# Closed ideals in the bidisc algebra

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## 0.

Let $\mathbf{D}$ be the open unit disc in $\mathbf{C}$, $\mathbf{T}=\partial\mathbf{D}$ the unit circle, and let $\mathbf{D}^n=\mathbf{D}\times\ldots\times\mathbf{D}$ be the $n$-dimensional unit polydisc. Let $H^\infty(\mathbf{D}^n)$ be the algebra of bounded analytic functions on $\mathbf{D}^n$, endowed with the uniform norm on $\mathbf{D}^n$. The polydisc algebra is the space $A(\mathbf{D}^n)=C(\overline{\mathbf{D}}^n)\cap H^\infty(\mathbf{D}^n)$, also given the uniform norm on $\mathbf{D}^n$. The spaces $A(\mathbf{D})$ and $A(\mathbf{D}^2)$ are known as the disc and bidisc algebras, respectively.

Let us introduce a weak-star topology on $H^\infty(\mathbf{D}^n)$. The space $L^\infty(\mathbf{T}^n)$ is the dual space of $L^1(\mathbf{T}^n)$, so it has a weak-star topology. One can think of $H^\infty(\mathbf{D}^n)$ as a subspace of $L^\infty(\mathbf{T}^n)$ via radial limits, and as such it is weak-star closed. We define the weak-star topology on $H^\infty(\mathbf{D}^n)$ by saying that a set $U\subset H^\infty(\mathbf{D}^n)$ is open if there is a weak-star open set $V\subset L^\infty(\mathbf{T}^n)$ with $U=V\cap H^\infty(\mathbf{D}^n)$.

For a collection $\mathscr{F}$ of functions in $A(\mathbf{D}^n)$, associate the zero set

$$Z(\mathscr{F})=\{z\in\overline{\mathbf{D}}^n\colon f(z)=0 \text{ for all } f\in\mathscr{F}\},$$

and if $E\subset\overline{\mathbf{D}}^n$, introduce the closed ideal

$$\mathscr{I}(E)=\{f\in A(\mathbf{D}^n)\colon f=0 \text{ on } E\}.$$

In this paper, we will try to describe the closed ideals of the bidisc algebra. The result we obtain is the following. Every closed ideal $I$ in the bidisc algebra $A(\mathbf{D}^2)$ has the form

$$\begin{aligned} I = {} & \mathscr{I}(Z(I)\cap\mathbf{T}^2) \\ & \cap\{f\in A(\mathbf{D}^2)\colon f(\alpha,\cdot)\in u_\alpha H^\infty(\mathbf{D}) \text{ and } f(\cdot,\alpha)\in v_\alpha H^\infty(\mathbf{D}) \text{ for all } \alpha\in\mathbf{T}\} \\ & \cap[I]_{w*}, \end{aligned}$$

where $u_\alpha$ and $v_\alpha$ are inner functions in $H^\infty(\mathbf{D})$ for each $\alpha\in\mathbf{T}$, and $[I]_{w*}$ is the weak-star closure of $I$ in $H^\infty(\mathbf{D}^2)$. It is easy to see that $[I]_{w*}$ is a (weak-star closed) ideal

[1] This research was supported by the National Science Foundation (NSF).

in $H^\infty(\mathbf{D}^2)$. Unfortunately, there is no concrete description of the weak-star closed in $H^\infty(\mathbf{D}^2)$.

The above description of the closed ideals in $A(\mathbf{D}^2)$ can be generalized to $A(\mathbf{D}^n)$. The result is the following, which we mention without proof. For $\alpha\in\overline{\mathbf{D}}$ and $1\leqq j\leqq n$, let $\mathcal{R}_{j,\alpha}\colon A(\mathbf{D}^n)\to A(\mathbf{D}^{n-1})$ be the restriction operator

$$\mathcal{R}_{j,\alpha}f(z_1, \ldots, z_n) = f(z_1, \ldots, z_{j-1}, \alpha, z_{j+1}, \ldots, z_n), \quad f\in A(\mathbf{D}^n).$$

Every closed ideal in $A(\mathbf{D}^n)$ has the form

$$I = \{f\in A(\mathbf{D}^n)\colon \mathcal{R}_{j,\alpha}f\in\mathcal{R}_{j,\alpha}I \text{ for all } 1\leqq j\leqq n \text{ and } \alpha\in\mathbf{T}\}\cap[I]_{w^*},$$

where $[I]_{w^*}$ is the weak-star closure of $I$ in $H^\infty(\mathbf{D}^n)$. Also, $\mathcal{R}_{j,\alpha}I$ is a closed ideal in $A(\mathbf{D}^{n-1})$ for all $1\leqq j\leqq n$ and $\alpha\in\mathbf{T}$. For the unit ball in $\mathbf{C}^n$, the author has obtained a corresponding result, with much less effort [Hed].

## 1.

Let $\lambda$ be arc length measure on $\mathbf{T}$, normalized so that $\lambda(\mathbf{T})=1$, and set $\lambda_n=\lambda\times\ldots\times\lambda$ ($n$-times), which is the $n$-dimensional volume measure on $\mathbf{T}^n$, normalized so that $\lambda_n(\mathbf{T}^n)=1$. Let $M(\mathbf{T}^n)$ be the space of finite Borel measures on $\mathbf{T}^n$. We may regard $M(\mathbf{T}^n)$ as the dual space of $C(\mathbf{T}^n)$ via the dual action

$$\langle f, \mu\rangle = \int_{\mathbf{T}^n} f\,d\mu, \quad f\in C(\mathbf{T}^n), \quad \mu\in M(\mathbf{T}^n).$$

We will write $f\mu$ for the Borel measure with $d(f\mu)=f\,d\mu$.

A representing measure for 0 is a Borel probability measure on $\mathbf{T}^n$ such that

$$f(0) = \int_{\mathbf{T}^n} f\,d\varrho, \quad f\in A(\mathbf{D}^n).$$

We will denote by $M_0(\mathbf{T}^n)$ the convex set of representing measures for the origin. A measure $\varrho\in M_0(\mathbf{T}^n)$ is a Jensen measure if

$$\log|f(0)| \leqq \int_{\mathbf{T}^n} \log|f|\,d\varrho, \quad f\in A(\mathbf{D}^n).$$

A *band of measures* $\mathcal{B}$ on $\mathbf{T}^n$ is a closed subspace of $M(\mathbf{T}^n)$ such that if $\mu\in\mathcal{B}$, $\nu\in M(\mathbf{T}^n)$, and $\nu$ is absolutely continuous with respect to $\mu$, then $\nu\in\mathcal{B}$.

## 2.

We shall need some results on measures that annihilate $A(\mathbf{D}^2)$. Let

(1) $\mathscr{B}_0$ be the band generated by the representing measures at 0;

(2) $\mathscr{B}_1$ be the band generated by measures $\mu\in A(\mathbf{D}^2)^\perp$ such that $\mu$ is carried by a set $E\times\mathbf{T}$, where $E$ is a Borel set with arc length zero;

(3) $\mathscr{B}_2$ be the band generated by measures $\mu\in A(\mathbf{D}^2)^\perp$ such that $\mu$ is carried by a set $\mathbf{T}\times E$, where $E$ is a Borel set with arc length zero;

(4) $\mathscr{B}_s$ be the band of measures singular to all the measures in $\mathscr{B}_0$, $\mathscr{B}_1$, and $\mathscr{B}_2$.

*Remark 2.1.* The band $\mathscr{B}_0$ consists of those measures which are absolutely continuous with respect to some representing measure for 0.

The following result was obtained by Brian Cole in the early 1970's (see [Gam, pp. 143—146], [Bek]).

**Theorem 2.2.** *Every measure* $\mu\in M(\mathbf{T}^2)$ *has a unique decomposition*

$$\mu = \mu_0+\mu_1+\mu_2+\mu_s,$$

*where* $\mu_0\in\mathscr{B}_0$, $\mu_1\in\mathscr{B}_1$, $\mu_2\in\mathscr{B}_2$, *and* $\mu_s\in\mathscr{B}_s$. *If* $\mu\in A(\mathbf{D}^2)^\perp$, *then* $\mu_s=0$, *and* $\mu_0, \mu_1, \mu_2\in A(\mathbf{D}^2)^\perp$; *moreover,*

$$d\mu_1(z_1, z_2) = g(z_1, z_2)\, d\sigma(z_1)\, d\lambda(z_2),$$

$$d\mu_2(z_1, z_2) = h(z_1, z_2)\, d\lambda(z_1)\, d\tau(z_2),$$

*where* $\sigma$ *and* $\tau$ *are two Borel probability measures on* $\mathbf{T}$ *carried by a set of zero arc length,* $g\in L^1(\sigma\times\lambda)$, $h\in L^1(\lambda\times\tau)$, *and* $g(\alpha, \cdot), h(\cdot, \alpha)\in H_0^1(\mathbf{T})$ *for every* $\alpha\in\mathbf{T}$. *Here,* $H_0^1(\mathbf{T})$ *is the restriction to* $\mathbf{T}$ *of the functions in* $H_0^1(\mathbf{D})=\{f\in H^1(\mathbf{D}): f(0)=0\}$, *where* $H^1(\mathbf{D})$ *is the usual Hardy space on the disc.*

We shall need the following related lemma.

**Lemma 2.3.** *A measure* $g\cdot(\sigma\times\lambda)$, *where* $\sigma$ *is a singular Borel probability measure on* $\mathbf{T}$, *and* $g\in L^1(\sigma\times\lambda)$, *annihilates* $A(\mathbf{D}^2)$ *if and only if* $g(\alpha, \cdot)\in H_0^1(\mathbf{T})$ *for* $\sigma$*-almost all* $\alpha\in\mathbf{T}$.

*Proof.* If $g(\alpha, \cdot)\in H_0^1(\mathbf{T})$, then if $f\in A(\mathbf{D}^2)$, we have

$$\int_{\mathbf{T}^2} f(z_1, z_2)g(z_1, z_2)\, d\sigma(z_1)\, d\lambda(z_2) = \int_{\mathbf{T}^2} f(z_1, z_2)g(z_1, z_2)\, d\lambda(z_2)\, d\sigma(z_1) = 0,$$

so that $g\cdot(\sigma\times\lambda)\perp A(\mathbf{D}^2)$. If, on the other hand, $g\cdot(\sigma\times\lambda)\perp A(\mathbf{D}^2)$, we have for integers $n, m\geqq 0$,

$$\int_{\mathbf{T}^2} z_1^n z_2^m g(z_1, z_2)\, d\lambda(z_2)\, d\sigma(z_1) = 0,$$

from which we can conclude, since $\sigma$ is singular to $\lambda$, that

$$\int_{\mathbf{T}} z_2^m g(z_1, z_2)\, d\lambda(z_2) = 0$$

for $\sigma$-almost every $z_1 \in \mathbf{T}$. It follows that $g(z_1, \cdot) \in H_0^1(\mathbf{T})$ for $\sigma$-almost every $z_1 \in \mathbf{T}$.

A sequence $\{f_j\}_{j=1}^{\infty}$ of functions in $A(\mathbf{D}^2)$ is said to be a *Montel sequence* if $\sup_j \|f_j\| < \infty$ and $f_j(z) \to 0$ as $j \to \infty$ for every $z \in \mathbf{D}^2$. Following G. M. Henkin [Hen], we say that a measure $\mu \in M(\mathbf{T}^2)$ is an *A-measure* if

$$\int_{\mathbf{T}^2} f_j\, d\mu \to 0 \quad \text{as} \quad j \to \infty$$

for every Montel sequence $\{f_j\}_{j=1}^{\infty}$. Observe that this definition does not agree with that of Bekken [Bek]. Also observe that the $A$-measures form a closed subspace of $M(\mathbf{T}^2)$. The proof of the following theorem is identical to that of Valskiĭ's theorem in [Rud, p. 187].

**Theorem 2.4.** *If $\mu \in M(\mathbf{T}^2)$ is an A-measure, then there exist $\nu \in A(\mathbf{D}^2)^{\perp}$ and $g \in L^1(\lambda_2) = L^1(\mathbf{T}^2)$ such that $\mu = \nu + g\lambda_2$.*

The following result is a reformulation of Corollary 3.3 in [Bek].

**Theorem 2.5.** *For a measure $\mu \in M(\mathbf{T}^2)$, the following are equivalent:*

(a) *$\mu \in \mathscr{B}_0$, that is, $\mu$ is absolutely continuous with respect to some representing measure for* 0,

(b) *every $\nu \in M(\mathbf{T}^2)$ with $\nu \ll \mu$ is an A-measure.*

In other words, $\mathscr{B}_0$ is the biggest band contained within the set of $A$-measures. On the other hand, the smallest band containing the $A$-measures is $\mathscr{B}_0 \oplus \mathscr{B}_1 \oplus \mathscr{B}_2$.

We now state our main result.

**Theorem 2.6.** *Every closed ideal $I$ in $A(\mathbf{D}^2)$ has the form*

$$I = \mathscr{I}(Z(I) \cap \mathbf{T}^2)$$
$$\cap \{f \in A(\mathbf{D}^2) : f(\alpha, \cdot) \in u_\alpha H^\infty(\mathbf{D}) \text{ and } f(\cdot, \alpha) \in v_\alpha H^\infty(\mathbf{D}) \text{ for all } \alpha \in \mathbf{T}\}$$
$$\cap [I]_{w*},$$

*where $u_\alpha, v_\alpha \in \mathscr{U} \cup \{0\}$ for each $\alpha \in \mathbf{T}$, and $[I]_{w*}$ is the weak-star closure of $I$ in $H^\infty(\mathbf{D}^2)$. Here, $\mathscr{U}$ denotes the collection of inner functions in $H^\infty(\mathbf{D})$.*

*Proof.* For $\alpha \in \mathbf{T}$, let $I(\alpha, \cdot)$ and $I(\cdot, \alpha)$ denote the ideals $\{f(\alpha, \cdot) \in A(\mathbf{D}) : f \in I\}$ and $\{f(\cdot, \alpha) : f \in I\}$, respectively; observe that these ideals are closed in $A(\mathbf{D})$ because $\{\alpha\} \times \overline{\mathbf{D}}$ and $\overline{\mathbf{D}} \times \{\alpha\}$ are peak sets for the bidisc algebra. Consider the weak-star closure of $I(\alpha, \cdot)$ in $H^\infty(\mathbf{D})$. By the well-known description of the weak-star closed ideals in $H^\infty(\mathbf{D})$ [Gar, p. 85], it has the form $u_\alpha H^\infty(\mathbf{D})$, where $u_\alpha$ is either an inner function in $H^\infty(\mathbf{D})$, or vanishes identically on $D$. This determines the functions $u_\alpha$ up to unimodular constant factors. These are the $u_\alpha$'s mentioned in the theorem. The functions $v_\alpha$ are defined similarly. Observe that with this choice of $u_\alpha$ and $v_\alpha$ we have

$$I \subset \mathscr{I}(Z(I) \cap \mathbf{T}^2) \\ \cap \{f \in A(\mathbf{D}^2)\colon f(\alpha, \cdot) \in u_\alpha H^\infty(\mathbf{D}) \text{ and } f(\cdot, \alpha) \in v_\alpha H^\infty(\mathbf{D}) \text{ for all } \alpha \in \mathbf{T}\} \\ \cap [I]_{w^*},$$

so what remains to be shown is the reverse inclusion.

Let $\varphi \in A(\mathbf{D}^2)^*$ annihilate $I$. By the Hahn—Banach theorem, there is a measure $\mu \in M(\mathbf{T}^2)$ such that

$$\langle f, \varphi \rangle = \langle f, \mu \rangle = \int_{\mathbf{S}_n} f\, d\mu, \quad f \in A(\mathbf{D}^2);$$

then $\mu \perp I$. If we can show that

$$\mu \perp \mathscr{I}(Z(I) \cap \mathbf{T}^2) \\ \cap \{f \in A(\mathbf{D}^2)\colon f(\alpha, \cdot) \in u_\alpha H^\infty(\mathbf{D}) \text{ and } f(\cdot, \alpha) \in v_\alpha H^\infty(\mathbf{D}) \text{ for all } \alpha \in \mathbf{T}\} \\ \cap [I]_{w^*},$$

the assertion will follow, again by the Hahn—Banach theorem. By Theorem 2.2,

$$\mu = \mu_0 + \mu_1 + \mu_2 + \mu_s,$$

where $\mu_0 \in \mathscr{B}_0$, $\mu_1 \in \mathscr{B}_1$, $\mu_2 \in \mathscr{B}_2$, and $\mu_s \in \mathscr{B}_s$. We intend to show that $\mu_0 \perp [I]_{w^*} \cap A(\mathbf{D}^2)$, $\mu_s \perp \mathscr{I}(Z(I) \cap \mathbf{T}^2)$,

$$\mu_1 \perp \{f \in \mathscr{I}(Z(I) \cap \mathbf{T}^2)\colon f(\alpha, \cdot) \in u_\alpha H^\infty(\mathbf{D}) \text{ for all } \alpha \in \mathbf{T}\},$$

and

$$\mu_2 \perp \{f \in \mathscr{I}(Z(I) \cap \mathbf{T}^2)\colon f(\cdot, \alpha) \in v_\alpha H^\infty(\mathbf{D}) \text{ for all } \alpha \in \mathbf{T}\}.$$

Let $f \in I$ be arbitrary. Then $f\mu \perp A(\mathbf{D}^2)$, and

$$f\mu = f\mu_0 + f\mu_1 + f\mu_2 + f\mu_s \in \mathscr{B}_0 \oplus \mathscr{B}_1 \oplus \mathscr{B}_2 \oplus \mathscr{B}_s$$

is the unique decomposition of $f\mu$. By Theorem 2.2, $f\mu_s = 0$, and $f\mu_j \in A(\mathbf{D}^2)^\perp$ for $j = 0, 1, 2$. It follows that $\mu_s$ is supported on $Z(f) \cap \mathbf{T}^2$, and by varying $f \in I$, we realize that $\mu_s$ is supported on $Z(I) \cap \mathbf{T}^2$, so that $\mu_s \perp \mathscr{I}(Z(I) \cap \mathbf{T}^2)$; we also get $\mu_j \perp I$ for $j = 0, 1, 2$. We will now show that $\mu_0 \perp [I]_{w^*} \cap A(\mathbf{D}^2)$. By Theorem 2.5, $\mu_0$ is an $A$-measure, so that by Theorem 2.4, $\mu_0 = \nu + \varphi\lambda_2$, where $\nu \in A(\mathbf{D}^2)^\perp$ and $\varphi \in L^1(\lambda_2)$. Since $\mu_0 \perp I$, we also have $\varphi\lambda_2 \perp I$. Now $\varphi\lambda_2 \perp [I]_{w^*}$, because $\varphi\lambda_2$ is in the predual of $L^\infty(\mathbf{T}^2)$, so we get $\mu_0 \perp [I]_{w^*} \cap A(\mathbf{D}^2)$. It remains for us to show that $\mu_1$ annihilates

$$I_u = \{f \in \mathscr{I}(Z(I) \cap \mathbf{T}^2)\colon f(\alpha, \cdot) \in u_\alpha H^\infty(\mathbf{D}) \text{ for all } \alpha \in \mathbf{T}\};$$

the verification process for the related assertion concerning $\mu_2$ is identical. The measure $\mu_1$ has a decomposition $\mu_1 = \mu_1^a + \mu_1^b$, where $\mu_1^b$ is supported on the set $Z(I) \cap \mathbf{T}^2$, and $|\mu_1^a|(Z(I) \cap \mathbf{T}^2) = 0$. Since $\mu_1^b$ annihilates $\mathscr{I}(Z(I) \cap \mathbf{T}^2)$ and $\mu_1 \perp I$, we obtain

$\mu_1^a \perp I$. If we can show that $\mu_1^a \perp I_u$, the assertion $\mu_1 \perp I_u$ follows. In what follows, we shall write $\mu_1$ instead of $\mu_1^a$. Just as before, let $f \in I$ be arbitrary. By Theorem 2.2, $f\mu_1$ has the form

$$f(z_1, z_2)\, d\mu_1(z_1, z_2) = g_f(z_1, z_2)\, d\sigma_f(z_1)\, d\lambda(z_2),$$

where $\sigma_f$ is a Borel probability measure on $\mathbf{T}$ carried by a (Borel) set $E_f$ of arc length $0, g_f \in L^1(\sigma_f \times \lambda)$, and $g_f(\alpha, \cdot) \in H_0^1(\mathbf{T})$ for all $\alpha \in \mathbf{T}$; we use subscripts on $\sigma$ and $g$ to indicate that they may depend on $f$. We will now show that we can choose $\sigma$ independently of $f$. Since $A(\mathbf{D}^2)$ is a separable Banach space, the closed subspace $I$ is separable as well, which means that $I$ has a countable dense subset $\mathcal{F}$. Let $\{\varepsilon_f\}_{f \in \mathcal{F}}$ be a sequence such that $\varepsilon_f > 0$ for all $f \in \mathcal{F}$ and $\sum_{f \in \mathcal{F}} \varepsilon_f = 1$. Moreover, let $\sigma$ be the Borel probability measure $\sigma = \sum_{f \in \mathcal{F}} \varepsilon_f \sigma_f$, which is carried by the Borel set $\bigcup_{f \in \mathcal{F}} E_f$, which has arc length 0. Then $f\mu_1 \ll \sigma \times \lambda$ if $f \in \mathcal{F}$, so that

$$f(z_1, z_2)\, d\mu_1(z_1, z_2) = G_f(z_1, z_2)\, d\sigma(z_1)\, d\lambda(z_2),$$

where $G_f \in L^1(\sigma \times \lambda)$. We conclude that

$$\mu_1|_{\mathbf{T}^2 \setminus Z(f)} = (G_f/f)(\sigma \times \lambda)|_{\mathbf{T}^2 \setminus Z(f)}$$

for every $f \in \mathcal{F}$, so that in particular, $G_f/f \in L^1(\mathbf{T}^2 \setminus Z(f), \sigma \times \lambda)$, and two quotients $G_{f_1}/f_1$ and $G_{f_2}/f_2$, where $f_1, f_2 \in \mathcal{F}$, are equal $\sigma \times \lambda$-almost everywhere on $\mathbf{T}^2 \setminus (Z(f_1) \cup Z(f_2))$. Since $\mathcal{F}$ was countable, this means that we can find a Borel measurable function $\Phi$ on $\mathbf{T}^2$, which is 0 on $Z(\mathcal{F}) \cap \mathbf{T}^2 (Z(\mathcal{F}) = Z(I)$ because $\mathcal{F}$ is dense in $I)$, and for every $f \in \mathcal{F}$, equals $G_f/f$ $\sigma \times \lambda$-almost everywhere on $\mathbf{T}^2 \setminus Z(f)$. From the estimate

$$\int_{\mathbf{T}^2 \setminus Z(f)} |G_f/f|\, d(\sigma \times \lambda) \leqq |\mu_1|(\mathbf{T}^2)$$

and Lebesgue's monotone convergence theorem, it follows that $\Phi \in L^1(\sigma \times \lambda)$; in fact,

$$\|\Phi\|_{L^1(\sigma \times \lambda)} = |\mu_1|(\mathbf{T}^2 \setminus Z(I)) = |\mu_1|(\mathbf{T}^2).$$

Moreover, since $|\mu_1|(Z(I) \cap \mathbf{T}^2) = 0$, we have

$$\mu_1 = \Phi(\sigma \times \lambda).$$

If $f \in I$, we get

$$f\mu_1 = \Phi f(\sigma \times \lambda) \in A(\mathbf{D}^2)^\perp,$$

so by Lemma 2.3, it follows that $(\Phi f)(\alpha, \cdot) \in H_0^1(\mathbf{T})$ for $\sigma$-almost every $\alpha \in \mathbf{T}$. Since $\Phi \in L^1(\sigma \times \lambda)$, we have $\Phi(\alpha, \cdot) \in L^1(\lambda)$ for $\sigma$-almost all $\alpha \in \mathbf{T}$. Let $\alpha \in \mathbf{T}$ be such that $\Phi(\alpha, \cdot) \in L^1(\lambda)$ and $(\Phi f)(\alpha, \cdot) \in H_0^1(\mathbf{T})$ for every $f \in \mathcal{F}$; observe that the collection of such $\alpha$ has full $\sigma$-measure. Then $\Phi(\alpha, \cdot)\lambda \perp f(\alpha, \cdot) H^\infty(\mathbf{T})$ for every $f \in \mathcal{F}$, where $H^\infty(\mathbf{T})$ is the restriction to $\mathbf{T}$ of the space $H^\infty(\mathbf{D})$, and since $\mathcal{F}$ was dense in $I$ and $\Phi(\alpha, \cdot) \in L^1(\mathbf{T})$, we find that $\Phi(\alpha, \cdot)\lambda$ annihilates the weak-star closure $u_\alpha H^\infty(\mathbf{T})$ of $I(\alpha, \cdot)$ in $H^\infty(\mathbf{T})$, so that

$$u_\alpha \Phi(\alpha, \cdot) \in H_0^1(\mathbf{T}).$$

Now let $\varphi \in I_u$ be arbitrary. Since $\varphi(\alpha, \cdot) \in u_\alpha H^\infty(\mathbf{T})$, we have

$$\varphi(\alpha, \cdot)\Phi(\alpha, \cdot) \in H_0^1(\mathbf{T}),$$

and because this happens for $\sigma$-almost every $\alpha \in \mathbf{T}$, Lemma 2.3 says that

$$\varphi\mu_1 = (\varphi\Phi)(\sigma \times \lambda) \in A(\mathbf{D}^2)^\perp.$$

In particular, $\mu_1 \perp \varphi$, and the assertion $\mu_1 \perp I_u$ follows. The proof is complete.

Let us make the following observation. It has some implications concerning the choice of $u_\alpha$'s and $v_\alpha$'s in Theorem 2.6.

**Proposition 2.7.** *Let $f \in A(\mathbf{D}^2)$, and assume $f(\alpha, \cdot) \in uH^\infty(\mathbf{D})$ for all $\alpha \in E$, where $u$ is an inner function in $H^\infty(\mathbf{D})$, and $E$ is a subset of $\mathbf{T}$. If $\lambda(\bar{E}) > 0$, then $f(\alpha, \cdot) \in uH^\infty(\mathbf{D})$ for all $\alpha \in \mathbf{T}$.*

*Proof.* Let $\mu \in M(\mathbf{T})$ be such that $\mu \perp uH^\infty(\mathbf{D}) \cap A(\mathbf{D})$, and introduce the function $F_\mu \colon \bar{\mathbf{D}} \to \mathbf{C}$ defined by the relation

$$F_\mu(\alpha) = \langle f(\alpha, \cdot), \mu \rangle, \quad \alpha \in \bar{\mathbf{D}}.$$

Now because $f \in A(\mathbf{D}^2)$, it is easy to see that $F_\mu \in A(\mathbf{D})$. By assumption, $F_\mu = 0$ on $E$, and by continuity on $\bar{E}$, and since $\lambda(\bar{E}) > 0$, we may conclude that $F_\mu = 0$ on all of $\bar{\mathbf{D}}$. If we vary $\mu$ and apply the Hahn—Banach theorem, the assertion follows.

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*Received January 19, 1989*

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