On essential maximality of linear pseudo-differential operators

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1. Introduction

We consider linear pseudo-differential operators $L(x, D)$ of the Beals and Fefferman type. $L(x, D)$ maps the Schwartz class S into itself. Furthermore, the formal transpose $L'(x, D)$ of $L(x, D)$ exists and $L'(x, D)$ maps S into itself, as well. This enables us to define the minimal closed realization $L_k^{\sim}$ and the maximal closed realization $L_k'^{\#}$ of $L(x, D)$ in the appropriate Hilbert space H_k (cf. the Subsections 2.1 and 2.2). In the case when $k=1$, we see that $H_k=L_2(\mathbf{R}^n)$. We write $L_k^{\sim}=L^{\sim}$ and $L_k'^{\#}=L'^{\#}$ when $k=1$.

Our aim is to give sufficient criteria for the equality $L^{\sim}=L'^{\#}$, that is, for the essential maximality of $L(x, D)$ in $L_2(\mathbf{R}^n)$. One knows classes of operators $L(x, D)\colon S\to S$ which are essentially maximal in $L_2(\mathbf{R}^n)$ (cf. [2], [3], [6] and [7]) We consider also the bijectivity of $L^{\sim}+aI$ when a is large enough. The bijectivity of $L^{\sim}+aI$ and $L'^{\#}+aI$ implies the equality $L^{\sim}=L'^{\#}$.

Employing the convolution theory we, at first, show the essential maximality of $L(x, D)$, when $L(\cdot, \cdot)$ belongs to the Beals and Fefferman class $S_{\Phi,\varphi}^{1,1}$ of symbols (cf. Theorem 3.6). After that we apply our theory on the class $S_{\Phi,\varphi}^{M,m}$, $M, m\in\mathbf{R}$ of, operators. When the solutions of $L'^{\#}u=f$ belong to H_q with a suitable $q(\cdot)\in S_{\Phi,\varphi}^{M-1,m-1}$, the essential maximality of $L(x, D)$ is verified (cf. Theorem 3.7). In Chapter 4 we deal with the bijectivity of $L^{\sim}+aI$ and $L'^{\#}+aI$. In addition, we obtain an algebraic criterion for the essential maximality and for the inclusion

$$D(L'^{\#})\subset H_q \tag{1.1}$$

where $q(\cdot)$ is suitably chosen from $S_{\Phi,\varphi}^{M-1,m-1}$.

Especially, applying our theory on the Hörmander class of operators we obtain: Suppose that $L(\cdot, \cdot)\in S_{\varrho,\delta}^{m}$; $\delta<\varrho$, $m\geqq 0$ such that with a constant $t\in]m-(\varrho-\delta), m]$

one has

$$\operatorname{Re} L(x, \xi) \geqq c(1+|\xi|)^t \quad \text{for all} \quad |\xi| \geqq E. \tag{1.2}$$

Then the corresponding pseudo-differential operator $L(x, D)$ is essentially maximal in $L_2(\mathbf{R}^n)$ and $L^{\sim}+aI$: $L_2(\mathbf{R}^n)\to L_2(\mathbf{R}^n)$ is a bijection when a is large enough.

2. General background

2.1. Denote by K' the totality of continuous weight functions k: $\mathbf{R}^n\to\mathbf{R}$ such that

$$c(1+|\xi|^2)^{-r/2} \leqq k(\xi) \leqq C(1+|\xi|^2)^{R/2} =: Ck_R(\xi), \quad \text{for} \quad \xi\in\mathbf{R}^n, \tag{2.1}$$

where c, C, r and R positive constants. When k is in K' one sees that the functions k^s and $k^{\check{}}$ defined by $k^s(\xi)=(k(\xi))^s$ and $k^{\check{}}(\xi)=k(-\xi)$ are also elements of K'. Let S denote the Schwartz class of smooth functions φ: $\mathbf{R}^n\to\mathbf{C}$ and let S' be the dual of S (cf. [4], pp. 1—33). We define a scalar product $\langle\cdot, \cdot\rangle_k$ in S by the requirement

$$\langle\varphi, \psi\rangle_k = (2\pi)^{-n}\int_{\mathbf{R}^n}(F\varphi)(\xi)\overline{(F\psi)(\xi)}k^2(\xi)\,d\xi, \tag{2.2}$$

where F: $S\to S$ is the Fourier transform. The completion of S with respect to the scalar product $\langle\cdot, \cdot\rangle_k$ is denoted by H_k. Suppose that u belongs to H_k. Choose a representative $\{\varphi_n\}\subset S$ of u. Applying the Banach—Steinhaus Theorem and the fact that by the Parseval formula one has

$$|(\varphi, \psi)| := \left|\int_{\mathbf{R}^n}\varphi(x)\psi(x)\,dx\right| \leqq \|\varphi\|_k\,\|\psi\|_{1/k^{\check{}}}, \tag{2.3}$$

we see that the linear mapping λ defined by

$$(\lambda u)(\varphi) = \lim_{n\to\infty}(\varphi_n, \varphi) \quad \text{for} \quad \varphi\in S,\, u\in H_k \tag{2.4}$$

maps H_k injectively onto a subspace $\lambda(H_k)$ of S'. In the sequel we denote the space $\lambda(H_k)$ by H_k, as well. A familiar characterization of the H_k-space is the following one: A distribution $T\in S'$ belongs to H_k if and only if $FT\in L_1^{\mathrm{loc}}(\mathbf{R}^n)$ and

$$\|T\|_k := \left((2\pi)^{-n}\int_{\mathbf{R}^n}|(FT)(\xi)k(\xi)|^2\,d\xi\right)^{1/2} < \infty. \tag{2.5}$$

Here F: $S'\to S'$ denotes the Fourier transform. Furthermore, one knows that $\|T\|_k=\|\lambda^{-1}(T)\|_{H_k}$ and in the following H_k is equipped with this norm. H_k is also a Hilbert space. The scalar product $\langle\cdot, \cdot\rangle_k$ is given by

$$\langle u, v\rangle_k = (2\pi)^{-n}\int_{\mathbf{R}^n}(Fu)\overline{(Fv)(\xi)}\,k^2(\xi)\,d\xi. \tag{2.6}$$

As a Hilbert space H_k is a reflexive Banach space. In addition, the next characterization of the dual H_k^* of H_k is valid

Lemma 2.1. *Suppose that l^* is in H_k^*. Then there exists a unique element $l\in H_{1/k^\vee}$ such that*

$$l^*\varphi = l(\varphi) \quad \text{for all } \varphi\in S. \tag{2.7}$$

On the other hand, suppose that l is in $H_{1/k^\vee}$. Then $l\colon S\to\mathbf{C}$ has a unique continuous extension l^* on H_k. The linear mapping $\lambda_k\colon H_k^*\to H_{1/k^\vee}$ defined by $\lambda_k(l^*)=l$ is an isometrical isomorphism. □

2.2. Let L be a linear operator $S\to S$. We suppose that the formal transpose L' of L exists, in other words, there exists a linear operator $L'\colon S\to S$ such that

$$(L\varphi,\psi) := \int_{\mathbf{R}^n} (L\varphi)(x)\psi(x)\,dx = (\varphi, L'\psi) \quad \text{for} \quad \varphi,\psi\in S. \tag{2.8}$$

This assumption enables us to define a dense linear operator $L'^\#\colon L_2:=L_2(\mathbf{R}^n)\to L_2$ by the requirement

$$\begin{cases} D(L'^\#) = \{u\in L_2 \mid \text{there exists } f\in L_2 \text{ such that } u(L'\varphi) = f(\varphi) \text{ for all } \varphi\in S\}, \\ L'^\# u = f. \end{cases} \tag{2.9}$$

Here we denoted $g(\varphi)=\int_{\mathbf{R}^n} g(x)\varphi(x)\,dx$. One sees that $L'^\#$ is a closed operator.

Furthermore, we define a dense linear operator $L_0\colon L_2\to L_2$ by

$$\begin{cases} D(L_0) = S, \\ L_0\varphi = L\varphi \quad \text{for} \quad \varphi\in S. \end{cases} \tag{2.10}$$

In virtue of (2.8) one sees that L_0 is closable in L_2. Denote by $L^\sim\colon L_2\to L_2$ the smallest closed extension of L_0 (cf. [8], pp. 76—79). One obtains that $L^\sim\subset L'^\#$. The operator $L^\sim$ ($L'^\#$, respectively) is called the minimal realization of L in L_2 (and the maximal realization of L in L_2, respectively). When the equality $L^\sim = L'^\#$ holds, we say that L is essentially maximal in L_2.

Remark 2.2. A) Let $L^*\colon L_2^*\to L_2^*$ be the dual operator of L_0 and let $L^{**}\colon L_2^{**}\to L_2^{**}$ be the dual operator of L^*. Since L_2 is reflexive, one knows that

$$L^\sim = J^{-1}\circ L^{**}\circ J, \tag{2.11}$$

where $J\colon L_2\to L_2^{**}$ is the canonical isometrical isomorphism (cf. [5], p. 168).

B) The operators $L'^\#$ and L'^* (here L'^* is the dual of $(L')_0$) obey the relation

$$L'^* = \lambda^{-1}\circ L'^\#\circ\lambda, \tag{2.12}$$

where $\lambda\colon L_2\to L_2^*$ is the isometrical isomorphism announced in Lemma 2.1 (one must note that $H_k=L_2$, when $k=1$).

3. On essential maximality

3.1. Let Φ and φ: $\mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ form a pair of continuous weight functions in the sence of Beals and Fefferman [1], that is, the following criteria hold: There exist constants $c>0$, $C>0$ and $\varepsilon>0$ such that

(ia) $c \leqq \Phi(x, \xi) \leqq C(1+|\xi|)$, for all $x, \xi \in \mathbf{R}^n$,

(ib) $c(1+|\xi|)^{\varepsilon-1} \leqq \varphi(x, \xi) \leqq C$, for all $x, \xi \in \mathbf{R}^n$,

(ii) $\Phi(x, \xi)\varphi(x, \xi) \geqq c$ for all $x, \xi \in \mathbf{R}^n$,

(iii) For each $r>0$ there exists $q_r>0$ such that

$$\frac{\Phi(x, \xi)}{\varphi(x, \xi)} \cdot \frac{\varphi(y, \eta)}{\Phi(y, \eta)} + \frac{\Phi(y, \eta)}{\varphi(y, \eta)} \cdot \frac{\varphi(x, \xi)}{\Phi(x, \xi)} \leqq q_r$$

for all $(x, y, (\xi, \eta)) \in \mathbf{R}^n \times \mathbf{R}^n \times \{(\xi, \eta) \in \mathbf{R}^n \times \mathbf{R}^n | (|\xi|/|\eta|) + (|\eta|/|\xi|) \leqq r\}$.

(iv) For each $(x, \xi) \in \mathbf{R}^n \times \mathbf{R}^n$ there exists a constant $C>0$ such that

$$\frac{\Phi(y, \eta)}{\Phi(x, \xi)} + \frac{\Phi(x, \xi)}{\Phi(y, \eta)} \leqq C \quad \text{and} \quad \frac{\varphi(y, \eta)}{\varphi(x, \xi)} + \frac{\varphi(x, \xi)}{\varphi(y, \eta)} \leqq C$$

for all $(y, \eta) \in U_{x,\xi} := \{(y, \eta) \in \mathbf{R}^n \times \mathbf{R}^n | |y-x| < c\varphi(x, \xi) \text{ and } |\eta-\xi| < c\Phi(x, \xi)\}$.

For example, the functions Φ and φ defined by $\Phi(x, \xi) = (1+|\xi|)^{\varrho}$ and $\varphi(x, \xi) = (1+|\xi|)^{-\delta}$; $0 \leqq \delta \leqq \varrho \leqq 1$, $\delta < 1$ form a pair of weight functions.

Choose M and m from $\mathbf{R}$. Then we say that the function $L(x, \xi) \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ is in the class $S^{M,m}_{\Phi,\varphi}$ provided that for each pair $(\alpha, \beta) \in \mathbf{N}_0^n \times \mathbf{N}_0^n$ there exists a constant $C_{\alpha,\beta}>0$ such that

$$|(D_x^\alpha D_\xi^\beta L)(x, \xi)| \leqq C_{\alpha,\beta} \Phi^{M-|\beta|}(x, \xi) \varphi^{m-|\alpha|}(x, \xi) \tag{3.1}$$

for all $(x, \xi) \in \mathbf{R}^n \times \mathbf{R}^n$.

Suppose that $L(x, \xi) \in S^{M,m}_{\Phi,\varphi}$. Define a linear pseudo-differential operator $L(x, D)$ by

$$(L(x, D)\varphi)(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} L(x, \xi)(F\varphi)(\xi) e^{i(\xi, x)} \, d\xi, \tag{3.2}$$

where $\varphi \in S$. In [1] one has proved that $L(x, D)$ maps S into S and the the formal transpose $L'(x, D)$: $S \to S$ of $L(x, D)$ exists. In addition, $L(x, D)$ and $L'(x, D)$: $S \to S$ are continuous. In [1] one has developed a fertile calculus for the pseudo-differential operators (3.2), where $L(\cdot, \cdot)$ belongs to $\bigcup_{M,m \in \mathbf{R}} S^{M,m}_{\Phi,\varphi}$ (the elements of $\bigcup_{M,m \in \mathbf{R}} S^{M,m}_{\Phi,\varphi}$ are called symbols). In the sequel we shall apply this calculus and, in addition, the following two theorems

Theorem 3.1. *Suppose that* $L(x,\xi)\in S^{0,0}_{\Phi,\varphi}$. *Then there exists a constant* $C(L(x,\xi))>0$ *such that*

$$\text{(3.3)}\qquad \|L(x,D)\varphi\|:=\|L(x,D)\varphi\|_{L_2}\leqq C(L(x,\xi))\|\varphi\|,\quad \textit{for all}\quad \varphi\in S.$$

Furthermore, let A be a subset of $S^{0,0}_{\Phi,\varphi}$ *such that*

$$p^{0;0}_{\alpha;\beta}(Q(x,\xi)):=\sup_{x,\xi}\Phi^{|\beta|}(x,\xi)\varphi^{|\alpha|}(x,\xi)\,|(D^\alpha_x D^\beta_\xi Q)(x,\xi)|\leqq C_{\alpha,\beta}<\infty$$

for all $Q(x,\xi)\in A$. *Then the constant* $C(Q(x,\xi))$ *can be chosen to be independent of* $Q(x,\xi)$ *on A.* □

For the proof cf. the proof of Theorem 3.1 given in [1], pp. 12—17.

Theorem 3.2. *Suppose that* $L(x,\xi)\in S^{M,m}_{\Phi,\varphi}$ *such that*

$$\text{(3.4)}\qquad L(x,\xi)\geqq 0\quad \textit{for all}\quad x,\xi\in\mathbf{R}^n.$$

Then there exists a symbol $l(x,\xi)\in S^{M-1,m-1}_{\Phi,\varphi}$ *such that*

$$\text{(3.5)}\qquad \operatorname{Re}\langle (L(x,D)+l(x,D))\varphi,\varphi\rangle\geqq 0\quad \textit{for all}\quad \varphi\in S,$$

where $\langle\cdot,\cdot\rangle$ *denotes the* L_2 *scalar product.* □

For the proof cf. Theorem 3.2 showed in [1], p. 19.

3.2. Let $\theta\in C_0^\infty:=C_0^\infty(\mathbf{R}^n)$ such that $\theta(x)=1$ for $x\in B(0,1):=\{x\in\mathbf{R}^n\,|\,|x|<1\}$. Define $\theta':=(2\pi)^{-n}\theta$, $\theta_l:=\theta(x/l)$ and $\theta'_l:=\theta'(x/l)$. Then one sees that

$$(F\theta'_l)(\eta)=\int_{\mathbf{R}^n}\theta'(x/l)\,e^{-i(x,\eta)}\,dx=l^n(F\theta')(l\eta),$$

where

$$\int_{\mathbf{R}^n}(F\theta')(\xi)\,d\xi=\int_{\mathbf{R}^n}(F\theta')(\xi)e^{i(0,\xi)}\,d\xi=(2\pi)^n\theta'(0)=1.$$

Write $\psi_l:=F\theta'_l$. Then we obtain for any $u\in L_2$

$$\text{(3.6)}\qquad \|\theta_l u-u\|\to 0\quad\text{with}\quad l\to\infty$$

and

$$\text{(3.7)}\qquad \|\psi_l * u-u\|\to 0\quad\text{with}\quad l\to\infty.$$

The next lemma is easily proved

Lemma 3.3. *Suppose that* $L(x,\xi)\in S^{M,m}_{\Phi,\varphi}$. *Then there exist constants* $C>0$ *and* $N\in\mathbf{R}$ *such that*

$$\text{(3.8a)}\qquad \|L(x,D)\varphi\|\leqq C\|\varphi\|_{k_N}$$

and

(3.8b) $$\|L(x, D)\varphi\|_{k_{-N}} \leqq C\|\varphi\| \quad \text{for all} \quad \varphi \in S. \quad \square$$

Here we denoted as above $k_s(\xi)=(1+|\xi|^2)^{s/2}$.

Since $\psi_l * u \in \bigcap_{k\in K'} H_k$ one sees by (3.8a) that

(3.9) $$\psi_l * u \in D(L^{\sim}) \quad \text{for any} \quad l \in \mathbf{N} \quad \text{and} \quad u \in L_2.$$

For $M=m=1$ we have

Lemma 3.4. *Suppose that* $L(x, \xi) S^{1,1}_{\Phi,\varphi}$. *Then for any* $(j, l) \in \mathbf{N}\times\mathbf{N}$ *there exists* $R_{j,l}(x, \xi) \in S^{0,0}_{\Phi,\varphi}$ *such that*

(3.10) $$\psi_l*(\theta_j L(x, D)\varphi) = \theta_j L(x, D)(\psi_l*\varphi)+R_{j,l}(x, D)\varphi$$

for all $\varphi \in S$ *and that*

(3.11) $$\|R_{j,l}(x, D)\varphi\| \leqq C\|\varphi\| \quad \text{for all} \quad \varphi \in S,$$

where the constant $C>0$ *is independent of* $j, l \in \mathbf{N}$.

Proof. A) Define a pseudo-differential operator $\hat{\psi}_l(D)$ by

$$(\hat{\psi}_l(D)\varphi)(x) = (2\pi)^{-n}\int_{\mathbf{R}^n} (F\psi_l)(\xi)(F\varphi)(\xi)e^{i(\xi,x)}\, d\xi.$$

Furthermore, write $L_j(x, \xi)=\theta_j L(x, \xi)$. Trivially one has $\hat{\psi}_l(\xi) \in S^{0,0}_{\Phi,\varphi}$ and so we obtain

(3.12) $$[\psi_l*(\theta_j L(x, D)\varphi)](x) = (2\pi)^{-n}\int_{\mathbf{R}^n} F(\psi_l*(\theta_j L(x, D)\varphi))(\xi)e^{i(\xi,x)}\, d\xi$$

$$= (2\pi)^{-n}\int_{\mathbf{R}^n} (F\psi_l)(\xi)F(L_j(x, D)\varphi)(\xi)e^{i(\xi,x)}\, d\xi$$

$$= [(\hat{\psi}_l(D)\circ L_j(x, D))\varphi](x) =: [(\hat{\psi}_l\circ L_j)(x, D)\varphi](x).$$

In addition, we obtain (cf. [1], p. 5)

(3.13) $$(\hat{\psi}_l\circ L_j)(x, \xi) = (F\psi_l)(x, \xi)L_j(x, \xi)+R_{j,l}(x, \xi),$$

where

$$R_{j,l}(x, \xi) = \sum_{|\gamma|=1}(1/\gamma!)\int_0^1\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \partial^\gamma(F\psi_l)(\xi+t(\eta-\xi))(D_y^\gamma L_j)(y, \xi)e^{i(x-y,\eta-\xi)}\, dy.$$

The symbol $(\hat{\psi}_l L_j)(x, \xi):=(F\psi_l)(\xi)L_j(x, \xi)$ induces the operator

(3.14) $$((\hat{\psi}_l L_j)(x, D)\varphi)(x) = (2\pi)^{-n}\int_{\mathbf{R}^n} (F\psi_l)(\xi)L_j(x, \xi)(F\varphi)(\xi)e^{i(\xi,x)}\, d\xi$$

$$= (2\pi)^{-n}\int_{\mathbf{R}^n} L_j(x, \xi)F(\psi_l*\varphi)(\xi)e^{i(\xi,x)}\, d\xi = \theta_j(x)(L(x, D)(\psi_l*\varphi))(x)$$

and then our task reduces to show that $R_{j,l}(x,\xi)\in S^{0,0}_{\Phi,\varphi}$ and that the estimate (3.11) holds. Since $\hat{\psi}_l(\xi)\in S^{0,0}_{\Phi,\varphi}$ and since $L_j(x,\xi)\in S^{1,1}_{\Phi,\varphi}$ we know that $R_{j,l}(x,\xi)\in S^{0,0}_{\Phi,\varphi}$.

B) Since $\psi_l=F\theta'_l$, one has $F\psi_l=(2\pi)^n(\theta'_l)^\vee=\theta_l^\vee$. Because $\theta\in C_0^\infty\subset S$ we can (for any $\beta\in\mathbf{N}_0^n$) choose a constant $C_\beta>0$ such that

$$|(D^\beta\theta)(\xi)|\leqq C_\beta(1+|\xi|)^{-|\beta|}\quad\text{for all}\quad \xi\in\mathbf{R}^n.$$

Furthermore, one has (recall that $\Phi(x,\xi)\leqq C(1+|\xi|)$)

$$(1+|\xi|)^{-|\beta|}\leqq C^{|\beta|}\Phi^{-|\beta|}(x,\xi)\quad\text{for all}\quad x,\xi\in\mathbf{R}^n,$$

and so we obtain

$$\text{(3.15)}\quad p^{0,0}_{0,\beta}(\hat{\psi}_l(\xi)):=\sup_{x,\xi}\left(\Phi^{|\beta|}(x,\xi)\,|D_\xi^\beta(\hat{\psi}_l)(\xi)|\right)=\sup_{x,\xi}\left(\Phi^{|\beta|}(x,\xi)\,l^{-|\beta|}\,|(D^\beta\theta)(-\xi/l)|\right)$$

$$\leqq C_\beta\sup_{x,\xi}\Phi^{|\beta|}(x,\xi)\,l^{-|\beta|}(1+|\xi/l|)^{-|\beta|}\leqq C_\beta\sup_{x,\xi}\left(\Phi^{|\beta|}(x,\xi)(1+|\xi|)^{-|\beta|}\right)=C_\beta C^{|\beta|}.$$

Hence the sequence $\{\hat{\psi}_l(\xi)\}_l$ is bounded in $S^{0,0}_{\Phi,\varphi}$ (for the definition of the Frechet space topology in $S^{M,m}_{\Phi,\varphi}$ we refer to [1], p. 3).

C) Since for any $j\in\mathbf{N}$ and $\alpha\in\mathbf{N}_0^n$ one has

$$|(D^\alpha\theta_j)(x)|=j^{-|\alpha|}|(D^\alpha\theta)(x/j)|\leqq\sup_x|(D^\alpha\theta)(x)|,$$

one sees by the Leibniz rule

$$\text{(3.16)}\qquad |(D_x^\alpha D_\xi^\beta L_j)(x,\xi)|\leqq\sum_{u\leqq\alpha}\binom{\alpha}{u}|(D^u\theta_j)(x)|\,|(D_x^{\alpha-u}D_\xi^\beta L)(x,\xi)|$$

$$\leqq\sum_{u\leqq\alpha}\binom{\alpha}{u}\left(\sup_x|(D^u\theta)(x)|\right)C_{\alpha-u,\beta}\,\Phi^{1-|\beta|}(x,\xi)\varphi^{1-|\alpha|+|u|}(x,\xi)$$

$$\leqq\sum_{u\leqq\alpha}\binom{\alpha}{u}C^{|u|}\left(\sup_x|(D^u\theta)(x)|\right)C_{\alpha-u,\beta}\,\Phi^{1-|\beta|}(x,\xi)\varphi^{1-|\alpha|}(x,\xi)$$

and so

$$p^{1,1}_{\alpha,\beta}(L_j(x,\xi)):=\sup_{x,\xi}\left(\Phi^{-1+|\beta|}(x,\xi)\,\varphi^{-1+|\alpha|}(x,\xi)|(D_x^\alpha D_\xi^\beta L_j)(x,\xi)|\right)\leqq C'_{\alpha,\beta}.$$

Hence the sequence $\{L_j(x,\xi)\}_j$ is bounded in $S^{1,1}_{\Phi,\varphi}$.

D) Due to Theorem 1 of [1], p. 4, one obtains that (3.13) holds and that the sequence $\{R_{j,l}(x,\xi)\}_{j,l}$ is bounded in $S^{0,0}_{\Phi,\varphi}$ (cf. also the proof of Theorem 1 of [1]), that is,

$$p^{0,0}_{\alpha,\beta}(R_{j,l}(x,\xi))\leqq C_{\alpha,\beta}\quad\text{for all}\quad l,j\in\mathbf{N}.$$

Thus by Theorem 3.1 the estimate (3.11) is valid. This completes the proof. □

Lemma 3.5. *Suppose that* $L(x, \xi) S^{1,1}_{\Phi,\varphi}$. *Let* u *be in* $D(L'^{\#})$. *Then one has*

$$\|L'^{\#}(\psi_l * u)\| \leqq C(\|L'^{\#}u\| + \|u\|), \tag{3.17}$$

where C *is independent of* l.

Proof. In virtue of (3.10)—(3.11) we get

$$\begin{aligned} \|\theta_j L(x, D)(\psi_l * \varphi)\| &\leqq \|\psi_l * (\theta_j L(x, D)\varphi)\| + \|R_{j,l}(x, D)\varphi\| \\ &\leqq \|\psi_l * \theta_j L(x, D)\varphi\| + C\|\varphi\|. \end{aligned} \tag{3.18}$$

Since $\|\theta_j L(x, D)(\psi_l * \varphi)\| \to \|L(x, D)(\psi_l * \varphi)\|$ with $j \to \infty$ and since (cf. [4], p. 39)

$$\begin{aligned} &\|\psi_l * (\theta_j L(x, D)\varphi) - \psi_l * (L(x, D)\varphi)\| \\ &\leqq \|\psi_l\|_{\infty,1} \|\theta_j L(x, D)\varphi - L(x, D)\varphi\| \to 0 \quad \text{with} \quad j \to \infty \end{aligned} \tag{3.19}$$

we obtain that

$$\|L(x, D)(\psi_l * \varphi)\| \leqq \|\psi_l * L(x, D)\varphi\| + C\|\varphi\|. \tag{3.20}$$

Choose a sequence $\{\varphi_n\} \subset C_0^\infty$ such that $\|\varphi_n - u\| \to 0$ with $n \to \infty$. Furthermore, choose $N \in \mathbf{N}$ such that (3.8b) holds. Then we obtain (cf. [4], p. 39)

$$\|\psi_l * (L(x, D)\varphi)\| \leqq \|L(x, D)\varphi\|_{k_{-N}} \|\psi_l\|_{\infty, k_N} \leqq C\|\varphi\| \|\psi_l\|_{\infty, k_N} \tag{3.21}$$

and so $\|\psi_l * L(x, D)\varphi_n - \psi_l * L'^{\#}u\| \to 0$ with $n \to \infty$, (this follows from the fact that by (3.21) $\{\psi_l * L(x, D)\varphi_n\}_n$ is a Cauchy sequence in L_2 and that

$$(\psi_l * L(x, D)\varphi_n)(x) = \varphi_n(L'(x, D)\psi_l(x - (\cdot))) \to (\psi_l * L'^{\#}u(x)).$$

Thus by (3.20) $\{L(x, D)(\psi_l * \varphi_n)\}_n$ is a Cauchy sequence in L_2. Since

$$(L(x, D)(\psi_l * \varphi_n))(\varphi) = (\psi_l * \varphi_n)(L'(x, D)\varphi) \to (\psi_l * u)(L'(x, D)\varphi) = (L'^{\#}(\psi_l * u))(\varphi),$$

we obtain that

$$\|L(x, D)(\psi_l * \varphi_n) - L'^{\#}(\psi_l * u)\| \to 0 \quad \text{with} \quad n \to \infty.$$

This implies finally (together with (3.20)) that

$$\|L'^{\#}(\psi_l * u)\| \leqq \|\psi_l * L'^{\#}u\| + C\|u\| = \|\psi_l\|_{\infty,1} \|L'^{\#}u\| + C\|u\|,$$

where

$$\|\psi_l\|_{\infty,1} = \sup_\xi |(F\psi_l)(\xi)| = \sup_\xi |\theta_l^{\check{}}(\xi)| = \sup_\xi |\theta(\xi)| < \infty.$$

This proves the assertion (3.17). □

We are now ready to establish

Theorem 3.6. *Suppose that* $L(x, \xi) \in S^{1,1}_{\Phi,\varphi}$. *Then one has*

$$L^{\sim} = L'^{\#}. \tag{3.22}$$

Proof. Let u be in $D(L'^{\#})$ and let $L'^{\#}u=f$. Then by (3.17) $\{L'^{\#}(\psi_l * u)\}_l$ is bounded in L_2 and so we find a subsequence $\{L'^{\#}(\psi_{l_j} * u)\}_j$ such that

$$\left\|(1/r)\sum_{j=1}^{r} L'^{\#}(\psi_{l_j} * u)-g\right\| \to 0 \quad \text{with} \quad r\to\infty$$

where $g\in L_2$ (cf. the Banach—Saks Theorem). Since

$$(3.23) \qquad (1/r)\sum_{j=1}^{r} L'^{\#}(\psi_{l_j} * u) = L'^{\#}\big((1/r)\sum_{j=1}^{r}(\psi_{l_j} * u)\big),$$

$$\left\|(1/r)\sum_{j=1}^{r}(\psi_{l_j} * u)-u\right\| \to 0, \quad \text{with} \quad r\to\infty$$

and since by (3.9) $(1/r)\sum_{j=1}^{r}\psi_{l_j} * u)\in D(L^{\sim})$, we get that $u\in D(L^{\sim})$ and that $L^{\sim}u=g$. Because $L^{\sim}\subset L'^{\#}$, we get that $g=f$. Hence $u\in D(L^{\sim})$ and $L^{\sim}u=f$ and so $L'^{\#}\subset L^{\sim}$. This finishes the proof. □

3.3. From Theorem 3.6 we obtain the following criterion for $L^{\sim}=L'^{\#}$, when $L(x,\xi)\in S^{M,m}_{\Phi,\varphi}$; $M, m\in\mathbf{R}$.

Theorem 3.7. *Suppose that* $L(x,\xi)\in S^{M,m}_{\Phi,\varphi}$ *and that there exists a symbol* $q(\xi)\in S^{M-1,m-1}_{\Phi,\varphi}$ *(which is independent of* x*) such that* $q(\xi)\geqq 1$ *and*

$$(3.24) \qquad q(\xi)\geqq c\Phi^{M-1}(x,\xi)\varphi^{m-1}(x,\xi)$$

and that

$$(3.25) \qquad D(L'^{\#}) < H_q.$$

Then the relation $L^{\sim}=L'^{\#}$ *holds.*

Proof. Choose u in $D(L'^{\#})$ and denote $L'^{\#}u=f$. In virtue of (3.24) one observes that $q^{-1}(\xi)\in S^{-M+1,-m+1}_{\Phi,\varphi}$ and so $(L\circ q^{-1})(x,\xi)\in S^{1,1}_{\Phi,\varphi}$ (here we denoted $q^{-1}(\xi)=(q(\xi))^{-1}$). Furthermore, we obtain (we denote $q(x,D)=q(D)$)

$$(q'^{\#}u)\big((L\circ q^{-1})'(x,D)\varphi\big) = (q'^{\#}u)\big(((q^{-1})'\circ L')(x,D)\varphi\big) = u\big(L'(x,D)\varphi\big) = f(\varphi)$$

and so

$$(L\circ q^{-1})'^{\#}(q'^{\#}u) = f$$

(note that $H_q\subset D(q'^{\#})$). Due to Theorem 3.6 one has, $q'^{\#}u\in D\big((L\circ q^{-1})^{\sim}\big)$ and $(L\circ q^{-1})^{\sim}u=f$. Choose a sequence $\{\varphi_n\}\subset S$ such that $\|\varphi_n-q'^{\#}u\|\to 0$ and that $\|(L\circ q^{-1})(x,D)\varphi_n-f\|\to 0$ with $n\to\infty$. Then $\{q^{-1}(D)\varphi_n\}\subset S$ is a sequence such that $\|q^{-1}(D)\varphi_n-u\|+\big\|L(x,D)\big(q^{-1}(D)\varphi_n\big)-f\big\|\to 0$ with $n\to\infty$ (note that $q\geqq 1$). Thus $u\in D(L^{\sim})$ and $L^{\sim}u=f$, which completes the proof. □

In the next Chapter 4 we shall establish a sufficient condition for the inclusion (3.25). Also the essential maximality will be considered.

Remark 3.8. A) The proof of Lemma 3.4 shows also the following fact: Suppose that $L(x, \xi)\in S_{\Phi,\varphi}^{M,m}$. Then for any $(j, l)\in\mathbf{N}^2$ there exists $R_{j,l}(x, \xi)\in S_{\Phi,\varphi}^{M-1,m-1}$ so that

$$\psi_l*(\theta_j L(x, D)\varphi) = \theta_j L(x, D)(\psi_l*\varphi)+R_{j,l}(x, D)\varphi,$$

where

$$p_{\alpha,\beta}^{M-1,m-1}(R_{j,l}(x, \xi)) := \sup_{x,\xi}(\Phi^{-M+1+|\beta|}(x, \xi)\varphi^{-m+1+|\alpha|}(x, \xi)|D_x^\alpha D_\xi^\beta R_{j,l}(x, \xi)|)$$
$$\leqq C_{\alpha,\beta}<\infty \quad \text{for all } (j, l)\in\mathbf{N}^2.$$

B) Suppose that $q(\xi)\in S_{\Phi,\varphi}^{M-1,m-1}$ so that (3.24) holds. Then one has for $L(x, \xi)\in S_{\Phi,\varphi}^{M,m}$

(3.26) $$\|\psi_l*L(x, D)\varphi-L(x, D)(\psi_l*\varphi)\| \leqq \|\varphi\|_q \quad \text{for all} \quad \varphi\in S,$$

where C is independent of l.

The proof of (3.26) follows by applying Lemma 3.4 to $L(x, D)\circ q^{-1}(D)$.

4. On bijectivity of minimal realizations

4.1. In this chapter we shall deal with the bijectivity of $L^\sim+aI$: $L_2\to L_2$. Also the essential maximality is considered. When (Φ, φ) forms a pair of weight functions, one sees that also $(\Phi^\vee, \varphi^\vee)$ forms a pair of weight functions, where $\Phi^\vee(x, \xi)=\Phi(x, -\xi)$ and $\varphi^\vee(x, \xi)=\varphi(x, -\xi)$. We need

Lemma 4.1. *Suppose that* $L(x, \xi)\in S_{\Phi,\varphi}^{M,m}$ *such that*

(4.1) $$L_{\mathrm{Re}}(x, \xi) := \mathrm{Re}\, L(x, \xi) \geqq 0 \quad \textit{for all} \quad x, \xi\in\mathbf{R}^n.$$

Then there exists a $a(\cdot, \cdot)\in S_{\Phi,\varphi}^{M-1,m-1}$ *such that*

(4.2) $$\mathrm{Re}\langle(L(x, D)+a(x, D))\varphi, \varphi\rangle \geqq 0 \quad \textit{for all} \quad \varphi\in S.$$

Proof. Due to Theorem 3.2 there exists $l(\cdot, \cdot)\in S_{\Phi,\varphi}^{M-1,m-1}$ so that

(4.3) $$\mathrm{Re}\langle(L_{\mathrm{Re}}(x, D)+l(x, D))\varphi, \varphi\rangle \geqq 0 \quad \text{for all} \quad \varphi\in S.$$

Furthermore, we know that (cf. [1], Theorem 1)

$$L'(x, \xi) = L(x, -\xi)+b(x, -\xi),$$

where $b(\cdot, \cdot)\in S_{\Phi,\varphi}^{M-1,m-1}$ and so

(4.4) $$\mathrm{Re}\,\langle L(x, D)\varphi, \varphi\rangle = (1/2)\langle L(x, D)\varphi+\overline{L'(x, D)\bar{\varphi}}, \varphi\rangle$$
$$\mathrm{Re}\,\langle L_{\mathrm{Re}}(x, D)\varphi, \varphi\rangle+(1/2)\,\mathrm{Re}\,\langle\bar{b}(x, D)\varphi, \varphi\rangle,$$

where $\bar{b}(\cdot, \cdot)=\overline{b(\cdot, \cdot)}$. Here we noted that

$$\overline{(L'(x, D)\bar{\varphi})(x)} = (2\pi)^{-n} \overline{\int_{\mathbf{R}^n} L'(x, \xi)(F\bar{\varphi})(\xi) e^{i(\xi, x)}\, d\xi}$$

$$= (2\pi)^{-n} \int_{\mathbf{R}^n} \overline{L'(x, -\xi)}(F\varphi)(\xi) e^{i(\xi, x)}\, d\xi$$

$$= (2\pi)^{-n} \int_{\mathbf{R}^n} \overline{L(x, \xi)}(F\varphi)(\xi) e^{i(\xi, x)} + (\bar{b}(x, D)\varphi)(x).$$

Thus the assertion follows from (4.3) by choosing $a(x, \xi)=l(x, \xi)-(1/2)\bar{b}(x, \xi)$. □

Suppose that $Q(\xi)\in C^\infty(\mathbf{R}^n)$ obeys the estimate

$$|(D_\xi^\beta Q)(\xi)| \leqq C_\beta \Phi^{M-|\beta|}(x, \xi)\varphi^m(x, \xi). \tag{4.5}$$

Then the mapping $\hat{Q}(x, \xi)$ defined by $\hat{Q}(x, \xi)=Q(\xi)$ belongs to $S^{M,m}_{\Phi,\varphi}$ and we denote (as above) $\hat{Q}(x, D)=Q(D)$, $\hat{Q}(x, \xi)=Q(\xi)$. Suppose that $Q(\xi)$ is real-valued and that with $c>0$

$$Q(\xi) \geqq c\Phi^M(x, \xi)\varphi^m(x, \xi). \tag{4.6}$$

Then the mappings $Q^s(x, \xi)$ defined by $Q^s(x, \xi)=(Q(\xi))^s$ lie in $S^{Ms,ms}_{\Phi,\varphi}$ for any $s\in\mathbf{R}$. The corresponding operators are denoted by $Q^s(D)$. It is easy to see that $Q\in K'$, when (4.5)—(4.6) hold. The following lemmas are needed

Lemma 4.2. *Suppose that* $L(x, \xi)\in S^{M,m}_{\Phi,\varphi}$ *and that there exists* $Q(\xi)\in S^{-M,-m}_{\Phi,\varphi}$. *Then there exists a constant* $C>0$ *such that*

$$\|L(x, D)\varphi\|_Q \leqq C\|\varphi\| \tag{4.7}$$

and

$$\|L'(x, D)\varphi\|_{Q^\vee} \leqq C\|\varphi\| \quad \textit{for all} \quad \varphi\in S. \tag{4.8}$$

Proof. The composite operator $Q(D)\circ L(x, D)$ belongs to $L^{0,0}_{\Phi,\varphi}$ and so by Theorem 3.1 there exists a constant $C>0$ such that

$$\|L(x, D)\varphi\|_Q = \|(Q(D)\circ L(x, D))\varphi\| \leqq C\|\varphi\| \quad \text{for all} \quad \varphi\in S.$$

Here we utilized the fact that by the Fourier inversion formula

$$F(Q(D)\varphi)(\xi) = Q(\xi)(F\varphi)(\xi) \tag{4.9}$$

(note that by (4.5), $Q(\cdot)F\varphi\in S$). Since $L'(x, \xi)\in S^{M,m}_{\Phi^\vee,\varphi^\vee}$ and $Q^\vee(\xi)\in S^{-M,-m}_{\Phi^\vee,\varphi^\vee}$, the inequality (4.8) is similarly shown. □

Remark. Suppose that $Q(\xi)\in S^{M,m}_{\Phi,\varphi}$ such that (4.6) holds. Then $Q^{-1}(\xi)\in S^{-M,-m}_{\Phi,\varphi}$.

Lemma 4.3. *Suppose that* $P(\xi)\in S^{M,m}_{\Phi,\varphi}$ *and that* $q(\xi)\in S^{M-1,m-1}_{\Phi,\varphi}$ *such that*

$$P(\xi)>0 \tag{4.10}$$

$$q(\xi)\geqq c\Phi^{M-1}(x,\xi)\varphi^{m-1}(x,\xi) \tag{4.11}$$

and

$$q(\xi)/P(\xi)\to 0 \quad with \quad |\xi|\to\infty. \tag{4.12}$$

Then for any $l(x,\xi)\in S^{M-1,m-1}_{\Phi,\varphi}$, $\varepsilon>0$ *and* $N\in\mathbf{N}$ *there exists a constant* $C>0$ *such that*

$$|\langle l(x,D)\varphi,\varphi\rangle|\leqq\varepsilon\|\varphi\|^2_{P^{1/2}}+C\|\varphi\|^2_{k_{-N}} \quad for\ all \quad \varphi\in S. \tag{4.13}$$

Proof. The composite operator $q^{-1/2}(D)\circ l(x,D)\circ q^{-1/2}(D)$ is a pseudo-differential operator with a symbol in $S^{0,0}_{\Phi,\varphi}$. Hence due to the Theorem 3.1 one has

$$\begin{aligned}(4.14)\qquad &|\langle l(x,D)\circ q^{-1/2}(D)\varphi,\ q^{-1/2}(D)\varphi\rangle|\\ &=\left|\left\langle\left(q^{-1/2}(D)\circ l(x,D)\circ q^{-1/2}(D)\varphi,\ \varphi\right\rangle\right|\leqq C\|\varphi\|^2.\end{aligned}$$

Since $q^{1/2}(D)\varphi\in S$ when $\varphi\in S$ we obtain from (4.14)

$$\begin{aligned}&\langle l(x,D)\varphi,\varphi\rangle\leqq C\|q^{1/2}(D)\varphi\|^2\\ &\leqq C(2\pi)^{-n}\int_{|\xi|\geqq R}\varepsilon P(\xi)|(F\xi)(\xi)|^2\,d\xi\\ &+C(2\pi)^{-n}\int_{|\xi|\leqq R}\big(\sup_{|\xi|\leqq R} q(\xi)k^2_N(\xi)\big)|(F\varphi)(\xi)k_{-N}(\xi)|^2\,d\xi\leqq C\|\varphi\|^2_{P^{1/2}}+C'\|\varphi\|^2_{k_{-N}},\end{aligned}$$

where R is so large that $q(\xi)/P(\xi)\leqq\varepsilon$ for $|\xi|\geqq R$. This proves the assertion. □

From Lemma 4.3 we obtain

Theorem 4.4. *Suppose that* $L(x,\xi)\in S^{M,m}_{\Phi,\varphi}$ *and that* $k(\xi)\in S^{M',m'}_{\Phi,\varphi}$ *such that*

$$k(\xi)\geqq c\Phi^{M'}(x,\xi)\varphi^{m'}(x,\xi). \tag{4.15}$$

Furthermore, assume that there exist $P(\xi)\in S^{M,m}_{\Phi,\varphi}$ *and* $q(\xi)\in S^{M-1,m-1}_{\Phi,\varphi}$ *such that* (4.10)—(4.12) *hold and that*

$$\operatorname{Re}L(x,\xi)\geqq cP(\xi)\quad \text{for}\quad x,\xi\in\mathbf{R}^n. \tag{4.16}$$

Then for any $N\in\mathbf{N}$ *there exists a constant* $C>0$ *such that*

$$\operatorname{Re}\left\langle\left(L(x,D)\circ k^2(D)\right)\varphi,\varphi\right\rangle\geqq(c/2)\|\varphi\|^2_{kP^{1/2}}-C\|\varphi\|^2_{kk_{-N}}. \tag{4.17}$$

Proof. The composite operator $A(x,D)$ defined by $A(x,D)=L(x,D)\circ k^2(D)$ belongs to $L^{M+2M',m+2m'}_{\Phi,\varphi}$. Similarly, one sees that the symbols $B(\xi):=P(\xi)k^2(\xi)$

(and $b(\xi):=q(\xi)k^2(\xi)$) belong to $S_{\Phi,\varphi}^{M+2M',m+2m'}$ (and to $S_{\Phi,\varphi}^{M-1+2M',m-1+2m'}$, resp.). Furthermore, one has

$$B(\xi)>0, \tag{4.18}$$

$$b(\xi)=q(\xi)k^2(\xi)\geqq c^3\Phi^{M-1+2M'}(x,\xi)\varphi^{m-1+2m'}(x,\xi), \tag{4.19}$$

$$b(\xi)/B(\xi)=q(\xi)/P(\xi)\to 0 \quad \text{with} \quad |\xi|\to\infty \tag{4.20}$$

and

$$\operatorname{Re}A(x,\xi)=\operatorname{Re}L(x,\xi)k^2(\xi)\geqq cP(\xi)k^2(\xi)=cB(\xi). \tag{4.21}$$

Define $T(x,\xi)=A(x,\xi)-cB(\xi)$. Then $T(x,\xi)\in S_{\Phi,\varphi}^{M+2M',m+2m'}$ and $\operatorname{Re}T(x,\xi)\geqq 0$. Due to Theorem 4.1 there exists $\lambda(x,\xi)\in S_{\Phi,\varphi}^{M-1+2M',m-1+2m'}$ such that

$$\operatorname{Re}\big\langle\big(T(x,D)+\lambda(x,D)\big)\varphi,\varphi\big\rangle\geqq 0. \tag{4.22}$$

Furthermore, in virtue of Lemma 4.3 there exists $C>0$ such that

$$|\langle\lambda(x,D)\varphi,\varphi\rangle|\leqq(c/2)\|\varphi\|_{B^{1/2}}^2+C\|\varphi\|_{k_{-N'}}^2,$$

where $N'\in\mathbf{N}$ such that

$$k_{-N'}\leqq Ckk_{-N}: \tag{4.23}$$

Hence we obtain from (4.22)

$$\begin{aligned}\operatorname{Re}\langle A(x,D)\varphi,\varphi\rangle&=\operatorname{Re}\langle T(x,D)\varphi,\varphi\rangle+c\|\varphi\|_{B^{1/2}}^2\\&\geqq c\|\varphi\|_{B^{1/2}}^2-|\langle\lambda(x,D)\varphi,\varphi\rangle|\\&\geqq(c/2)\|\varphi\|_{B^{1/2}}^2-C\|\varphi\|_{k_{-N'}}^2\end{aligned}$$

and so we finally have by (4.23)

$$\begin{aligned}\operatorname{Re}\big\langle\big(L(x,D)\circ k^2(D)\big)\varphi,\varphi\big\rangle&=\operatorname{Re}\langle A(x,D)\varphi,\varphi\rangle\\&\geqq(c/2)\|\varphi\|_{kP^{1/2}}^2-C\|\varphi\|_{kk_{-N}}^2,\end{aligned}$$

as desired. □

Corollary 4.5. *Let* $L(x,\xi)$, $k(\xi)$, $P(\xi)$ *and* $q(\xi)$ *be as in Theorem* 4.4. *Then there exists a constant* $a_0\geqq 0$ *such that for any* $a\geqq a_0$ *the estimates*

$$\big\|\big(L(x,D)+aI\big)\varphi\big\|_k\geqq\|\varphi\|_k \tag{4.24}$$

and

$$\big\|\big(L'(x,D)+aI\big)\varphi\big\|_{k^{\vee}}=\|\varphi\|_{k^{\vee}} \quad \textit{for all} \quad \varphi\in S \tag{4.25}$$

hold.

Proof. A) From (4.17) we get (with $N=0$) for $a'\geqq 1$

$$(4.26)\qquad \|\varphi\|_k^2 \leqq (a'+C)\|\varphi\|_k^2+\mathrm{Re}\langle (L(x,D)\circ k^2(D))\varphi,\varphi\rangle$$
$$=(C+a')\|\varphi\|_k^2+\mathrm{Re}\langle k^2(D)\varphi, L'(x,D)\bar\varphi\rangle=\mathrm{Re}\langle k^2(D)\varphi,(L'(x,D)+(C+a')I)\bar\varphi\rangle$$
$$=\mathrm{Re}\langle k(D)\varphi, k(D)(\overline{(L'(x,D)+(C+a')I)\bar\varphi})\rangle \leqq \|\varphi\|_k\|(L'(x,D)+(C+a')I)\bar\varphi\|_{k^\vee},$$

where we observed that $\|\bar\varphi\|_k=\|\varphi\|_{k^\vee}$. Hence the assertion (4.25) follows.

B) To prove the inequality (4.24) we observe that

$$(4.27)\qquad \mathrm{Re}\langle L'(x,D)\circ(k^\vee)^2(D)\varphi,\varphi\rangle$$
$$=\mathrm{Re}\langle (k^\vee)^2(D)\varphi, L(x,D)\bar\varphi\rangle=\mathrm{Re}\langle (k^\vee)^2(D)\varphi,\overline{L^\vee}(x,D)\varphi\rangle,$$

where $\overline{L^\vee}(x,\xi):=\overline{L(x,-\xi)}$. Applying Theorem 4.4 to the case, where $L(x,\xi)$ is replaced by $\overline{L^\vee}(x,\xi)\in S^{M;m}_{\Phi^\vee,\varphi^\vee}$, $P(\xi)$ is replaced by $P^\vee(\xi)$, $q(\xi)$ is replaced by $q^\vee(\xi)$ and where $k(\xi)$ is replaced by $(k^\vee)^{-1}(\xi)$, we find that

$$(4.28)\quad \mathrm{Re}\langle \overline{L^\vee}(x,D)\circ(k^\vee)^{-2}(D)\varphi,\varphi\rangle \geqq (c/2)\|\varphi\|^2_{(k^\vee)^{-1}(P^\vee)^{1/2}}-C'\|\varphi\|^2_{(k^\vee)^{-1}k_{-N}}$$

for all $\varphi\in S$. Since $k^2(-D)\varphi$ belongs to S when φ belongs to S, we obtain by (4.27)—(4.28) that

$$\mathrm{Re}\langle L'(x,D)\circ(k^\vee)^2(D)\varphi,\varphi\rangle\geqq (c/2)\|\varphi\|^2_{k^\vee(P^\vee)^{1/2}}-C'\|\varphi\|^2_{k^\vee k_{-N}},$$

and then (4.24) can be verified as (4.25) (cf. the Part A)). □

4.2. We shall now prove the bijectivity of $L^\sim+aI$ and $L'^\#+aI$ for a large enough. The key is the following lemma

Lemma 4.6. *Suppose that* $L(x,\xi)\in S^{M,m}_{\Phi,\varphi}$ *and that there exists* $Q(\xi)\in S^{-M,-m}_{\Phi,\varphi}\cap S^{M',m'}_{\Phi,\varphi}$ *such that*

$$(4.29)\qquad Q(\xi)\geqq c\Phi^{M'}(x,\xi)\varphi^{m'}(x,\xi)\quad \text{and}\quad Q(\xi)\leqq 1.$$

Furthermore, assume that there exist $a\in\mathbf{C}$, $c>0$ *and* $N\in\mathbf{N}$ *so that*

$$(4.30)\qquad \|(L(x,D)+aI)\varphi\|\ \geqq c\|\varphi\|$$

$$(4.31)\qquad \|(L(x,D)+aI)\varphi\|_Q\geqq c\,\|\varphi\|_{k_{-N}}$$

and

$$(4.32)\qquad \|(L'(x,D)+aI)\varphi\|_{Q^\vee}\geqq c\|\varphi\|_{k_{-N}}\quad \textit{for all}\quad \varphi\in S.$$

Then one has

$$(4.33)\qquad R(L^\sim+aI)=L_2\quad \text{and}\quad N(L'^\#+aI)=\{0\}.$$

Proof. A) Let u be in $N(L'^\#+aI)$ and choose a sequence $\{\varphi_n\}\subset S$ such that $\|\varphi_n-u\|\to 0$. Then by (4.7) one sees that $\{L(x,D)\varphi_n\}$ is a Cauchy sequence in H_Q.

Choose $g\in H_Q$ so that $\|L(x, D)\varphi_n - g\|_Q \to 0$. Since one has

$$g(\varphi) = \lim_{n\to\infty} \big(L(x, D)\varphi_n\big)(\varphi) = \lim_{n\to\infty} \varphi_n\big(L'(x, D)\varphi\big),$$

$$u\big(L(x, D)\varphi\big) = (L'^{\#} u)(\varphi),$$

we obtain that $g = L'^{\#} u$ and so (note that $Q \leqq 1$)

$$\big\|\big(L(x, D) + aI\big)\varphi_n\big\|_Q = \big\|\big(L(x, D) + aI\big)\varphi_n - L'^{\#} u - au\big\|_Q \to 0$$

with $n\to\infty$. Due to (4.31) one has $\|\varphi_n\|_{k_{-N}} \to 0$ with $n\to\infty$ and so $u=0$. This shows that

$$N(L'^{\#} + aI) = \{0\}.$$

Similarly one finds from (4.32) and (4.8) that (here $L^{\#}$ is the maximal realization of $L'(x, D)$)

(4.34) $$N(L^{\#} + aI) = \{0\}.$$

B) Let U be in $N(L^* + aI^*) \subset L_2^* (= H_k^*$ with $k=1)$. Then there exists $u\in L_2$ such that (cf. Lemma 2.1)

$$U\varphi = u(\varphi) \quad \text{and} \quad \|U\| = \|u\|$$

(this follows also from Riesz theorem). Since one has

$$u\big((L(x, D) + aI)\varphi\big) = U\big((L_0 + aI)\varphi\big) = 0,$$

we obtain by (4.34) that $u=0$ and then $U=0$. Thus $N(L^* + aI^*) = \{0\}$. Since by (4.30) $R\big((L^* + aI^*)^*\big) = R(L^{\sim} + aI)$ is closed and since

$$N(L^* + aI^*) = \{0\}$$

one sees that $R(L^{\sim} + aI) = L_2$ (cf. [5], p. 234). This completes the proof. □

Combining Corollary 4.5 and Lemma 4.6 we get

Theorem 4.7. *Suppose that* $L(x, \xi)\in S_{\Phi,\varphi}^{M,m}$ *and that* $Q(\xi)\in S_{\Phi,\varphi}^{-M,-m}\cap S_{\Phi,\varphi}^{M',m'}$ *such that*

(4.35) $$Q(\xi) \geqq c\Phi^{M'}(x, \xi)\varphi^{m'}(x, \xi) \quad \text{and} \quad Q(\xi) \leqq 1.$$

Furthermore, assume that there exist $P(\xi)\in S_{\Phi,\varphi}^{M,m}$ *and* $q(\xi)\in S_{\Phi,\varphi}^{M-1,m-1}$ *such that*

$$P(\xi) > 0,$$

$$q(\xi) \geqq c\Phi^{M-1}(x, \xi)\varphi^{m-1}(x, \xi),$$

$$q(\xi)/P(\xi) \to 0 \quad \textit{with} \quad |\xi| \to \infty$$

and

$$\operatorname{Re} L(x, \xi) \geqq cP(\xi) \quad \textit{for all} \quad x, \xi \in \mathbf{R}^n.$$

Then there exists a constant $a_0 \geqq 0$ *such that*

$$R(L^{\sim} + aI) = L_2 \quad \textit{and} \quad N(L'^{\#} + aI) = \{0\} \quad \textit{for} \quad a \geqq a_0. \tag{4.36}$$

Proof. The application of Corollary 4.5 with $k=1(\in S^{0,0}_{\Phi,\varphi})$ gives (4.30). The application with $k=Q$ implies (4.31)—(4.32). Hence Lemma 4.6 proves the assertion. □

Corollary 4.8. *Let* $L(x, \xi)$, $Q(\xi)$, $P(\xi)$ *and* $q(\xi)$ *be as in Theorem* 4.7. *Then the relation*

$$L^{\sim} = L'^{\#}$$

holds.

Proof. Choose a such that (4.36) is valid. Let u be in $D(L'^{\#})$ and let $L'^{\#}u=f$. Then one has $L'^{\#}u+au=(L^{\sim}+aI)w$ with some $w \in D(L^{\sim})$. Since $N(L'^{\#}+aI)=\{0\}$ and since $L^{\sim} \subset L'^{\#}$ one sees that $u=w \in D(L^{\sim})$, which proves that $L'^{\#} \subset L^{\sim}$. □

4.3. Let $L_k^{\sim}$ (and $L_k'^{\#}$): $H_k \to H_k$ be the minimal realization (the maximal realization, resp.) of $L(x, D)$ in H_k. The definition of $L_k^{\sim}$ and $L_k'^{\#}$ is given as the definition of $L^{\sim}$ and $L'^{\#}$ (cf. Section 2.2).

Theorem 4.9. *Suppose that* $L(x, \xi) \in S^{M,m}_{\Phi,\varphi}$ *and that* $k(\xi) \in S^{M',m'}_{\Phi,\varphi}$ *such that*

$$k(\xi) \geqq c\Phi^{M'}(x, \xi)\, \varphi^{m'}(x, \xi). \tag{4.37}$$

Furthermore, assume that

$$(k \circ L \circ k^{-1})'^{\#} = (k \circ L \circ k^{-1})^{\sim}. \tag{4.38}$$

Then the relation

$$L_k^{\sim} = L_k'^{\#} \tag{4.39}$$

holds.

Proof. Let u be in $D(L_k'^{\#})$ and let $L_k'^{\#}u=f$. Then one has (here $k(D)u$ and $k(D)f \in L_2$; $k(D)u$ is defined by $u(k(-D)\varphi)=(k(D)u)(\varphi)$)

$$\begin{aligned} &(k(D)u)\big((k(D) \circ L(x, D) \circ k^{-1}(D))' k(D)\varphi\big) \\ &= (k(D)u)\big(((k^{\vee})^{-1}(D) \circ L'(x, D) \circ k^{\vee}(D))\varphi\big) \\ &= u\big(L'(x, D)(k^{\vee}(D)\varphi)\big) = f(k^{\vee}(D)\varphi) = (k(D)f)(\varphi) \end{aligned}$$

and then $(k(D)\circ L(x, D)\circ k^{-1}(D))'^{\#}(k(D)u)=k(D)f$. Choose a sequence $\{\varphi_n\}\subset S$ such that (cf. (4.38))

$$\|\varphi_n-k(D)u\|=\|k(D)\circ L(x,D)\circ k^{-1}(D)\varphi_n-k(D)f\| \to 0.$$

Then one sees that

$$\|k^{-1}(D)\varphi_n-u\|_k+\|L(x,D)\circ k^{-1}(D)\varphi_n-f\|_k \to 0 \quad \text{with} \quad n\to\infty,$$

which proves that $u\in D(L_k^{\sim})$ and that $L_k^{\sim}u=f$, as desired. □

Remark 4.10. A) Let $L(x, \xi)$, $k(\xi)$, $P(\xi)$ and $q(\xi)$ be as in Theorem 4.4. With the similar computation as presented in the proof of Theorem 4.4 one sees that

$$\mathrm{Re}\left\langle P^{1/2}(D)\circ L'(x,D)\circ(P^{\check{}})^{-1/2}(D)\circ(k^{\check{}})^2(D)\varphi, \varphi\right\rangle$$
$$\geqq (c/4)\|\varphi\|^2_{k^{\check{}}(P^{\check{}})^{1/2}}-C\|\varphi\|^2_{k^{\check{}}k_{-N}}$$

for any $N\in\mathbf{N}$. Hence one has

$$\mathrm{Re}\langle L'(x,D)\circ(k^{\check{}})^2(D)\varphi, (P^{\check{}})(D)\varphi\rangle$$
$$\geqq (c/4)\|(P^{\check{}})^{1/2}\varphi\|^2_{(kP^{\check{}})^{1/2}}-C\|(P^{\check{}})^{1/2}\varphi\|_{k^{\check{}}k_{-N}} = (c/4)\|\varphi\|^2_{k^{\check{}}P^{\check{}}}-C\|\varphi\|^2_{k^{\check{}}k_{-N}(P^{\check{}})^{-1/2}}.$$

Thus one gets (cf. (4.26))

$$\left\|\left(L(x,D)+aI\right)\varphi\right\|_k \geqq (c/4)\|\varphi\|_{kP}$$

for a large enough. This implies finally

$$D(L_k^{\sim})\subset H_{kP}\subset H_{kq} \tag{4.40}$$

and then the assumptions of Theorem 4.4 imply (3.25).

B) Since one has

$$(k\circ L\circ k^{-1})(x,\xi) = L(x,\xi)+\lambda(x,\xi), \tag{4.41}$$

where $\lambda(x,\xi)\in S^{M-1,m-1}_{\Phi,\varphi}$, one sees that the assumptions of Theorem 4.7 imply (4.39) for any $k(\xi)\in S^{M'',m''}_{\Phi,\varphi}$, which obeys (4.37).

4.4. Let δ and ϱ be non-negative numbers such that $0\leqq\delta<\varrho\leqq 1$. Denote by $S^m_{\varrho,\delta}$, $m\geqq 0$ the class of $C^\infty(\mathbf{R}^n\times\mathbf{R}^n)$-functions $L(x,\xi)$ such that for any $(\alpha,\beta)\in\mathbf{N}_0^2$ there exists a constant $C_{\alpha,\beta}>0$ with which

$$|D_x^\alpha D_\xi^\beta L(x,\xi)| \leqq C_{\alpha,\beta}(1+|\xi|)^{m-\varrho|\beta|+\delta|\alpha|} \quad \text{for all} \quad x,\xi\in\mathbf{R}^n. \tag{4.42}$$

One sees that the functions Φ and φ defined by $\Phi(x,\xi)=(1+|\xi|)^\varrho$ and $\varphi(x,\xi)=$

$(1+|\xi|)^{-\delta}$ form a pair of weight functions in the sense of [1]. Furthermore, one has

$$(4.43)\qquad (1+|\xi|)^{m-\varrho|\beta|+\delta|\alpha|} = \Phi^{(m/\varrho)-|\beta|}(x,\xi)\,\varphi^{-|\alpha|}(x,\xi)$$

and the $S^m_{\varrho,\delta}=S^{(m/\varrho),0}_{\Phi,\varphi}$.

Corollary 4.11. *Let $L(x,\xi)$ be in $S^m_{\varrho,\delta}$; $0\leqq\delta<\varrho\leqq 1$ such that with some constants $c>0$, $E\geqq 0$ and $t\in]m-(\varrho-\delta), m]$ one has*

$$(4.44)\qquad \operatorname{Re} L(x,\xi) \geqq c(1+|\xi|)^t \quad \text{for} \quad |\xi| \geqq E.$$

Then the relations

$$(4.45)\qquad R(L^{\sim}+aI)=L_2,\quad N(L'^{\#}+aI)=\{0\} \quad \text{and} \quad L^{\sim}=L'^{\#}$$

hold, when a is large enough.

Proof. Define functions $P(\xi)$, $q(\xi)$ by $P(\xi)=(1+|\xi|^2)^{t/2}$ and $q(\xi)=(1+|\xi|^2)^{(m-\varrho+\delta)/2}$. Then one has $P(\xi)\in S^t_{1,0}\subset S^m_{\varrho,\delta}=S^{(m/\varrho),0}_{\Phi,\varphi}$ and

$$|D^\beta_\xi q(\xi)| \leqq C_\beta(1+|\xi|)^{m-\varrho+\delta-|\beta|} \leqq C_\beta(1+|\xi|)^{m-\varrho+\delta-\varrho|\beta|}$$
$$= C_\beta\,\Phi^{(m/\varrho)-1-|\beta|}(x,\xi)\,\varphi^{-1}(x,\xi).$$

Thus $q(\xi)\in S^{(m/\varrho)-1,-1}_{\Phi,\varphi}$. Furthermore, we get

$$P(\xi)>0,$$

$$q(\xi) \geqq \varkappa(1+|\xi|)^{(m-\varrho+\delta)} = \varkappa\Phi^{(m/\varrho)-1}(x,\xi)\,\varphi^{-1}(x,\xi)$$

and

$$q(\xi)/P(\xi) = (1+|\xi|^2)^{(m-t+\delta-\varrho)/2} \to 0 \quad \text{with} \quad |\xi|\to\infty.$$

Let C be a positive number such that $C=2C_{0,0}(1+R)^m$. Then one sees by (4.42) and (4.44) that with some $\varkappa>0$

$$\operatorname{Re}\big(L(x,\xi)+C\big) \geqq \varkappa P(\xi) \quad \text{for all} \quad x,\xi\in\mathbf{R}^n.$$

Since $m\geqq 0$ we have that $L(\cdot,\cdot)+C\in S^m_{\varrho,\delta}=S^{(m/\varrho),0}_{\Phi,\varphi}$. By virtue of Theorem 4.7 we obtain (choose $Q(\xi)=(1+|\xi|^2)^{-m/2}$)

$$R(L^{\sim}+aI)=L_2,\quad N(L'^{\#}+aI)=\{0\} \quad \text{and} \quad L^{\sim}=L'^{\#},$$

as desired. □

Remark. The above method gives also that the operators $L(x,D)$ satisfying the assumptions of Corollary 4.11 are essentially maximal in the Sobolev spaces $H^s(\mathbf{R}^n)=H_{k_s}$ with $s\in\mathbf{R}$.

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