Global parametrices for fundamental solutions of first order pseudo-differential hyperbolic operators

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Dedicated to Lennart Carleson on his sixtieth birthday

Introduction

Global parametrices of fundamental solutions of hyperbolic differential operators were first constructed by Ludwig [7] and for pseudo-differential operators by Duistermaat and Hörmander [2] using Fourier integral operators and canonical relations.

The aim of this paper is to give an elementary construction of global parametrices for fundamental solutions of first order pseudo-differential operators. It is a simplified and corrected version of the construction given in my lectures 1985 at the Nankai university in Tianjin, China (Gårding [3]) and reported on in (Gårding [4]).

Consider a first order pseudo-differential or, more precisely, differential-pseudo-differential hyperbolic operator,

$$Q = D_t + P(t, x, D_x) \tag{1}$$

defined on the product of the real line and a paracompact manifold Ω of dimension n. Here $D_t = \partial_t/i$, $D = \partial_x/i$ with obvious ∂_t and ∂_x and $P(t, x, D_x)$ is a classical first order pseudo-differential operator with real principal symbol $p(t, x, \xi)$ defined on the product of the real line and the cotangent bundle $C = T^*(\Omega)\diagdown 0$ of Ω.

Let Y be a point in Ω with coordinates 0 in a system of coordinates y around Y. A parametrix of a fundamental solution of Q with pole at $(0, Y)$ is a distribution $E(t, x)$ satisfying

$$QE(t, x) \equiv \delta(t)\delta(y)$$

modulo smooth functions. It is sufficient to have $E(t, x) = 0$ when $t < 0$ and let

$E(t, x)$ solve Cauchy's problem

$$QE(t, x) \equiv 0, \quad E(0, y) \equiv \delta(y) \tag{2}$$

when $t>0$. Our construction of the parametrix can be described briefly as follows.

Let x, ξ denote canonical coordinates in C respecting the canonical form $\xi \cdot dx = \xi_1 dx_1 + \ldots + \xi_n dx_n$ and let the words Hamilton flow on C and bicharacteristic (path) refer to the principal symbol $p(t, x, \xi)$ and assume that outflow of a compact subset of Ω from $t=0$ to t stays over a compact part of Ω when $t>0$ is bounded.

Let y, η be canonical coordinates in a conical neighborhood $K(0)$ of Y in C and let $L(0)=(0, \mathbf{R}^n \setminus 0)$ be the fiber over Y. Let K and L be the outflows from $K(0)$ and $L(0)$ and let $K(t)$ and $L(t)$ be their restrictions to time t. By Hörmander's propagation of singularities theorem, the wave front set of the distribution $x \to E(t, x)$ equals $L(t)$ which is a manifold of dimension n. It is conical in an obvious sense and Lagrangian in the sense that the form $\xi \cdot dx$ vanishes on $L(t)$.

Let the coordinates $x(t), \xi(t)$ in $K(t)$ be images of y, η. The function $f(t, x, \xi)$ on K defined by

$$f(t, x(t), \xi(t)) = y\eta \tag{3}$$

will be called the *gauge*. It is going to serve as a kind of universal phase function in the constructions which follow.

When t is bounded, $E(t, x)$ is constructed in the form of a finite sum of distributions

$$I(t, x) = \int e^{if(t,x,\xi)} a(t, x, \xi)\, d\xi$$

where x, ξ are canonical coordinates in C covering the outflow $N(t)$ at time t of some small open conical part $N(0)$ of $K(0)$ and the amplitude $a(t, x, \xi)$ is a zero order classical amplitude supported in $N(t)$. It is understood that the space coordinates x at time t are chosen such that the gradient $f_x(t, x, \xi)$ does not vanish and that ξ parametrizes the Lagrangian manifold $L(t)$ in $N(t)$ where its equation then is $f_\xi(t, x, \xi)=0$. It will be show that this situation can be achieved when $N(0)$ is small enough. Here and in the sequel indices denote partial derivatives with respect to the corresponding variables.

In order to prolong an oscillatory integral $I(s, x)$ with phase function $f(s, x, \xi)$ to a time t slightly larger than s, the function $f(t, x, \xi)$ is written as a function $h(t, x, \eta)$ where y, η are connected to x, ξ by the flow from s to t. The function $h(t, x, \eta)$ then satisfies a weak form of the Hamilton—Jacobi equation,

$$h_t + p(t, x, h_x) = z(t, x, \eta) h_\eta(t, x, \eta)$$

where $z(t, x, \eta)$ is a smooth function of homogeneity 1 in η for large values of this variable. Using the function h as a phase, we can now write down an oscillatory integral

$$J(t, x) = \int e^{ih(t,x,\eta)} b(t, x, \eta)\, d\eta$$

and determine the amplitude $b(t, x, \eta)$ so that $QJ(t, x) \equiv 0$ and $J(t, x) = I(s, y)$ when $t=s$, $x=y$. This is done by recursive integrations of first order linear differential equations for the homogeneous parts of $b(t, x, \eta)$ with the initial condition $b(s, y, \eta) = a(s, y, \eta)$. Restoring the variable ξ at t gives $J(t, x)$ the form of an $I(t, x)$. When the neighborhood $N(0)$ shrinks to a ray over Y, our construction extends the distribution $I(t, x)$ indefinitely.

When $f(0, y, \eta) = y \cdot \eta$ the function $h(t, x, \eta)$ constructed above satisfies the ordinary Hamilton—Jacobi equation. In this case it was used by (Lax [6]) to write down a local parametrix for the solution of Cauchy's problem for strongly hyperbolic differential operators with data at time 0. His parametrix is a sum of oscillatory integrals of the type above defined for small times.

1. Phase functions in the Hamilton flow

This section introduces some phase functions to be used later. Consider the Hamilton flow

$$dx/dt = p_\xi(t, x, \xi), \quad d\xi/dt = -p_x(t, x, \xi).$$

Differentiation along the flow is given by the Lie operator

$$X = \partial_t + p_\xi(t, x, \xi)\partial_x - p_x(t, x, \xi)\partial_\xi.$$

The flow commutes with changes of variables $\xi \to \text{const}\,\xi$ and changes of space coordinates leaving the form $\xi \cdot dx$ invariant. A simple computation using that $p(t, x, \xi)$ is homogeneous of degree 1 in ξ shows that the Lie operator annihilates the differential form

$$\omega = \xi\, dx - p(t, x, \xi)\, dt$$

which shows that the form is invariant under the flow. From these three properties, it follows that the image $L(t)$ at time t of the fiber $L(0)$ over Y is a manifold of dimension n, that it is conical and that $\xi \cdot dx = 0$ on $L(t)$ since $\eta \cdot dy = 0$ on $L(0)$. It also follows that the gauge $f(t, x, \xi)$ is defined by (2) on the outflow K from $t=0$ of the neighborhood $K(0)$ of Y and that it is homogeneous of degree 1 in ξ.

In the sequel we let $N(0)$ be a neighborhood of a fixed ray $Z(0) = (0, \eta_0)$ in $L(0)$ over Y and let $Z(t)$ be the outflow of $Z(0)$ at time t.

A set of canonical coordinates x, ξ in a neighborhood M of the point $Z(r)$ are said to be *good* if the fiber coordinates ξ can be chosen as parameters on $M \cap L(t)$ when t is close enough to r. At $Z(0)$ with coordinates y, η we have $df(y, \eta) = \eta \cdot dy$

and hence

$$df(t, x, \xi) = f_x\, dx + f_\xi\, d\xi = \xi\, dx$$

when x, ξ is in $L(t)$. Since ξ is arbitrary there, it follows that $f_\xi(t, x, \xi) = 0$ so that $\xi = f_x(t, x, \xi)$ and $f_x = \xi$ on $L(t) \cap M$. It follows that df_ξ equals $dx + f_{\xi\xi}\, d\xi$ at $L(t)$ so that the n differentials df_ξ are linearly independent there.

Theorem 1. 1) *Let r be any time. There is choice of space coordinates at $Z(r)$ resulting in good canonical coordinates x, ξ in C which cover a neighborhood M of $Z(r)$.*

2) *Let $T = T(\varepsilon)$: $|t - r| \leqq \varepsilon$ be a time interval around a fixed time r and let the map $x, \xi \to y, \eta$ be induced by the Hamilton flow from a time $t \in T$ to the time $r - \varepsilon$. When ε is small enough, $Z(r) \times T$ has a neighborhood which is covered by the coordinates t, x, ξ and t, x, η. In this neighborhood, the function h defined by $h(t, x, \eta) = f(t, x, \xi)$ satisfies a weak Hamilton—Jacobi equation*

$$h_t + p(t, x, h_x) = z(t, x, \eta) h_\eta(t, x, \eta),$$

where $z(t, x, \eta)$ is a smooth vector, homogeneous of degree 1 in η.

Note. When $s = 0$, $f(0, y, \eta) = y \cdot \eta$ and the function $h(t, x, \eta)$ is a generating function of the homogeneous canonical map $y, \eta \to x, \xi$ from time 0 to time t (see Caratheodory [1], p. 98). It then satisfies the ordinary Hamilton—Jacobi equation. The mistake in Gårding [3] and [4] was to assume this also in the later steps of the construction. It did not influence the final result.

Proof. 1) Let y, η be canonical coordinates around $Z(r)$ such that $y = 0$, $\eta = (1, 0, \dots, 0)$ at $Z(r)$. This can be achieved by an affine change from arbitrary canonical coordinates. We can then find a number k such that the differentials

$$d\eta_1, \dots, d\eta_k, \quad dy_{k+1}, \dots, dy_n$$

are linearly independent when restricted to $L(r)$ in a neighborhood of $Z(r)$ for r fixed. Now make a change of space coordinates,

$$x_1 = y_1 + z, \quad x_2 = y_2, \dots, x_n = y_n$$

where

$$z = a(y_{k+1}^2 + \dots + y_n^2)/2.$$

The rule $\eta \cdot dy = \xi \cdot dx$ then gives the new dual coordinates

$$\xi_j = \eta_j \quad \text{when} \quad j \leqq k, \quad \xi_j = \eta_j + \eta_1 a y_j \quad \text{when} \quad j > k.$$

Hence, at $Z(r)$ we have

$$d\xi_j = d\eta_j \quad \text{when} \quad j \leqq k, \quad d\xi_j = d\eta_j + a\, dy_j \quad \text{when} \quad j > k.$$

Taking a large enough, it follows that $d\xi_1, \dots, d\xi_n$ are linearly independent on $L(r)$ at $Z(r)$. Hence they are linearly independent close to $Z(r)$ and $L(r)$.

2) The Hamilton map $x, \xi \to y, \eta$ from time t to $s=r-\varepsilon$ is the identity when $t=s$. Hence the required neighborhood of $Z(r)\times T$ exists when ε is small enough. The differentials of the two coordinates systems are related by a linear system of equations,

$$dx = x_y\, dy + x_\eta\, d\eta, \quad d\xi = \xi_y\, dy + \xi_\eta\, d\eta.$$

If $f_\xi(t, x, \xi)=0$ and $dt=0$, then $df=\xi\cdot dx=h_x dx+h_\eta d\eta$ depends only on dx and hence $f_\xi(t, x, \xi)$ and $h_\eta(t, x, \eta)$ vanish at the same time, so that $L(t)$ has the equation $h_\eta(t, x, \eta)=0$. Since the differentials df_ξ are linearly independent on $L(t)$ the same is true of the differentials dh_η close to $Z(r)\times T$ when ε is small enough.

When x, ξ is in $L(t)$ and the corresponding y, η is in $L(s)$ and we have

$$dh = \eta\, dy = \xi\, dx - p(t, x, \xi)\, dt = h_x\, dx + h_t\, dt$$

by the invariance of the differential form ω. Hence $\xi=h_x$ and $h_t=-p(t, x, \xi)$ which means that $h(t, x, \eta)$ satisfies the Hamilton—Jacobi equation $h_t+p(t, x, h_x)=0$ on $L(t)$. Since $h_\eta(t, x, \eta)=0$ is the equation of $L(t)$ and the differentials $dh_\eta(t, x, \eta)$ are linearly independent, it follows that $h(t, x, \eta)$ satisfies the weak Hamilton—Jacobi equation stated above. This finishes the proof.

2. Construction of a global parametrix

To prepare for the main result we need to put Lax's result in a general setting (Lemma and Theorem 2 below).

Let us consider a pseudo-differential operator

$$Q = D_t + P(t, x, D_x)$$

defined on $\Omega\times\mathbf{R}$. We assume that P is a classical pseudo-differential operator of order 1 which means that the symbol $P(t, x, \xi)$ of P is smooth in all variables and has an asymptotic expansion for large ξ,

$$P(t, x, \xi) \sim \sum p_k(t, x, \xi), \quad k = 1, 0, -1, \dots$$

with p_k smooth when $\xi\neq 0$ and homogeneous of degree k in ξ. The principal symbol $p=p_1$ is supposed to be real. We shall let P operate on oscillatory integrals

$$I(t, x) = \int e^{ig(t,x,\eta)} c(t, x, \eta)\, d\eta$$

where the amplitude $c(t, x, \eta)$ and the phase $g(t, x, \eta)$ are defined in some open set O of t, x, η-space which is conical in η. The amplitude is supposed to be classical of order 0 with conically compact support. The term classical means that $c(t, x, \eta)$

is a smooth with a uniform expansion for large η,

$$c(t, x, \eta) \sim \sum_k^{-\infty} c_j(t, x, \eta)$$

where c_j is smooth and of pure order j, i.e. it is homogeneous of degree j in $\eta \neq 0$. The maximal pure order of the non-vanishing terms of $c(t, x, \eta)$ is called its order. We shall also assume that the phase function $g(t, x, \eta)$ is regular in the sense that it has pure order 1 and is smooth for $\eta \neq 0$ and that it has the property that

$$g_x(t, x, \eta) \neq 0$$

everywhere. It is well-known that the integral I then defines a distribution.

Example. The function $h(t, x, \eta)$ of Theorem 1 has this property close to $Z(r)$. In fact, $\xi = h_x(t, x, \eta)$ on $L(t)$.

Lemma. *Modulo smooth functions, the distribution $PI(t, x)$ equals an oscillatory integral*

$$K(t, x) = \int e^{ig(t,x,\eta)} b(t, x, \eta)\, d\eta$$

where the amplitude b has the expansion

$$\sum P^{(\alpha)}(t, x, g_x(t, x, \eta))(D_y + r_y(t, x, y, \eta))^\alpha c(t, y, \eta)_{y=x}, \tag{4}$$

where

$$g(t, x, \eta) - g(t, y, \eta) = g_x(t, x, \eta)(x-y) - r(t, x, y, \eta).$$

Note. The leading term of $b(t, y, \eta)$ is $p(t, x, g_x(t, x, \eta))c_k(t, x, \eta)$. Since $r_x = 0$ when $x = y$, the coefficients of $(D_y + ir_y)^\alpha$ have order at most $[|\alpha|/2]$ in η after putting $y = x$. Hence the order of a term in (4) with $c(t, y, \eta)$ replaced by $c_j(t, y, \eta)$ is at most $j - |\alpha|/2$. This guarantees an expansion according to homogeneities of $b(t, x, \eta)$. The lemma is implicit in the classical calculus of pseudo-differential operators and there is a proof in (Gårding [3]). For the convenience of the reader we give a sketch of it.

Sketch of the proof. Changing the variable x to y in the integral, its Fourier transform is

$$\int e^{-iy\xi + ig(t,y,\eta)} c(t, y, \eta)\, dy\, d\eta.$$

Hence $K(t, x)$ is the oscillatory integral

$$\int e^{i(x-y)\xi + ig(t,y,\eta)} P(t, x, \xi) c(t, y, \eta)\, dy\, d\eta\, d\xi$$

so that

$$b(t, x, \eta) = \int e^{i(x-y)\xi + i(g(t,y,\eta) - g(t,x,\eta))} P(t, x, \xi) c(t, x, \eta)\, dy\, d\xi.$$

Hence, by the definition of $r(t, x, y, \eta)$,

$$b(t, x, \eta) = \int e^{i(x-y)(\xi - g_x(t,x,\eta)) + ir(t,x,y,\eta)} P(t, x, \xi) c(t, x, \eta)\, dy\, d\xi.$$

The next step is to expand $\xi \to P(t, x, \xi)$ in terms of $\xi - g_x(t, x, \eta)$,

$$P(t, x, \xi) = \sum P^{(\alpha)}(t, x, g_x(t, x, \eta))(\xi - g_x(t, x, \eta))^\alpha.$$

This calls for an integration by parts with respect to y with the result that

$$b(t, x, \eta) = \int \sum e^{i(x-y)\xi} P^{(\alpha)}(t, x, g_x(t, x, \eta)) D_y^\alpha e^{ir(t,x,y,\eta)} c(t, y, \eta)\, dy\, d\xi.$$

An integration with respect to ξ completes the result.

The next theorem is a corollary of of the Lemma and Theorem 1.

Theorem 2. *Let $h(t, x, \eta)$ be the phase function Theorem 1 constructed in a neighborhood M of the product of $Z(r)$ and a time interval T from $s = r - \varepsilon$ to $r + \varepsilon$. Let $b(s, x, \xi)$ be a classical amplitude of order 1 supported in the outflow $Z(s)$ of a neighborhood $N(0)$ of $Z(0)$. If $Z(0)$ and ε are small enough, there are amplitudes $b(t, x, \eta)$ with conically compact supports for fixed t such that the oscillatory integral*

$$J(t, x) = \int e^{ih(t,x,\eta)} b(t, x, \eta)\, d\eta$$

has the property that QJ is smooth.

Note. Changing the variables x, η to x, ξ in the integral, we can write it as

$$I(t, x) = \int e^{if(t,x,\xi)} a(t, x, \xi)\, d\xi$$

where f is the gauge and a is a classical amplitude of order zero. It may not be supported in $N(t)$. But this property which is desirable under repeated applications of Theorem 2 may be achieved. In fact, Hörmander's propagation of singularities theorem shows that the wave front set of the distribution $t \to I(t, x)$ is contained in the outflow from s to t of the intersection of $L(s)$ and the support of $a(s, x, \xi)$ and hence is contained in $N(t)$. Hence, if we multiply $a(t, x, \xi)$ by a smooth zero order function supported in $N(t)$ and equal to 1 close to the wave front set of $I(t, x)$, this integral changes only by a smooth function.

Proof. Suppose first that the amplitude $b(t, x, \eta)$ is of degree 0 and supported in $N(t)$. Then, by the lemma above, $QI(t, x)$ equals an oscillatory integral $K(t, x)$ as above with amplitude $b(t, x, \eta)$ with main term

$$(h_t + p(t, x, h_x))\, b_0(t, x, \eta)$$

of pure order 1 and a term

$$(D_t + p_\xi(t, x, h_x) D_x + p_0(t, x, h_x))\, b_0(t, x, \eta)$$

of pure order 0. Using Theorem 1, the integral of the term of pure order 1, namely

$$\int e^{ih(t,x,\eta)} z(t, x, \eta)\, h_\eta(t, x, \eta)\, d\eta$$

reduces by an integration by parts to

$$-\int e^{ih(t,x,\eta)} D_\eta\big(z(t,x,\eta)\big) b_0(t,x,\eta)\, d\eta$$

which is the integral of a homogeneous amplitude of pure order 0. Collecting terms and putting

$$L = L(t, x, \eta, D_t, D_x, D_\eta) = D_t + p_\xi(t, x, h_x) D_x - D_\eta z(t, x, \eta) + p_0(t, x, h_x),$$

we can make $QJ(t, x)$ an oscillatory integral whose amplitude has order <0 provided $b_0(t, x, \eta)$ satisfies the equation

$$Lb_0(t, x, \eta) = 0,$$

and reduces to $b(s, x, \eta)$ at time s.

The other terms are treated similarly. The result is a set of equations

$$Lb_k(t, x, \eta) + F(k, t, x, \eta) = 0, \quad b_{-k} = b_{-k}(s, x, \eta) \quad \text{when} \quad t = s,$$

for the vanishing of the amplitudes of pure order $k = 0, -1, -2, \ldots$ of $QJ(t, x)$. Here the second term only depends on the previously computed terms $b_0, b_{-1}, \ldots, b_{k+1}$.

When solving the differential equations above, we do not get out of the domain of definition of the coordinates x, η provided that ε and $N(s)$ are small enough. Putting

$$b(t, x, \eta) = \sum \big(1 - \chi(\varepsilon_j \eta)\big) b_j(t, x, \eta)$$

with χ infinitely differentiable with compact support and 1 close to the origing and ε_j tending to zero sufficiently fast (Hörmander [5], III, p. 66) completes the construction.

Note. In the case $s=0$, $b(y, \eta) = 1/(2\pi)^n$ and $h(0, y, \eta) = y\eta$, we get $I(0, y) = \delta(y)$ and our construction essentially reduces to that of Lax [6]. Theorem 2 is a general version of Lax's construction.

The global parametrix

It is now easy to construct a parametrix of Cauchy's problem (2) of the introduction. Consider instead of (2) a Cauchy problem

$$QF(t, x) \equiv 0, \quad F(0, y) = \int e^{iy,\eta} a(\eta)\, d\eta$$

where $a(\eta)$ is a zero order amplitude with very small conical support $N(0)$ around a given ray $Z(0)$. By Theorem 1 and Theorem 2 there is a phase function $h(t, x, \eta)$ and an amplitude $b(t, x, \eta)$ such that

$$F(t, x) = I(t, x) = \int e^{ih(t,x,\eta)} b(t, x, \eta)\, d\eta$$

solves the problem for small times. Also, changing coordinates from x, η to x, ξ gives to $I(t, x)$ the form

$$I(t, x) = \int e^{if(t,x,\xi)} a(t, x, \xi)\, d\xi \tag{5}$$

where $a(t, x, \xi)$ is a classical amplitude which, by the note to Theorem 2, may be taken to supported in the outflow $N(t)$ of $N(0)$. This works as long as the ξ variables parametrize the Lagrangian manifold $L(t)$ in $N(t)$. But, close to such a point, say $t=r$, we can shrink $N(0)$ and try to continue and, if this does not work, Theorem 2 shows that it is possible to shrink $N(0)$ and switch space coordinates at $Z(r)$ so that the first step can be repeated to times $>r$. Hence, shrinking $N(0)$ and repeatedly switching space coordinates, we can solve the problem (5) up to any time provided $N(0)$ is close enough to $Z(0)$.

To finish the construction of a global parametrix, we only need to cover $\mathbf{R}^n \setminus 0$ by a finite number of neighborhoods of chosen rays which permit solutions of the corresponding Cauchy problems up to time T. If we make the initial amplitudes $a(\eta)$ add up to $(2\pi)^{-n}$ for large η, the sum

$$\sum F(t, x)$$

of the corresponding solutions is a global parametrix.

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