{M_1}}{\partial x_1^{M_1}} \cdots \frac{\partial^{M_r}}{\partial x_r^{M_r}}.$$

Set

$$\mathscr{P} = [\mathscr{W}(x)\cdot]\circ\mathscr{P}_1\circ[\mathscr{W}(x)^{-1}\cdot],$$

where the zero degree operators in brackets mean multiply the indicated function.

Because $\mathscr{P}_1$ is a constant coefficient operator and the tensor product in (4.14) is over $\mathcal{O}_{X'}$, it is clear that (4.14) holds. Indeed, one evidently has using the definition of the right action on $\Omega_{X'}$ in terms of the lie derivative $-\mathscr{L}_{\partial/\partial x_j}$,

$$[(\prod x_i^{m_i})\, dx \otimes_{\mathcal{O}_{X'}} (\prod x_i^{M_i})^{s+1}]\, \mathscr{P}_1 = b^*_{F,p}(s)\, [\prod x_i^{m_i})\, dx \otimes_{\mathcal{O}_{X'}} (\prod x_i^{M_i})^s]. \quad \blacksquare$$

Remarks. 1) Note that no assumption has been made about the nature of the singular point x (Theorem 5 is evident if x is nonsingular). This is because Kashiwara's proof is both local and for any singular point of f.

2) It would be interesting to know if the following is true. In the notation of (4.1), let $\mathscr{E}=\{D_i\colon D_i$ is an irreducible component of the exceptional locus for the resolution $\pi\}$. So, the strict transform does not appear in $\mathscr{E}$. Let

$$\varrho_f = \max_{D_i \in \mathscr{E}} \{-(m_i+1)/M_i\}.$$

Then, is ϱ_f a root of $b_f(s)$? (If $\varrho_f > -1$, it is well known that the answer is yes.)

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Recently (in the American J. of Mathematics) F. Loeser has extended Theorem 1 to the case where f is reducible at 0.