Local regularity of solutions to nonlinear Schrödinger equations

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In P. Sjögren and P. Sjölin [4] we studied the regularity of solutions to the Schrödinger equation $i\partial u/\partial t = -Pu + Vu$ in a half-space $\{(x, t) \in \mathbf{R}^n \times \mathbf{R}_+\}$. Here P is an elliptic self-adjoint constant-coefficient operator in x of order $m \geqq 2$ and $V = V(x)$ a real-valued potential. We assumed that $V \in C^\infty(\mathbf{R}^n)$ and that $D^\alpha V$ is bounded for every α, where $D = (D_1, \ldots, D_n)$ and $D_k = -i\partial/\partial x_k$.

To state the results in [4] we introduce Sobolev spaces $H_s = H_s(\mathbf{R}^n)$ and mixed Sobolev spaces $H_{\varrho,r}$ for $\varrho \geqq 0, r \geqq 0$. We set $H_{\varrho,r} = H_{\varrho,r}(\mathbf{R}^n \times \mathbf{R}) = (G_\varrho \otimes G_r) * L^2(\mathbf{R}^{n+1})$, where G_ϱ and G_r are Bessel kernels in $\mathbf{R}^n$ and $\mathbf{R}$, respectively.

For $f \in L^2(\mathbf{R}^n)$ we let u denote the solution to the above Schrödinger equation with $u(x, 0) = f(x)$. We also set

$$\mathscr{A} = \{\varphi \in C^\infty(\mathbf{R}^n);\ \text{there exists } \varepsilon > 0 \text{ such that}$$

$$|D^\alpha \varphi(x)| \leqq C_\alpha (1+|x|)^{-1/2-\varepsilon} \text{ for every } \alpha\}$$

and

$$Sf(x, t) = \varphi(x)\psi(t)u(x, t)$$

where $\varphi \in \mathscr{A}$ and $\psi \in C_0^\infty(\mathbf{R})$. The following result was proved in [4].

Theorem A. *If* $\varrho \geqq 0$, $r \geqq 0$, *then*

$$\|Sf\|_{H_{\varrho,r}} \leqq C\|f\|_{H_{\varrho+mr-(m-1)/2}}, \quad f \in \mathscr{S},$$

where the constant C depends on φ *and* ψ.

Theorem A expresses a local smoothing property for the Schrödinger equation. Setting $I = [0, T]$, $T > 0$, we observe that it follows from the above estimate with $r = 0$ that

$$\|\varphi u\|_{L^2(I;\, H_{\varrho+(m-1)/2}(\mathbf{R}^n))} \leqq C\|f\|_{H_\varrho}$$

for $\varrho \geqq -(m-1)/2$.

We shall here consider analogues of this estimate for solutions to the nonlinear

Schrödinger equation

$$i\partial u/\partial t = -\Delta u + F(u), \quad t \geqq 0, \quad x \in \mathbf{R}^n.$$

Our results are based on the work of Kato [3] on this equation.

We introduce some notation. We let p satisfy $1<p<\infty$ for $n=1, 2$ and $1<p<(n+2)/(n-2)$ for $n \geqq 3$. Then set $r=4(p+1)/n(p-1)$ so that $2<r<\infty$. We write $\partial=(\partial_1, \ldots, \partial_n)$ where $\partial_j=\partial/\partial x_j$ and set $\partial^2=(\partial_i\partial_j)_{i,j=1}^n$.

Bessel potential spaces are denoted L_s^q, $1 \leqq q<\infty$, $s \geqq 0$, so that $H_s=L_s^2$, and we set $L^{q,s}=L^s(I; L^q(\mathbf{R}^n))$, $1 \leqq s \leqq \infty$, $1 \leqq q<\infty$.

We assume $F \in C^1(\mathbf{R}^2)$, F complex-valued, $F(0)=0$, and

$$|D^\alpha F(\zeta)| \leqq C|\zeta|^{p-1} \quad \text{for} \quad |\zeta| \geqq 1 \quad \text{and} \quad |\alpha| = 1. \tag{1}$$

Then assume $f \in H_1(\mathbf{R}^n)$.

Kato [3] has proved that there exists a $T>0$ such that the nonlinear Schrödinger equation

$$i\partial_t u = -\Delta u + F(u), \quad t \geqq 0, \quad x \in \mathbf{R}^n, \tag{2}$$

has a unique solution $u \in C(I; H_1)$ with $u(0)=f$. Also $\partial u \in L^r(I; L^{p+1})$. Here Δ denotes the Laplace operator in the x-variable and $F(u)(x, t)=F(u(x, t))$. We shall first prove the following theorem.

Theorem 1. *Assume p and F are as above and let $f \in H_1(\mathbf{R}^n)$, $\varphi \in \mathscr{A}$. Let u denote the above solution to the equation* (2). *Then the following holds.*

In the case $n=1$ or 2 $\varphi u \in L^2(I; H_{3/2})$ for $1<p<\infty$.

In the case $3 \leqq n \leqq 5$ $\varphi u \in L^2(I; H_{3/2})$ for $1<p<p_1$, where

$$p_1 = \frac{n+4+\sqrt{n^2+24n+16}}{2n}.$$

In the case $n \geqq 6$ set

$$\delta(p) = \frac{p(3-n)+n+3}{2(p+1)}$$

for $1 \leqq p \leqq (n+1)/(n-1)$. Then $\varphi u \in L^2(I; H_{\delta(p)})$ for $1<p<(n+1)/(n-1)$.

We remark that $2<p_1<3$ and $p_1<(n+2)/(n-2)$ for $3 \leqq n \leqq 5$ and also that $\delta(p)$ is a decreasing function of p on the interval $[1, (n+1)/(n-1)]$ with $\delta(1)=3/2$ and $\delta((n+1)/(n-1))=1$.

Kato has also proved that if $u(0)=f \in H_2$ then the solution u of (2) belongs to $C(I; H_2)$. We shall prove the following result.

Theorem 2. *Assume that $1 \leqq n \leqq 7$ and that p and F are as above. Also assume that $F \in C^2(\mathbf{R}^2)$ and that $|D^\alpha F(\zeta)| \leqq C|\zeta|^{\max(p-2,0)}$ for $|\zeta| \geqq 1$ and $|\alpha|=2$. Assume that $f \in H_2$ and $\varphi \in \mathscr{A}$. Then the above solution u of* (2) *satisfies $\varphi u \in L^2(I; H_{5/2})$ if $T>0$ is sufficiently small.*

We remark that in the case $n \leqq 3$ Theorem 2 was essentially proved by Constantin and Saut [2].

Following Kato [3] we introduce the following spaces:

$$X_0 = L^{2,\infty} \cap L^{p+1,\infty}$$
$$\bar{X} = C(I;\ L^2) \cap L^{p+1,r}$$
$$X = L^{2,\infty} \cap L^{p+1,r}$$
$$X' = L^{2,1} + L^{1+1/p,r'}$$
$$\bar{Y} = \{v \in \bar{X};\ \partial v \in \bar{X}\}$$
$$Y = \{v \in X;\ \partial v \in X\}$$
$$Y' = \{v \in X';\ \partial v \in X'\}$$
$$Y_0 = \{v \in X_0;\ \partial v \in X_0\}.$$

We also set

$$\bar{W} = \{v \in \bar{X};\ \partial v \in \bar{X},\ \partial^2 v \in \bar{X}\},$$
$$W = \{v \in X;\ \partial v \in X,\ \partial^2 v \in X\}$$

and

$$W' = \{v \in X';\ \partial v \in X',\ \partial^2 v \in X'\}.$$

The norms in these spaces are defined in the obvious way (cf. [3]).

We shall need the following well-known estimates (Sobolev's theorem).

Lemma. (i) *If* $1<p<q<\infty$, $s>0$ *and* $1/q \geqq 1/p - S/n$ *then*

$$\|f\|_q \leqq C\|f\|_{L^p_s}.$$

(ii) *If* $1<p<\infty$, $p>n/k$ *and* $k \geqq 1$ *then*

$$\|f\|_\infty \leqq C\|f\|_{L^p_k}.$$

Choose $\psi \in C_0^\infty(\mathbf{R}^2)$ so that $\psi=1$ in a neighbourhood of the origin. Set $F_1=\psi F$ and $F_2=(1-\psi)F$ so that

$$F = F_1 + F_2.$$

We shall now prove the theorems.

Proof of Theorem 1. According to the proof of Theorem I in [3], p. 120, we have $u \in \bar{Y} \subset Y$ i.e. u and $\partial u \in \bar{X}$. It follows that

(3) $$u \in C(I;\ L^2) \cap L^{p+1,r}$$

and

(4) $$\partial u \in C(I;\ L^2) \cap L^{p+1,r}.$$

According to Lemma 2.2 in [3] $u \in Y$ implies $F(u) \in Y'$ i.e. $F(u) \in X'$ and $\partial(F(u)) \in X'$.

The proof of Lemma 2.2 really shows that

(5) $$F_1(u) \ \text{ and } \ \partial(F_1(u)) \in L^{2,1}$$

and

(6) $$F_2(u) \text{ and } \partial(F_2(u)) \in L^{1+1/p, r'}.$$

Now

$$u(t) = e^{it\Delta}f - i\int_0^t e^{i(t-\tau)\Delta}F(u(\tau))\,d\tau$$

([3], Lemma 1.1). With $\varphi \in \mathscr{A}$ and $s \geqq 1$ we obtain

$$\|\varphi u(t)\|_{H_s} \leqq \|\varphi e^{it\Delta}f\|_{H_s} + \int_0^t \big\|\varphi e^{i(t-\tau)\Delta}F(u(\tau))\big\|_{H_s}\,d\tau.$$

Hence

$$\|\varphi u\|_{L^2(I;\,H_s)} \leqq \|\varphi e^{it\Delta}f\|_{L^2(I;\,H_s)} + \int_0^T \Big(\int_0^T \big\|\varphi e^{it\Delta}e^{-i\tau\Delta}F(u(\tau))\big\|_{H_s}^2\,dt\Big)^{1/2}\,d\tau.$$

From Sjögren and Sjölin [4] it follows that

$$\|\varphi u\|_{L^2(I;\,H_s)} \leqq C\|f\|_{H_{s-1/2}} + C\int_I \big\|F(u(t))\big\|_{H_{s-1/2}}\,dt.$$

Since $f \in H_1$ it follows that for $1 < s \leqq 3/2$

(7) $$\varphi u \in L^2(I;\ H_s) \quad \text{if} \quad F(u) \in L^1(I;\ H_{s-1/2}).$$

We conclude from (5) that $F_1(u) \in L^1(I; H_{s-1/2})$ (assuming $1 < s \leqq 3/2$) and it remains to consider $F_2(u)$. We shall use (6). We have

(8) $$\big\|F_2(u(t))\big\|_{H_{1-\varepsilon}} \leqq C\big\|F_2(u(t))\big\|_{L_1^{1+1/p}},$$

where

$$1 - \frac{n}{1+1/p} = 1 - \varepsilon - \frac{n}{2}$$

([1], p. 153), and hence

$$\varepsilon = \varepsilon(p) = \frac{n(p-1)}{2(p+1)}.$$

It follows from the conditions on p that $0 < \varepsilon(p) < 1$ and hence $F_2(u) \in L^1(I; L^2)$ according to (6). We shall now estimate $\big\|\partial(F_2(u))\big\|_{L^{2,1}}$. We write $u = u_1 + iu_2$ where u_j real. If u is smooth the chain rule yields

(9) $$\partial_j(F_2(u)) = \frac{\partial F_2}{\partial x_1}(u)\partial_j u_1 + \frac{\partial F_2}{\partial x_2}(u)\partial_j u_2.$$

Choose $\varphi_0 \in C_0^\infty(\mathbf{R}^n)$ such that $\varphi_0 \geqq 0$, $\int \varphi_0\,dx = 1$. Set $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi_0(x/\varepsilon)$ and $u_m(t) = \varphi_{1/m} * (u(t))$, $m = 3, 4, 5, \ldots$, where $*$ denotes convolution in $\mathbf{R}^n$. Then (9) holds with u replaced by u_m.

For a.e. $t \in I$ we have (because of (3) and (4))

(10) $$u(t) \in L^2 \cap L^{p+1} \quad \text{and} \quad \partial u(t) \in L^2 \cap L^{p+1}.$$

We fix a t such that (10) holds. To prove (9) we shall prove that

(11) $$F_2(u_m(t)) \to F_2(u(t)), \quad m \to \infty,$$

and

(12) $$\frac{\partial F_2}{\partial x_1}(u_m(t))\partial_j u_{m,1}(t) \to \frac{\partial F_2}{\partial x_1}(u(t))\partial_j u_1(t), \quad m \to \infty,$$

in the sense of distributions in $\mathbf{R}^n$. In proving (11) and (12) we write u and u_m instead of $u(t)$ and $u_m(t)$.

It is clear that $F_2(u_m) \to F_2(u)$ a.e. in $\mathbf{R}^n$ since $u_m \to u$ a.e. Also

$$|F_2(u_m)| \leqq C|u_m|^p \leqq C(Mu)^p,$$

where Mu denotes the Hardy—Littlewood maximal function of u. Then $Mu \in L^{p+1}(\mathbf{R}^n)$ and hence $(Mu)^p \in L^1(B(0; R))$ where $B(0; R)$ denotes a ball in $\mathbf{R}^n$. It is then clear that

$$\int_{B(0;r)} |F_2(u_m) - F_2(u)|\, dx \to 0, \quad m \to \infty,$$

according to Lebesgue's theorem on dominated convergence (for every $R>0$) and hence (11) follows. To prove (12) we observe that

$$\frac{\partial F_2}{\partial x_1}(u_m)\partial_j u_{m,1} = \frac{\partial F_2}{\partial x_1}(u_m)(\varphi_{1/m} * (\partial_j u_1)) \to \frac{\partial F_2}{\partial x_1}(u)\partial_j u_1$$

a.e. and

$$\left|\frac{\partial F_2}{\partial x_1}(u_m)\partial_j u_{m,1}\right| \leqq C|u_m|^{p-1}M(\partial_j u_1) \leqq C[M(|u|+|\partial u|)]^p.$$

It then follows from (10) that $[M(|u|+|\partial u|)]^p \in L^1(B(0, R))$ and (12) follows from an application of Lebesgue's theorem on dominated convergence as above. Hence (9) is proved and it follows that

(13) $$|\partial(F_2(u))| \leqq C|u|^{p-1}|\partial u|.$$

Then define α by $2/(p+1)+1/\alpha=1$ so that $\alpha=(p+1)/(p-1)$. Hölder's inequality yields

$$\int_{\mathbf{R}^n} |\partial(F_2(u))|^2\, dx \leqq C\int_{\mathbf{R}^n} |u|^{2p-2}|\partial u|^2\, dx$$

$$\leqq C\Big(\int_{\mathbf{R}^n} |u|^{(2p-2)\alpha}\, dx\Big)^{1/\alpha}\Big(\int_{\mathbf{R}^n} |\partial u|^{p+1}\, dx\Big)^{2/(p+1)}.$$

Now $(2p-2)\alpha=2(p+1)$ and it follows that

(14) $$\|\partial(F_2(u))\|_2 \leqq C\|u\|_{2p+2}^{p-1}\|\partial u\|_{p+1},$$

where the norms are taken over $\mathbf{R}^n$ and we have written u instead of $u(t)$. It follows from (i) in the Lemma that

(15) $$\|u\|_{2p+2} \leqq C\|u\|_{L_1^{p+1}}$$

if $1/(2p+2) \geqq 1/(p+1)-1/n$, which is equivalent to

(16) $$p \geqq \frac{n}{2}-1.$$

Now assume $3 \leqq n \leqq 5$. Then (16) holds for $n=3$ and 4 and we may also assume that it holds for $n=5$ by increasing p (since $5/2-1<p_1$). A combination of (14) and (15) yields

(17) $$\|\partial(F_2(u))\|_2 \leqq C\|u\|^p_{L_1^{p+1}}.$$

Hence

$$\int_I \|\partial(F_2(u))\|_2 \, dt \leqq C \int_I \|u\|^p_{L_1^{p+1}} \, dt$$

and it follows from (3) and (4) that $\partial(F_2(u)) \in L^1(I; L^2)$ if $p \leqq r$. The last inequality is equivalent to

$$p^2 - \left(1+\frac{4}{n}\right)p - \frac{4}{n} \leqq 0,$$

which is easily seen to hold for $1<p \leqq p_1$. This completes the proof of Theorem 1 in the case $3 \leqq n \leqq 5$.

In the case $n=1$ or 2 we replace (15) by the inequality

(18) $$\|u\|_{2p+2} \leqq C\|u\|_{L_1^2},$$

which holds since $1/(2p+2) \geqq 1/2-1/n$ according to the Lemma. We can then replace (17) with

$$\|\partial(F_2(u))\|_2 \leqq C\|u\|^{p-1}_{L_1^2} \|u\|_{L_1^{p+1}}$$

and it follows from (3) and (4) that $\partial(F_2(u)) \in L^1(I; L^2)$.

It remains to study the case $n \geqq 6$. Because of (8) and (6) $F_2(u) \in L^1(I; H_{1-\varepsilon})$, where $\varepsilon=\varepsilon(p)=n(p-1)/2(p+1)$. According to (7) it then follows that

$$\varphi u \in L^2(I; H_{3/2-\varepsilon})$$

if $1<3/2-\varepsilon \leqq 3/2$ i.e. $0 \leqq \varepsilon < 1/2$ and this holds for $p<(n+1)/(n-1)$. It is easy to see that $3/2-\varepsilon(p)=\delta(p)$ and the proof of Theorem 1 is complete.

Proof of Theorem 2. We first assume that $1 \leqq n \leqq 5$ and $2<p<\infty$ for $n=1,2$ and $2<p<(n+2)/(n-2)$ for $3 \leqq n \leqq 5$. We set

$$Gv(t) = \int_0^t e^{i(t-s)\Delta} v(s) \, ds.$$

It then follows from Lemmas 1.2 and 2.1 in [3] that G is a bounded mapping from W' to $\overline{W} \subset W$ and that

(19) $$\|Gv\|_W \leqq C\|v\|_{W'}.$$

with C independent of T. We shall then prove that F maps W into W'. Therefore assume that $u\in W$ i.e.

(20) $$u,\ \partial u,\ \partial^2 u\in L^{2,\infty}\cap L^{p+1,r}.$$

It follows from Lemma 2.2 in [3] that F maps Y into Y' and

(21) $$\|F(v)\|_{Y'}\leqq C(T+T^{1-\alpha}\|v\|_Y^{p-1})\|v\|_Y,$$

where $\alpha=n(1/2-1/(p+1))$ so that $0<\alpha<1$. Thus

(22) $$\|F(u)\|_{X'}+\|\partial(F(u))\|_{X'}\leqq C(T+T^{1-\alpha}\|u\|_W^{p-1})\|u\|_W,$$

It remains to study $\|\partial^2(F(u))\|_{X'}$.

Defining u_1, u_2 and u_m as above we shall prove that

(23) $$\partial_i\partial_j(F(u))=\frac{\partial F}{\partial x_1}(u)\partial_i\partial_j u_1+\left(\frac{\partial^2 F}{\partial x_1^2}(u)\partial_i u_1+\frac{\partial^2 F}{\partial x_1\partial x_2}(u)\partial_i u_2\right)\partial_j u_1$$
$$+\frac{\partial F}{\partial x_2}(u)\partial_i\partial_j u_2+\left(\frac{\partial^2 F}{\partial x_1\partial x_2}(u)\partial_i u_1+\frac{\partial^2 F}{\partial x_2^2}(u)\partial_i u_2\right)\partial_j u_2.$$

This follows from the chain rule if u is smooth and the general case follows from an approximation argument of the type which led to (9). In fact, it is not hard to see that for instance

$$F(u_m(t))\to F(u(t)),\quad m\to\infty,$$

$$\frac{\partial F}{\partial x_1}(u_m(t))\partial_i\partial_j u_{m,1}(t)\to\frac{\partial F}{\partial x_1}(u(t))\partial_i\partial_j u_1(t),\quad m\to\infty,$$

and

$$\frac{\partial^2 F}{\partial x_1^2}(u_m(t))\partial_i\partial_{m,1}(t)\partial_j u_{m,1}(t)\to\frac{\partial^2 F}{\partial x_1^2}(u(t))\partial_i u_1(t)\partial_j u_1(t),\quad m\to\infty,$$

in the sense of distributions in $\mathbf{R}^n$ for a.e. t. This can be proved using Lebesgue's theorem on dominated convergence and the fact that

$$u(t),\ \partial u(t),\ \partial^2 u(t)\in L^2\cap L^{p+1}$$

for a.e. t. Similar convergence results hold for the other terms on the right-hand side of (23). We omit the details.

From (23) we obtain

(24) $$|\partial^2(F(u))|\leqq C(1+|u|^{p-1})|\partial^2 u|+C(1+|u|^{p-2})|\partial u|^2$$
$$=C|\partial^2 u|+C|u|^{p-1}|\partial^2 u|+C|\partial u|^2+C|u|^{p-2}|\partial u|^2=A_1+A_2+A_3+A_4.$$

We have

(25) $$\|A_1\|_{L^{2,1}}=C\int_0^T\left(\int_{\mathbf{R}^n}|\partial^2 u|^2\,dx\right)^{1/2}dt\leqq CT\|\partial^2 u\|_{L^{2,\infty}}\leqq CT\|u\|_W.$$

Using Hölder's inequality we also obtain

$$\int_{\mathbf{R}^n} |A_2|^{1+1/p}\,dx = C\Big(\int_{\mathbf{R}^n} |u|^{(p-1)(p+1)/p} |\partial^2 u|^{1+1/p}\,dx\Big)^{1/p}$$

$$\leqq C\Big(\int_{\mathbf{R}^n} |u|^{p+1}\,dx\Big)^{(p-1)/p}\Big(\int_{\mathbf{R}^n} |\partial^2 u|^{p+1}\,dx\Big)^{1/p} = C\,\|u\|_{p+1}^{(p+1)(p-1)/p}\|\partial^2 u\|_{p+1}^{(p+1)/p},$$

where we have written A_2 and u instead of $A_2(t)$ and $u(t)$. We have $1/(p+1) > 1/2-1/n$ and it follows that

$$\|u\|_{p+1} \leqq C\|u\|_{L_1^2}$$

and

$$\|A_2\|_{1+1/p} \leqq C\,\|u\|_{p+1}^{p-1}\|\partial^2 u\|_{p+1} \leqq C\,\|u\|_{L_1^2}^{p-1}\,\|\partial^2 u\|_{p+1}.$$

Invoking Hölder's inequality we obtain

$$\int_I \|A_2\|_{1+1/p}^{r'}\,dt \leqq C\big(\operatorname{ess\,sup}_I \|u\|_{L_1^2}^{p-1}\big)^{r'}\int_I \|\partial^2 u\|_{p+1}^{r'}\,dt$$

$$\leqq C\|u\|_W^{(p-1)r'}\Big(\int_I \|\partial^2 u\|_{p+1}^{r}\,dt\Big)^{r'/r} T^{1/q},$$

where q is defined by $r'/r+1/q=1$ so that $q=(r-1)/(r-2)$. Hence

(26) $$\|A_2\|_{L^{1+1/p,r'}} \leqq C\|u\|_W^{p-1}\|\partial^2 u\|_{L^{p+1,r}}T^{1/qr'} \leqq CT^{1-2/r}\|u\|_W^p.$$

To estimate A_3 we observe that

$$\|A_3\|_2 = C\|\partial u\|_4^2.$$

Then first assume $p+1<4$. According to the Lemma we have

$$\|\partial u\|_4 \leqq C\|\partial u\|_{L_1^{p+1}}$$

if $1/4 \geqq 1/(p+1)-1/n$ i.e. $4n \leqq (p+1)(n+4)$. However, this inequality holds since $p>2$ and $n \leqq 5$. Hence

$$\|\partial u\|_4 \leqq C(\|\partial u\|_{L_1^{p+1}} + \|\partial u\|_2)$$

and this inequality obviously holds also for $p+1>4$. Thus

(27) $$\|A_3\|_2 \leqq C(\|u\|_{L_2^{p+1}}^2 + \|\partial u\|_2^2)$$

and

(28) $$\|A_3\|_{L^{2,1}} \leqq C\int_I \|u\|_{L_2^{p+1}}^2\,dt + C\int_I \|\partial u\|_2^2\,dt$$

$$\leqq C\Big(\int_I \|u\|_{L_2^{p+1}}^r\,dt\Big)^{2/r}T^\gamma + CT\|u\|_W^2 \leqq CT^\gamma\|u\|_W^2 + CT\|u\|_W^2,$$

where $\gamma = 1-2/r$.

We then have

$$A_4 \leqq Cu_0^p$$

where $u_0=|u|+|\partial u|$. It is clear that

(29) $$\|u_0\|_{L^{p+1,\infty}} \leqq C\|u\|_W$$

since $L_1^2\subset L^{p+1}$. We have

$$\int_{\mathbf{R}^n} A_4^{1+1/p}\,dx \leqq C\int_{\mathbf{R}^n} u_0^{p+1}\,dx$$

and

$$\|A_4\|_{1+1/p} \leqq C\|u_0\|_{p+1}^p$$

and it follows that

(30) $$\|A_4\|_{L^{1+1/p,r'}} \leqq C\Big(\int_I \|u_0\|_{p+1}^{pr'}\,dt\Big)^{1/r'} \leqq C\|u\|_W^p T^{1/r'}.$$

Combining (22), (25), (26), (28) and (30) we obtain

$$\|F(u)\|_{W'} \leqq CT\|u\|_W + CT^{1-\alpha}\|u\|_W^p + CT^{\gamma}\|u\|_W^p + CT^{\gamma}\|u\|_W^2 + CT\|u\|_W^2 + CT^{1-1/r}\|u\|_W^p.$$

It follows that there exists a number β, $0<\beta<1$, such that

(31) $$\|F(u)\|_{W'} \leqq CT^{\beta}(\|u\|_W+\|u\|_W^p)$$

for $0<T<1$.

We introduce an operator G_0 by setting

$$G_0 f(t) = e^{it\Delta}f.$$

It then follows from Lemma 2.1 in [3] that G_0 maps H_2 into $\overline{W}$ and

(32) $$\|G_0 f\|_W \leqq C\|f\|_{H_2}$$

where C is independent of T.

Now fix $f\in H_2$ and set

(33) $$\Phi(v) = G_0 f - iGF(v), \quad v\in W.$$

Combining (19) and (31) we obtain

(34) $$\|GF(v)\|_W \leqq C\|F(v)\|_{W'} \leqq CT^{\beta}(\|v\|_W+\|v\|_W^p).$$

Then set $B_R(W)=\{v\in W;\ \|v\|_W\leqq R\}$. We choose $R>1$ and $v\in B_R(W)$ and then have

$$\|\Phi(v)\|_W \leqq C\|f\|_{H_2}+CT^{\beta}(\|v\|_W+\|v\|_W^p) \leqq C\|f\|_{H_2}+CT^{\beta}R^p.$$

We choose $R>C\|f\|_{H_2}$ and then T so small that

$$C\|f\|_{H_2}+CT^{\beta}R^p < R.$$

It is then clear that Φ maps $B_R(W)$ into $B_R(W)$. According to [3], p. 120, we also have

$$\|GF(v)-GF(w)\|_X \leqq C(T+T^{1-\alpha}R^{p-1})\|v-w\|_X \leqq d\|v-w\|_X,$$

where $0<d<1$, if v and $w\in B_R(W)$ and T is small. It is not hard to prove that $B_R(W)$ with the X-metric is a complete metric space and we have proved that Φ is a contraction on this space. The contraction theorem then implies that Φ has a fixed point $u\in W$ and we have $u=\Phi(u)\in \overline{W}$. It follows from Lemma 1.1 in [3] that u is a solution to the Schrödinger equation (2) with $u(0)=f$.

We have to prove that $\varphi u\in L^2(I; H_{5/2})$ and arguing as in the proof of Theorem 1 we see that it is sufficient to prove that $F(u)\in L^1(I; H_2)$.

The argument in the proof of Theorem 1 shows that $F(u)$ and $\partial(F_1(u))\in L^{2,1}$ and to study $\partial(F_2(u))$ we shall use (14). According to the Lemma we have

$$\|u\|_{2p+2} \leqq C\|u\|_{L_2^2} \tag{35}$$

if $1/(2p+2)\geqq 1/2-2/n$, which is equivalent to $4p+4\geqq np$. This inequality holds since $n\leqq 5$ and we obtain

$$\left\|\partial(F_2(u))\right\|_2 \leqq C\|u\|_{L_2^2}^{p-1}\|\partial u\|_{p+1}.$$

Invoking the facts that $u\in L^\infty(I; L_2^2)$ and $\partial u\in L^{p+1,r}$ we conclude that $\partial(F_2(u))\in L^{2,1}$.

It remains to prove that $\partial^2(F(u))\in L^{2,1}$. As above we have

$$\left|\partial^2(F(u))\right| \leqq A_1+A_2+A_3+A_4, \tag{36}$$

where

$$A_1 = C|\partial^2 u|,$$

$$A_2 = C|u|^{p-1}|\partial^2 u|,$$

$$A_3 = C|\partial u|^2,$$

and

$$A_4 = C|u|^{p-2}|\partial u|^2.$$

It is clear from (25) and (28) that A_1 and A_3 belong to $L^{2,1}$. We have

$$\int_{\mathbf{R}^n} |A_2|^2\,dx = C\int_{\mathbf{R}^n} |u|^{2p-2}|\partial^2 u|^2\,dx$$

and the argument which gave (14) now gives

$$\|A_2\|_2 \leqq C\|u\|_{2p+2}^{p-1}\|\partial^2 u\|_{p+1}.$$

Invoking (35) we obtain

$$\|A_2\|_2 \leqq C\|u\|_{L_2^2}^{p-1}\|\partial^2 u\|_{p+1}$$

and using the facts that $u\in L^\infty(I; L_2^2)$ and $\partial^2 u\in L^{p+1,r}$ we conclude that $A_2\in L^{2,1}$.

It remains to study A_4. We may assume that $A_4=C|u|^{p-2}|\partial u|^2\chi$ when χ is the characteristic function of the set where $|u|>1$. We first assume $1\leqq n\leqq 3$. We have $p+1>n$ and the Lemma yields

$$\|\partial u\|_\infty \leqq C\|\partial u\|_{L_1^{p+1}} \leqq C\|u\|_{L_2^{p+1}}.$$

Hence we obtain

$$\int_{\mathbf{R}^n} A_4^2\,dx = C\int_{\mathbf{R}^n} |u|^{2p-4}|\partial u|^4\chi\,dx \leqq C\|u\|_{L_2^{p+1}}^4 \int_{\mathbf{R}^n} |u|^{2p-4}\chi\,dx.$$

The Lemma also implies $\|u\|_\infty \leqq C\|u\|_{L_2^2}$ (since $2>n/2$) and setting $q=\max(2, 2p-4)$ we get

$$\int_{\mathbf{R}^n} |u|^{2p-4}\chi\,dx \leqq \int_{\mathbf{R}^n} |u|^q\,dx \leqq C(\|u\|_2+\|u\|_\infty)^q = C_u.$$

It follows that

$$\|A_4\|_2 \leqq C_u\|u\|_{L_2^{p+1}}^2$$

and since $u\in L^r(I; L_2^{p+1})$ where $r>2$, we obtain $A_4\in L^{2,1}$.

In the case $n=4$ or 5 we set $1/q_1=1/(p+1)-1/n$ and since $p+1<n$ we have $q_1<\infty$. The Lemma yields

$$\|\partial u\|_{q_1} \leqq C\|\partial u\|_{L_1^{p+1}}$$

and

$$q_1 = \frac{n(p+1)}{n-p-1} > \frac{3n}{n-3} > 4$$

since $n\leqq 5$.

Defining s by $4/q_1+1/s=1$ we obtain

$$\int_{\mathbf{R}^n} A_4^2\,dx = C\int_{\mathbf{R}^n} |u|^{2p-4}|\partial u|^4\chi\,dx \leqq C\Big(\int |u|^{(2p-4)s}\chi\,dx\Big)^{1/s}\Big(\int |\partial u|^{q_1}\,dx\Big)^{4/q_1}$$

and

$$\|A_4\|_2 \leqq C\Big(\int |u|^{(2p-4)s}\chi\,dx\Big)^{1/2s}\|u\|_{L_2^{p+1}}^2.$$

We have $u\in L^r(I; L_2^{p+1})$ where $r>2$ and to prove that $A_4\in L^{2,1}$ it is therefore sufficient to prove that

$$\int_{|u|>1} |u|^{(2p-4)s}\,dx \leqq C_u \tag{37}$$

for a.e. $t\in I$. We shall use the fact that $u\in L^\infty(I; L_2^2)$. In the case $n=4$ we have $1/2-2/n=0$ and it follows from the Lemma that $\|u\|_q \leqq C\|u\|_{L_2^2}$ for every q with $2\leqq q<\infty$. Hence (37) follows.

In the case $n=5$ we have $1/2-2/n=1/10$ and it follows that $u\in L^{10,\infty}$. It is therefore sufficient to prove that $(2p-4)s\leqq 10$. We have

$$s = \frac{n(p+1)}{(n+4)p+4-3n} = \frac{5(p+1)}{9p-11}$$

and

$$(2p-4)s = \frac{5(p+1)}{9p-11}(2p-4).$$

It is therefore sufficient to prove that $(p+1)(p-2)\leqq 9p-11$, i.e. $(p-1)(p-9)\leqq 0$, which is true since $2<p<7/3$. Hence Theorem 2 is proved in the case $1 \leqq n \leqq 5$ and $p>2$.

We shall then study the case $1 \leqq n \leqq 5$, $1<p \leqq 2$. Assume that F satisfies the conditions in Theorem 2. Choose p_1 so that $2<p_1<\infty$ for $n=1, 2$ and $2<p_1<(n+2)/(n-2)$ for $3 \leqq n \leqq 5$. Then F satisfies the conditions in Theorem 2 with p replaced by p_1. If f, φ and u are as in Theorem 2 the above argument therefore shows that $\varphi u \in L^2(I; H_{5/2})$.

We shall then study the case $n=6$ or 7. We have $1<p<(n+2)/(n-2)$ and $(n+2)/(n-2) \leqq 2$ so that $p<2$. We shall modify the above argument in the case $1 \leqq n \leqq 5$ and $p>2$.

We replace (24) with

$$|\partial^2(F(u))| \leqq A_1+A_2+A_3, \tag{38}$$

where A_i are as above. We obtain (25) and (26) as above. In the proof of (28) we need the inequality $4n \leqq (p+1)(n+4)$. Since we may replace p with a larger number p_1 as above it is sufficient to have

$$4n < \left(\frac{n+2}{n-2}+1\right)(n+4).$$

However this inequality holds since $n \leqq 7$.

We obtain (31) also in this case and the above argument gives a solution $u \in W$ to the Schrödinger equation (2). To prove that $\varphi u \in L^2(I; H_{5/2})$ we then have to prove that $F(u) \in L^1(I; H_2)$. As above we have $F(u) \in L^{2,1}$ and $\partial(F_1(u)) \in L^{2,1}$ and to estimate $\partial(F_2(u))$ we shall use (14). As in the case $1 \leqq n \leqq 5$ we can then apply the inequality (35) if $4 \geqq (n-4)p$. It is sufficient to have $4 \geqq (n-4)(n+2)/(n-2)$ and this holds for $n=6$. In the case $n=7$ we replace (35) with

$$\|u\|_{2p+2} \leqq C\|u\|_{L_2^{p+1}}, \tag{39}$$

which holds for $1/(2p+2) \geqq 1/(p+1)-2/7$ i.e. $p \geqq 3/4$. We obtain $\|\partial(F_2(u))\|_2 \leqq C\|u\|_{L_2^{p+1}}^p$. Since $u \in L^r(I; L_2^{p+1})$ we conclude that $\partial(F_2(u)) \in L^{2,1}$ if $p \leqq r$. However, in this case we have $p<2$ and $r>2$ and hence $\partial(F_2(u)) \in L^{2,1}$. It remains to prove that $\partial^2(F(u)) \in L^{2,1}$ and we shall use the estimate (38). The inequality (25) can be applied to A_1 and to estimate A_2 we can use (35) as above. In the proof of (35) we need $4 \geqq (n-4)p$ which holds for $n=6$ since $4=(n-4)(n+2)/(n-2)$ in this case.

In the case $n=7$ we replace (35) with (39) and obtain

$$\|A_2\|_2 \leqq C\|u\|_{L_2^{p+1}}^p.$$

It then follows that $A_2 \in L^{2,1}$ as in the above proof that $\partial(F_2(u)) \in L^{2,1}$.

To estimate A_3 we use (28). In the proof of (28) we need the inequality $4n \leqq (p+1)(n+4)$ and it is sufficient to have

$$4n < \left(\frac{n+2}{n-2}+1\right)(n+4).$$

However, this inequality holds for $n=6$ or 7. The proof of Theorem 2 is complete.

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