Traces of pluriharmonic functions on curves

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Abstract. We prove that, if γ is a simple smooth curve in the unit sphere in $\mathbf{C}^n$, the space o pluriharmonic functions in the unit ball, continuous up to the boundary, has a trace of finite cof dimension in the space of all continuous functions on the curve.

1. Introduction

Let B_n be the unit ball in $\mathbf{C}^n$, S its boundary and consider a simple smooth curve, γ in S. It has been known for some time that γ is an interpolating set for the ball algebra if and only if γ is complex tangential (see [2], [3], [5], [6] and [9]). In other words, γ has the property that any continuous function on γ extends to a continuous function on $\bar{B}_n$, holomorphic in B_n, if and only if at each point of γ its tangent vector lies in the complex tangent space of S at that point. In this paper we will treat the corresponding extension problem for pluriharmonic functions. We say that $E \subset S$ is a *set of pluriharmonic interpolation* if any continuous function on E can be extended to a continuous function on $\bar{B}_n$, which is pluriharmonic in B_n. E is said to be a *set of almost pluriharmonic interpolation* if the space of continuous functions on E with this property has finite codimension in the space of all continuous functions. The first paper to treat the pluriharmonic interpolation problem was [1]. There it is proved that γ is a set of almost pluriharmonic interpolation if γ is transversal to the complex structure. This means that at *no* point of γ its tangent vector should lie in the complex tangent plane of S. One instance when this condition is satisfied is when γ is the boundary of a complex variety in $\bar{B}_n$, which intersects S transversally, and the result can perhaps be thought of as a generalization of the solvability of the Dirichlet problem on such varieties. Indeed, the proof in [1] is similar to the way one solves the Dirichlet problem by double

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layer potentials. Thus, one constructs an approximative extension operator, L, which associates to every $\varphi \in C(\gamma)$ a pluriharmonic and continuous function $L(\varphi)$, so that on γ one has

$$L(\varphi) = \varphi + K(\varphi)$$

where the error term K defines a compact operator. This immediately implies the result by Fredholm theory. One should also note that positive codimension actually can occur if e.g. the variety has many singularities, or a complicated topological structure, see [1]. By the aforementioned result on holomorphic interpolation it is natural to conjecture that any smooth curve should be a set of almost pluriharmonic interpolation, since on the set where transversality fails one can even extend holomorphically. In this paper we will prove that this is indeed so.

Theorem 1. *Let γ be a smooth and simple curve in $S=\partial B_n$. Then the space of functions on γ that can be extended to continuous functions on $\bar{B}_n$, which are pluriharmonic in B_n, is a closed subspace of finite codimension in $C(\gamma)$.*

The proof of theorem 1 is along the same lines as in [1], but the points where γ changes from transversal to tangential cause additional problems. In fact, if one uses the same approximate extension operator as in [1] the error term is no longer compact. The main idea of our proof is that we use a different extension operator and combine the technique of [1] with the techniques of holomorphic interpolation.

It is worth mentioning that if γ is polynomially convex, then holomorphic polynomials are dense in $C(\gamma)$ (see the survey [3]). Since our subspace of finite codimension is closed we conclude

Corollary 2. *Suppose that γ is a smooth and simple curve in S which is polynomially convex. Then γ is a set of pluriharmonic interpolation.*

One case when we know that a smooth curve is polynomially convex is when it is an arc. Another case is provided by a result of Forstnerič [4], which says that if a smooth curve in S has a nontrivial hull, then this hull must be an analytic variety with at most a finite number of singularities. Moreover this variety intersects S transversally, so in particular γ is complex transversal in this case.

Corollary 3. *Suppose γ is an arc, or is complex tangential at at least one point. Then γ is a set of pluriharmonic interpolation.*

Recently, J.-P.Rosay has shown us a short direct proof that a curve which is complex-tangential at at least one point is polynomially convex. (See [7].)

An interesting problem which remains is to compute the codimension in case it is positive. Finally, we mention the result in [8], which says that a manifold in S

of dimension at least two can be a set of almost pluriharmonic interpolation only if it is complex-tangential. Thus results like theorem 1 hold only for curves.

We have made no attempt to optimize the regularity assumption on γ (but C^3 is surely enough). The proof below is written out for the case of a closed simple curve, which obviously implies the general case.

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2. Pseudodistances

Our curve γ will always be parametrized by arc length so that $|\dot\gamma|=1$. Since $\operatorname{Re}\gamma\cdot\bar{\dot\gamma}=0$ we can define a realvalued function T by

$$iT(t) = \gamma(t)\cdot\bar{\dot\gamma}(t).$$

Evidently $T(t)=0$ if and only if $\gamma(t)$ is a complex-tangential point on the curve, and in general T can be said to measure the transversality of γ. We have

$$\gamma(t)\cdot\bar{\ddot\gamma}(t) = i\dot T(t)-1,$$

and by Taylor's formula

(1) $$1-\bar\gamma(x)\gamma(t) = 1-\left(\bar\gamma(t)+\bar{\dot\gamma}(t)(x-t)+\tfrac{1}{2}\bar{\ddot\gamma}(t)(x-t)^2+\dots\right)\cdot\gamma(t)$$
$$= -iT(t)(x-t)+\tfrac{1}{2}\left(1-i\dot T(t)\right)(x-t)^2+\dots.$$

This implies

(2) $$|1-\bar\gamma(t)\cdot\gamma(x)| \approx |T(x)|\,|t-x|+|t-x|^2.$$

For $z\in B$ let $s=s(z)$ be such that

$$|1-\bar\gamma(s)\cdot z| = \min|1-\bar\gamma(t)\cdot z| \quad \text{over all } t.$$

Since the triangle inequality is satisfied by the expression $|1-\zeta\cdot z|^{1/2}$ it follows that

(3) $$|1-\bar\gamma(t)\cdot z| \approx |1-\bar\gamma(s)\cdot z|+|1-\bar\gamma(t)\cdot\gamma(s)| \approx |1-\bar\gamma(s)\cdot z|+|T(s)|\,|t-s|+|t-s|^2.$$

3. The kernel

The kernel used in [1] was essentially

$$K_0(t,z) = \frac{1}{\pi}\operatorname{Im}\frac{\gamma(t)\cdot\bar{\ddot\gamma}(t)}{z\cdot\bar\gamma(t)-1}.$$

The proof was based on the fact that when z approaches a point $\gamma(x)$ on the curve the measures $K_0(t, z)dt$ converge weakly to

$$\delta_x + L_0(t, x)\,dt$$

where δ_x is the Dirac measure at x and

$$L_0(t, x) := K_0(t, \gamma(x)) = \frac{1}{\pi} T(t) \frac{\operatorname{Re} \bar{\gamma}(x) \cdot \gamma(t) - 1}{|1 - \gamma(x) \cdot \bar{\gamma}(t)|^2}$$

is a bounded kernel for $x \neq t$. To see that L_0 is bounded one uses that by (1) and (2)

$$|L_0(t, x)| \approx \frac{|T(t)|\,|x-t|^2}{(|T(x)|\,|x-t| + |x-t|^2)^2}$$

which is bounded provided that T is bounded from below. In general however, this is not so, and hence we will modify the definition of the kernel. In order not to loose the delta-mass we will look for a modified kernel of the form

$$K(t, z) = \frac{1}{\pi} \operatorname{Im} \frac{\gamma(t) \cdot \bar{\dot{\gamma}}(t) + (z - \gamma(t)) \cdot \psi(t)}{z \cdot \bar{\gamma}(t) - 1}.$$

In this section we will look for the simplest vector-valued function ψ such that the corresponding

$$L(t, x) := K(t, \gamma(x))$$

is bounded for $t \neq x$. First note that since $|1 - \gamma(x) \cdot \bar{\gamma}(t)|$ dominates $(x-t)^2$ we can disregard all quadratic terms in the numerator. Denoting $A(t) = \dot{\gamma}(t) \cdot \psi(t)$ it is enough to estimate

$$\operatorname{Im} \frac{iT(t) + (x-t)A(t)}{\gamma(x) \cdot \bar{\gamma}(t) - 1} = \operatorname{Im} \frac{[iT(t) + (x-t)A(t)]\,[\bar{\gamma}(x) \cdot \gamma(t) - 1]}{|1 - \gamma(x) \cdot \gamma(t)|^2}.$$

Using the Taylor development (1), we find that the numerator above is

$$-T(t) \tfrac{1}{2} (x-t)^2 + T(t) \operatorname{Re} A(t)(x-t)^2$$

$$-\tfrac{1}{2} (x-t)^3 \operatorname{Im} [A(t)(1 - i\dot{T}(t))] + T(t) O((x-t)^3) + O((x-t)^4).$$

Here, the last two terms, when divided by $|1 - \gamma(x) \cdot \gamma(t)|^2$, are bounded because of (2). Thus, we need only choose A in such a way that the other terms vanish, which means that we require

$$\operatorname{Re} A(t) = \tfrac{1}{2}$$

and

$$\operatorname{Im} [A(t)(1 - i\dot{T}(t))] = 0.$$

This gives $A(t)=\frac{1}{2}(1+i\dot{T}(t))$, which will hold e.g. if we take

$$\psi(t)=\tfrac{1}{2}(1+i\dot{T}(t))\bar{\dot{\gamma}}(t).$$

Writing out the resulting kernel we obtain

$$K(t,z)=\frac{1}{2\pi}\operatorname{Im}\frac{(1+i\dot{T}(t))z\cdot\dot{\gamma}(t)+(1-i\dot{T}(t))\gamma(t)\cdot\bar{\dot{\gamma}}(t)}{z\cdot\bar{\gamma}(t)-1}. \tag{4}$$

Thus by construction we have:

Lemma 1. *If K is defined by* (4), *K is bounded for $z=\gamma(x)$ when $x\neq t$.*

When the curve γ is a slice, $\gamma(t)=e^{it}\zeta$, one gets

$$K=\frac{1}{2\pi}\frac{1-|z\bar{\gamma}(t)|^2}{|z\bar{\gamma}(t)-1|^2}$$

i.e. the usual Poisson kernel for the slice.

In the next section we will need:

Lemma 2.

$$\int|K(t,z)|\,dt\leqq C,\quad z\in B.$$

Proof. Let $\gamma(s)$ be the point on the curve closest to z as in Section 2.

$$2\pi K(t,z)=\operatorname{Im}\frac{\gamma(t)\cdot\bar{\dot{\gamma}}(t)+(z-\gamma(t))\cdot\psi(t)}{z\cdot\bar{\gamma}(t)-1}$$

$$=\operatorname{Im}\frac{(\gamma(t)\cdot\bar{\dot{\gamma}}(t)+(\gamma(s)-\gamma(t))\cdot\psi(t))(\bar{\gamma}(s)\cdot\gamma(t)-1)}{|1-\bar{\gamma}(t)\cdot z|^2}$$

$$+\operatorname{Im}\frac{(\gamma(t)\bar{\dot{\gamma}}(t)+(\gamma(s)-\gamma(t))\psi(t))(\bar{z}-\bar{\gamma}(s))\cdot\gamma(t)}{|1-\bar{\gamma}(t)\cdot z|^2}+\operatorname{Im}\frac{(z-\gamma(s))\cdot\psi(t)}{z\cdot\bar{\gamma}(t)-1}$$

$$=:F_1+F_2+F_3.$$

Using $|1-\bar{\gamma}(t)\cdot\gamma(s)|\lesssim|1-\bar{\gamma}(t)\cdot z|$, we see that F_1 is bounded by a constant times $K(t,\gamma(s))$, hence uniformly bounded by Lemma 1.

Let $\varrho=d(z,\gamma)=|1-z\cdot\bar{\gamma}(s)|$ and $m=|T(s)|$. Then

$$|z-\gamma(s)|^2\lesssim\operatorname{Re}(1-z\cdot\bar{\gamma}(s))\lesssim\varrho$$

and

$$|(\bar{z}-\bar{\gamma}(s))\cdot\gamma(t)|\leqq\varrho+|z-\gamma(s)|\,|\gamma(t)-\gamma(s)|\lesssim\varrho+\varrho^{1/2}|t-s|.$$

This gives

$$|F_2| \lesssim \frac{\big(|T(t)|+|s-t|\big)(\varrho+\varrho^{1/2}|t-s|)}{|1-\gamma(t)\cdot\bar{z}|^2} \lesssim \frac{\big(|T(s)|+|s-t|\big)(\varrho+\varrho^{1/2}|t-s|)}{(\varrho+|T(s)|\,|s-t|+|s-t|^2)^2}$$

$$\lesssim \frac{(m+|s-t|)(\varrho+\varrho^{1/2}|t-s|)}{(\varrho+m|s-t|+|s-t|^2)^2}$$

if we use (3). Finally

$$|F_3| \lesssim \frac{|z-\gamma(s)|}{|1-z\cdot\bar{\gamma}(t)|} \lesssim \frac{\varrho^{1/2}}{\varrho+m|s-t|+|s-t|^2}.$$

Putting $x=s-t$ we see that it suffices to estimate the integrals

$$\int_0^\infty \frac{\varrho^{1/2}\,dx}{\varrho+x^2} \quad\text{and}\quad \int_0^\infty \frac{(\varrho+\varrho^{1/2}x)(m+x)\,dx}{(\varrho+mx+x^2)^2}.$$

For the first one we need only substitute $x=\varrho^{1/2}u$. The second one is estimated by

$$\int_0^\infty \frac{\varrho^{1/2}(m+x)\,dx}{(\varrho+mx+x^2)^{3/2}},$$

which, with the change of variable $u=mx+x^2$, is like

$$\int_0^\infty \frac{\varrho^{1/2}\,du}{(\varrho+u)^{3/2}} = 2.$$

This proves the lemma.

4. Almost pluriharmonic interpolation

In the previous section we have seen that the limit of $K(t, z)$ as z approaches a point $\gamma(x)$ on the curve is bounded as long as $x\neq t$. Now we shall see that the contribution from $x=t$ is a Dirac measure if $\gamma(x)$ is a transverse point on the curve.

Given a continuous function ϕ on the curve, we define a pluriharmonic function

$$K\phi(z) = \int \phi(t)\,K(t, z)\,dt, \quad z\in B,$$

which clearly extends continuously to $\bar{B}$ off the curve. We shall now study the behaviour of $K\phi(z)$ as z approaches a point $\gamma(x)$ on the curve.

Lemma 3. *If* $T(x)\neq 0$, *then with* $s(x)=-\operatorname{sign} T(x)$ *and* $L(t, x)=K(t, \gamma(x))$

$$\lim_{z\to\gamma(x),\, z\in B} K\phi(z) = s(x)\,\phi(x)+\int \phi(t)\,L(t, x)\,dt.$$

Proof. With a fixed δ, we estimate the difference between the right-hand side and the left-hand side by

$$\int_{|x-t|\leqq\delta}|\phi(t)-\phi(x)|\,|K(t,z)|\,dt+|\phi(x)|\left|\int_{|x-t|\leqq\delta}K(t,z)\,dt-s(x)\right|$$
$$+\int_{|x-t|>\delta}|\phi(t)|\,|K(t,z)-L(t,x)|\,dt+\int_{|x-t|\leqq\delta}|\phi(t)|\,|L(t,x)|\,dt.$$

The principal part of $K(t,z)$ is $-\frac{1}{\pi}\operatorname{Im}\frac{z\cdot\bar{\dot{\gamma}}(t)}{1-z\cdot\bar{\gamma}(t)}$, i.e.

$$K(t,z)=-\frac{1}{\pi}\operatorname{Im}\frac{z\cdot\bar{\dot{\gamma}}(t)}{1-z\cdot\bar{\gamma}(t)}+O\left(\frac{|z-\gamma(t)|}{|1-z\cdot\bar{\gamma}(t)|}\right).$$

The last term is $O(|1-z\cdot\bar{\gamma}(t)|^{-1/2})$ which is bounded by $C(x)|s-t|^{-1/2}$ for z close to $\gamma(x)$ and $s=s(z)$. The integral of this term over $|x-t|\leqq\delta$ is maximal when $x=s$ and is thus dominated by $C(x)\delta^{1/2}$ for z close to $\gamma(x)$. Using lemma 2 for the first integral above, lemma 1 for the last one and dominated convergence in the third one, we get with $\omega(\phi,\delta)$ the modulus of continuity of ϕ

$$\limsup\left|K\phi(z)-\left(s(x)\phi(x)+\int\phi(t)L(t,x)\,dt\right)\right|$$
$$\leqq C\omega(\phi,\delta)+C\delta^{1/2}+C\limsup\left|\int_{x-\delta}^{x+\delta}\frac{1}{\pi}\operatorname{Im}\frac{z\cdot\bar{\dot{\gamma}}(t)}{z\cdot\bar{\gamma}(t)-1}\,dt-s(x)\right|+C\delta$$
$$\leqq C\omega(\phi,\delta)+C\delta^{1/2}+C\left(\frac{1}{\pi}\{\arg(1-\gamma(x)\cdot\bar{\gamma}(x+\delta))-\arg(1-\gamma(x)\cdot\bar{\gamma}(x-\delta))\}-s(x)\right).$$

Now we let $\delta\to 0$. Since

$$1-\gamma(x)\cdot\bar{\gamma}(x+\delta)=-iT(x)\delta+O(\delta^2)$$

and

$$1-\gamma(x)\cdot\bar{\gamma}(x-\delta)=iT(x)\delta+O(\delta^2)$$

the difference between the arguments is $-\pi$ if $0<T(x)$ and π if $T(x)<0$, so the lemma follows.

At points where $T(x)=0$, $K\phi(z)$ generally fails to have a limit, but the same proof shows:

Lemma 4. *If* $T(x)=0$ *and* $\phi(x)=0$ *then*

$$\lim_{z\to\gamma(x),\,z\in B}K\phi(z)=\int\phi(t)L(t,x)\,dt.$$

As a consequence of lemmas 3 and 4 it follows that if ϕ is continuous and $\phi=0$ whenever $T=0$ then the pluriharmonic function

$$P[\phi](z):=\int s(t)\phi(t)K(t,z)\,dt$$

is continuous on the closed ball and its value at the point $\gamma(x)$ is

$$\phi(x)+\int s(t)\phi(t)L(t,x)\,dt.$$

5. Holomorphic interpolation at the complex-tangential points

The set $E=\{\gamma(x);\ T(x)=0\}$ is known to be an interpolation set for the ball algebra, because it is locally contained in complex-tangential curves (see [10], p. 230). Actually, using the methods of [5], it is quite easy to show this directly, and moreover exhibit a linear interpolation operator. If $\frac{1}{2}<q<1$ and

$$h_q(z) = \int \frac{dt}{(1-\bar{\gamma}(t)\cdot z)^q}$$

then an elementary calculation using (3) shows that

$$|h_q(z)| \approx \int \frac{dt}{|1-\bar{\gamma}(t)\cdot z|^q} \approx \left(\varrho+T^2(s)\right)^{(1/2)-q} \tag{5}$$

where, as in section 3, $\gamma(s)$ is the point on γ closest to z and $\varrho=|1-\bar{\gamma}(s)\cdot z|$. We have also used the fact that since $|\arg(1-\bar{\gamma}(t)\cdot z)^{-q}|\leqq\frac{\pi q}{2}$, we can put the absolute value sign inside the integral. Now we let

$$M(t, z) = \frac{1}{h_q(z)}\,\frac{1}{(1-\bar{\gamma}(t)\cdot z)^q}.$$

Then $M(t, z)$ is holomorphic in z and satisfies

$$\int M(t, z)\,dt = 1 \quad\text{and}\quad \int |M(t, z)|\,dt = O(1). \tag{6}$$

The kernel M defines an operator for holomorphic interpolation in the following way: if $\phi\in C(\gamma)$ we consider ϕ as a function of t and put

$$I\phi(z) = \int \phi(t)\,M(t, z)\,dt.$$

If now z approaches a point $\gamma(x)$ on the curve where $T(x)=0$, then $\lim |h_q(z)|=\infty$ by (5). Therefore

$$\lim_{z\to\gamma(x)}\ \sup_{|t-x|\geqq\delta} |M(t, z)| = 0.$$

So $M(t, z)$ has all the properties of an approximate identity, and we get

$$\lim_{z\to\gamma(x)} I\phi(z) = \phi(x).$$

If on the other hand z approaches a point $\gamma(x)$ where $T(x)\neq 0$, then (5) shows that $\lim_{z\to\gamma(x)} h_q(z)=h_q(\gamma(x))\neq\infty$, so by dominated convergence

$$\lim_{z\to\gamma(x)} I\phi(z) = \frac{1}{h_q(\gamma(x))}\int \phi(t)\,M(t, \gamma(x))\,dt.$$

In particular, $I\phi$ has limits at any point in S, hence has a continuous extension to $\bar{B}_n$. Moreover, $I\phi$ is holomorphic in B_n and so defines a linear operator for holomorphic interpolation on E.

6. Conclusion of the proof of the main result

Given ϕ continuous we apply P to the function $s(\cdot)(\phi-I\phi)$, where $s=\text{sign}\, T$. Since this function vanishes on E, the result is a pluriharmonic function, continuous on $\bar{B}$. Its values on the curve are

$$\phi(x)-I\phi(x)+\int s(t)\big(\phi(t)-I\phi(t)\big)L(t,x)\,dt.$$

Thus the pluriharmonic function $P[\phi-I\phi]+I\phi$ has boundary values

$$\phi(x)+\int s(t)\big(\phi(t)-I\phi(t)\big)L(t,x)\,dt.$$

Now note that the last integral defines a compact linear operator on $C(\gamma)$. Namely, the operator that sends ϕ to $s(\phi-I\phi)$ is continuous on $C(\gamma)$, and integration against L defines a compact operator since L is bounded. Call this operator F. By Fredholm theory

$$R:=\text{range}\,(I+F)$$

is a closed subspace of finite codimension in $C(\gamma)$. Let Q be the space of all functions in $C(\gamma)$ that can be extended to functions in $C(\bar{B}_n)$ which are pluriharmonic in B_n. By the construction we have made R is a subspace of Q, so Q must also be of finite codimension. Moreover Q/R is closed in $C(\gamma)/R$ since the latter space is of finite dimension. Since R is closed the projection from $C(\gamma)$ to its quotient with R is continuous. Hence Q must also be closed in $C(\gamma)$. This proves theorem 1.

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