Degenerations of minimal ruled surfaces

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This paper studies degenerations of minimal ruled surfaces. With an additional assumption satisfied, after base change, by projective degenerations, we find that the minimal models of these degenerations are non-singular and particularly elementary: $\mathbf{P}^1$-bundles over minimal models of degenerations of the base curves.

Let $\pi: X \to \Delta$ be a proper flat holomorphic map from a non-singular complex threefold X onto Δ, the unit disk in $\mathbf{C}$, such that $X_t := \pi^{-1}(t)$ is a non-singular minimal ruled surface for $t \neq 0$ and X_0 is a union of non-singular surfaces meeting normally. In [11] p. 83, Persson finds an example of such a degeneration of minimal ruled surfaces for which X_0 is not algebraic, in fact, X_0 is a union of two Hopf surfaces, while the general fiber, X_t, is a minimal ruled elliptic surface. Of particular interest in this example is the fact that for S_t a section of X_t under its ruling, there is an analytic surface $S' \subset X \setminus X_0$ for which $S' \cap X_t = S_t$; however, S' does not extend to an analytic surface S over all of Δ.

In light of the above example, in this paper by *degeneration of minimal ruled surfaces*, we mean a map π as above for which $\pi = \pi_S \circ f$ where $f: X \to S$ is a holomorphic map, S is a non-singular complex surface, $\pi_S: S \to \Delta$ is a degeneration of curves for which $f|_{X_t}: X_t \to S_t$ gives X_t its structure of minimal ruled surface (see 1.1 for details). These degenerations are ruled degenerations, that is, X_t is being degenerated together with its structure as ruled surface. The central result of this paper is that, for X as above, X has a smooth minimal model which is a $\mathbf{P}^1$-fibration over S. The condition that such an S exists is both essential and quite natural: if $X \to \Delta$ is a degeneration of surfaces in the previous sense which is bimeromorphic over Δ to a projective degeneration, then, after a base-change and shrinking Δ, there exists an S as above (Persson [11], pp. 60 and 77). On the other hand, there are projective degenerations of minimal ruled surfaces for which no such S exist. For example, let Z be the blow-up of $\mathbf{P}^2$ at the 54 nodes of a nodal elliptic curve of degree 12. For C the resulting smooth elliptic curve, there is a projective conic bundle $\pi: X \to Z$ having C as its degeneration divisor and so X is not rational (see Sarkisov [13], p. 389 for details). On Z, consider $|C+C'|$ where C' is the pull-back to Z of a sextic curve on $\mathbf{P}^2$. Now $\dim|C+C'|\geqq 28$ (by Riemann—Roch, $h^0(\mathcal{O}_Z(C+C'))=28+h^1(\mathcal{O}_Z(C+C'))$ while $h^1(\mathcal{O}_Z(C+C'))\geqq h^1(\mathcal{O}_C(C+C'))$; on the other hand, $C+C'\equiv 2(K_Z+C)$, so $(C+C')|_C\equiv 2K_C\equiv 0$ and therefore $h^1(\mathcal{O}_C(C+C'))=h^1(\mathcal{O}_C)=1$), whereas $\dim|C'|=27$. Thus C is not a fixed component of $|C+C'|$ (in fact, $|C+C'|$ is base-point-free) and so the general element Σ is non-singular. For such a Σ, $\pi^{-1}(\Sigma)$ is a minimal ruled surface over Σ. Blowing up the base points of a general pencil in $|C+C'|$ and taking the fiber product with X gives a conic bundle $\tilde{\pi}\colon \tilde{Y}\to\tilde{Z}$, for $\tilde{Z}$ the blow-up of Z. $\tilde{Y}$ is a blow-up of X and so is still projective and not rational. Now $\tilde{Y}$ is also the total space of a degeneration of surfaces for which the general fiber is a minimal ruled surface. Were $\tilde{\pi}\colon \tilde{Y}\to\tilde{Z}$ to factor through a surface S as in our definition of degeneration of minimal ruled surfaces, S would be birational to $\tilde{Z}$ and so S would be rational. Thus $\tilde{Y}$ would be rational, contrary to construction.

Given a smooth threefold fibered over a curve by surfaces generically of a fixed type, an obvious question is when can the threefold be birationally modified, preserving the fibration, to a new smooth threefold having all the fibers of the given fixed type? This type of trivialization is known to exist if the threefold is fibered by $\mathbf{P}^2$'s or by rational surfaces if all local monodromies are trivial (Crauder [4]). The main theorem of this paper (theorem 1.3) is that any degeneration of minimal ruled surfaces (as defined above) may be birationally modified to $\tilde{X}$, a non-singular $\mathbf{P}^1$-bundle over S; moreover, the birational map from X to $\tilde{X}$ is an isomorphism off X_0 and $\tilde{X}_0$ (in fact, it is an isomorphism off the set of fibers of f which are not smooth irreducible reduced rational curves). In addition, if $\bar{S}$ is a non-singular minimal model of S over Δ, then $\tilde{X}$ may be further modified to $\bar{X}$, a non-singular $\mathbf{P}^1$-bundle over $\bar{S}$. In particular, note that the components of $\tilde{X}_0$ and $\bar{X}_0$ are all minimal ruled surfaces over the corresponding components of S_0 and $\bar{S}_0$. This gives a very simple standardized degeneration form for minimal ruled surfaces which is particularly nice: $\bar{X}$ is smooth, the number of components of $\bar{X}_0$ is the same as the number of components $\bar{S}_0$ and each component is a minimal ruled surface whose multiplicity and normal bundle information can be read off that for the corresponding component of $\bar{S}_0$. Thus, for example, if $S\to\Delta$ has a relatively minimal model with an irreducible smooth $\bar{S}_0$ (e.g. if the generic fiber is a rational curve), then the original fibration has a trivialization, since then all fibers of $\bar{X}\to\Delta$ are minimal ruled surfaces. One way to think of this result is that minimal ruled plus having a section S ensures that the monodromy of X is simply the pull-back of that of S, so if the monodromy of S is trivial, so is that of X and, as in the case for $\mathbf{P}^2$ and rational surfaces with trivial monodromy, we therefore expect X to be birational to a trivial degeneration.

Degenerations of ruled surfaces have also been studied by Persson, Kawamata and Bădescu. In [11], pp. 76—77, Persson showed that for $X \to \Delta$ having a section S as above, X is birational to $S \times \mathbf{P}^1$; however, the birational map is not an isomorphism off X_0 and so modifies already minimal fibers of f. Kawamata's results on semi-stable projective degenerations (theorem 10.1′, p. 145, [8]) apply to degenerations of any type of surface. For degenerations of minimal ruled surfaces, those results *a priori* yield a minimal model with terminal singularities which is a conic bundle over S. In [1], Bădescu considers projective degenerations for which X_0 is normal and so irreducible and X_t is either irregular ruled or rational with $b_2(X_t) \leqq 10$. The main result of [1] is to determine the singularities and plurigenera of X_0.

Our proof of theorem 1.3 has three main steps. In the first step, X is modified by smooth blow-ups and blow-downs over S to f': $X' \to S$ which is a $\mathbf{P}^1$-fibration off a finite set of points in S. In the second step, we apply flattening and desingularization theorems to f' to obtain a conic bundle g: $Y \to Z$, where Z is obtained from S by smooth blow-ups. By knowing that the degeneration divisor of the conic bundle lies in an exceptional curve on Z, we then show that g is in fact a $\mathbf{P}^1$-fibration. Finally, for the third step, we find a procedure for birationally modifying Y to blow Z back down to S without losing the $\mathbf{P}^1$-bundle structure. We thank R. Miranda for helpful discussions regarding this third step.

0. Definitions and basic facts

In this section, we recall facts about blow-ups and blow-downs in complex three-manifolds, various types of conic bundle structures and some miscellaneous facts about surfaces.

0.1. Definition. By *blow-up,* we mean the blow-up of a non-singular complex subspace of a non-singular complex space. By *blow-down,* we mean the inverse map to a blow-up (so, in particular, a blow-down is a holomorphic map).

0.2. Ruled surfaces and contractions. A ruled surface, S, in a complex three-manifold may be *smoothly contracted,* that is, S is the exceptional divisor of a blow-down if and only if S is a non-singular minimal ruled surface and $F \cdot S = -1$ for F a fiber on S (Nakano [10], main theorem, p. 484). Following Persson and Pinkham ([12], p. 51), we define S to be *generically contractible* if S is ruled and $F \cdot S = -1$ for F a fiber on S. Thus a ruled surface is smoothly contractible if and only if it is non-singular, minimal ruled and generically contractible.

0.3. Minimizing lemma. *Let $Y \subset X$ be a normal crossings divisor in a complex three-manifold (that is, Y is a union of non-singular surfaces meeting normally in X)*

and let $V \subset Y$ be a non-singular ruled surface. Then there is a birational mep (in fact, a composition of blow-ups and blow-downs) α: $X \dashrightarrow X'$ such that

i) the proper transform of V is non-singular and minimal ruled, and

ii) α is an isomorphism away from the reducible fibers on V.

The total transform of Y under α need not have normal crossings (or even local normal crossings); however, by ii) above, the total transform of Y does have normal crossings away from the reducible fibers of V. For details, see Crauder [3], lemma 1.1, p. 19 or Kulikov [9], p. 199.

0.4. Recall. We note the following facts about surfaces:

1. If f is a proper birational morphism of non-singular surfaces, then each fiber of f is either a single point or a tree of non-singular rational curves.

2. A reducible fiber on a non-singular ruled surface contains at least one exceptional curve.

0.5. Definition. A *conic bundle* is an irreducible three-dimensional complex space X with a morphism f: $X \to S$, for S a non-singular surface, whose generic fiber is a rational curve.

0.6. Notation. For a conic bundle, let $D(f)=\{s \in S \mid f^{-1}(s)$ is not an irreducible non-singular rational curve$\}$.

0.7. Definition. A *regular conic bundle* is a conic bundle for which X is non-singular and f is flat.

0.8. Regular conic bundles.

1. A regular conic bundle may be embedded as a conic over S in the projectivization of a rank three vector bundle over S. Thus our definition 0.7 is equivalent to the usual definition of conic bundle when $S=\mathbf{P}^2$ (cf. Beauville [2], p. 320, definition 1.1). The definitions we use are compatible with those of Sarkisov and Zagorskii ([13] and [14]-they allow f to be a rational map, whereas we restrict f to be a morphism; otherwise, our usages are identical), which are convenient for our uses (see 0.11 below).

2. For a regular conic bundle, $D:=D(f)$ is given as the zero-section of a line bundle over S and so is a divisor on S (in fact, $D=\emptyset$ or D is a reduced normal crossings curve on S). In addition, D has a natural 2:1 cover π: $\tilde{D} \to D$ for which π is étale over $D \setminus \operatorname{Sing} D$; moreover, by normalization, there is a smooth double cover π_i': $D_i' \to D_i$, over each irreducible component D_i of D, which is ramified exactly at the double points of D on D_i.

For details, see Sarkisov [13], § 1, and Beauville [2], chapters 0 and 1.

0.9. Definition. A *standard conic bundle* is a conic bundle for which $f^{-1}(G)$ is an irreducible surface on X if G is an irreducible curve on S.

0.10. Remarks.

1. A conic bundle is standard if and only if π_i': $D_i' \to D_i$ is a non-trivial double cover for all components D_i of D, i.e. each D_i' is irreducible.

2. If a conic bundle is not standard, then if D_j' is reducible, so is $f^{-1}(D_j)$- in fact, $f^{-1}(D_j)$ is a union of two non-singular ruled surfaces, ruled over D_j by f, either one of which may be smoothly blown-down (Sarkisov [13], 1.17).

0.11. Theorem. *If f: $X \to S$ is a conic bundle, then there is a standard conic bundle g: $Y \to Z$ such that*

$$\begin{array}{ccc} X & \overset{\beta}{\longleftarrow\cdots} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ S & \underset{\eta}{\longleftarrow} & Z \end{array}$$

where

1. *η is a composition of (smooth) blow-ups,*
2. *β is a composition of (smooth) blow-ups, blow-downs and blow-ups of fibers of f and*
3. *$D(g) \subset \eta^{-1}(D(f))$ and β is an isomorphism off $D(f)$.*

Proof. With the exception of 3, this is proved in Zagorskii [14] as theorem 1, p. 421. As for 3, in [14], Zagorskii first birationally replaces f: $X \to S$ with an embedded conic bundle over S and then changes base using resolution of singularities in order to make $D(f)$ have normal crossings. If instead Hironaka's global flattening theorem ([7], theorem 4.4, p. 536) is used, f can be flattened using only blow-ups on S with centers in $D(f)$ since only such blow-ups are "permissible" (see 4.3.4 and 4.4.3 of Hironaka [7]), after which we proceed following Zagorskii's proof (see pp. 421—422 of [14]). Zagorskii's proof turns on the fact that, beginning with f: $X \to S$ a flat, embedded conic bundle and $D(f)$ a normal crossings curve, singularities in X correspond to drops in the rank of the associated quadratic form: such singularities can be explicitly resolved by blow-ups on S and fiber product. To then go from the resulting regular conic bundle to a standard conic bundle, simply blow-down as in 0.10.2. Under this modification of Zagorskii's argument, β is an isomorphism over $S \setminus D(f)$, so $Z \setminus D(g) \subset \eta^{-1}(S \setminus D(f))$ yielding 3.

1. Statement of theorem and outline of proof

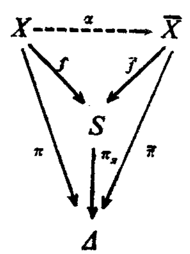
1.1. Definition. $\pi = \pi_S \circ f$: $X \to \Delta$ is a *degeneration of minimal ruled surfaces* if X is a three-dimensional complex manifold, Δ is the unit disk in $\mathbf{C}$, f: $X \to S$ is a holomorphic map onto S, a non-singular complex surface, π and π_S are flat holomorphic maps onto Δ, and $\pi = \pi_S \circ f$ such that

i) for $t \neq 0$, letting $X_t = \pi^{-1}(t)$ and $S_t = \pi_S^{-1}(t)$, then X_t and S_t are non-singular and all fibers of $f|_{X_t}: X_t \to S_t$ are non-singular rational curves, i.e. $f|_{X_t}$ gives X_t a structure of minimal ruled surface over S_t, and

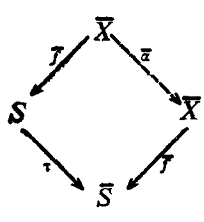
ii) $X_0 = \pi^{-1}(0)$ is a union of non-singular surfaces meeting normally and $S_0 = \pi_S^{-1}(0)$ is a union of non-singular curves meeting normally.

1.2. Note. Given $\pi = \pi_S \circ f$ as above, f is a generalized conic bundle (0.5) and $D = D(f) \subset (S_0)_{\mathrm{red}}$. Note that the components of S_0 have natural multiplicities from π_S whereas $D(f)$, for f a generalized conic bundle, is only defined as a set.

1.3. Theorem. *Given $\pi = \pi_S \circ f$ a degeneration of minimal ruled surfaces, there is a birational map $\alpha: X \dashrightarrow \tilde{X}$, which is an isomorphism off X_0 and $\tilde{X}_0$, composed of (smooth) blow-ups and blow-downs and blow-ups of fibers of f such that $\tilde{f} = f \circ \alpha^{-1}$ is a $\mathbf{P}^1$-bundle over S. Thus with $\tilde{\pi} = \pi_S \circ \tilde{f}$, we have*



where $\tilde{X}_0$ is a $\mathbf{P}^1$-bundle over S_0. In addition, if $\tau: S \to \bar{S}$ blows S down to $\bar{S}$, a non-singular minimal model of S over Δ, that is, S with all exceptional curves in S_0 contracted, then there is $\bar{\alpha}: \tilde{X} \dashrightarrow \bar{X}$ composed of (smooth) blow-ups and blow-downs such that $\bar{f} = \tau \circ \tilde{f} \circ \bar{\alpha}^{-1}: \bar{X} \to \bar{S}$ is a $\mathbf{P}^1$-bundle over $\bar{S}$:



with all maps defined over Δ.

1.4. Sketch of proof.

Step One. We use the minimizing lemma (0.3) to contract surfaces in X_0 mapping onto curves in D, thereby birationally modifying $f: X \to S$ to $f': X' \to S$ such that $D(f')$ is a union of points only.

Step Two. We use theorem 0.11 to make $f'\colon X'\to S$ into a standard conic bundle $g\colon Y\to Z$. The advantage of this is that we can show that g is in fact a $\mathbf{P}^1$-bundle by using the fact that $D(g)\subset\sigma^{-1}$ (points on S) where $\sigma\colon Z\to S$. The disadvantage is that we have blown up S, contrary to the statement of our theorem.

Step Three. Finally, we prove a lemma allowing us to birationally blow Z back down to S without losing the $\mathbf{P}^1$-bundle structure and, in fact, this same proposition allows us to continue the birational operation, blowing S down to $\bar{S}$.

2. STEP ONE: Reducing to Dim $(D)=0$

2.1. Notation. Given $\pi=\pi_S\circ f$ a degeneration of minimal ruled surfaces as in 1.1 and C a component of $(S_0)_{\text{red}}$, let P be a general point of C, Γ a germ of a non-singular curve in S such that $(C\cdot\Gamma)_S=P$. Let W be the closure of $f^{-1}(\Gamma\setminus\{P\})$ in X and V be an irreducible surface in X_0 such that $f(V)=C$. Note that W is irreducible, non-singular off $f^{-1}(P)$ and ruled near P by $f|_W\colon W\to\Gamma$ (if $Q\in\Gamma$, $Q\neq P$, then $f^{-1}(Q)$ is a non-singular rational curve because $\Gamma\setminus\{P\}\subset S\setminus S_0\subset S\setminus D(f)$).

2.2. Lemma.

1. *If E is an irreducible component of $V\cap W$, then E is non-singular and rational,*
2. *V is non-singular and ruled over $\hat{C}$, a non-singular finite cover of C; moreover, each component E of $V\cap W$ is an irreducible fiber of $V\to\hat{C}$ and*
3. *W is non-singular, ruled by $f|_W$ and meets V normally; moreover, $f|_W^{-1}(P)$ is all of $f^{-1}(P)$.*

Proof.

1. $f|_W\colon W\to\Gamma$ gives W a structure of ruled surface (albeit possibly singular ruled surface). Letting $\tilde{W}\to W$ be a desingularization of W, then the composed map $\tilde{W}\to\Gamma$ is a non-singular ruled surface whose fiber over P contains only rational curves, one of which is birational to E, so E is a rational curve. On the other hand, V is non-singular (since V is a component of X_0), so by generic smoothness (Sard's theorem) of $f|_V$ and our choice of P, $f^{-1}(P)\cap V$ is non-singular. Now $E\subset f^{-1}(P)\cap V$, so E is non-singular.

2. Since $f^{-1}(P)\cap V$ is non-singular, as we saw in 1, $f^{-1}(P)\cap V=\coprod_{i=1}^{a} E_i$ for E_i non-singular rational curves. Letting $V\to\hat{C}\to C$ be the Stein factorization of $f|_V$ (and normalization of $\hat{C}$), the fibers of $V\to\hat{C}$ are irreducible non-singular rational curves (in fact, they are the E_i above).

3. Let $Q\in V\cap W$, so $Q\in E$ where E is an irreducible fiber of $V\to\hat{C}$, by 2. Let Σ be a germ of a section of $V\to\hat{C}$ such that $Q\in\Sigma$; moreover, in a neighborhood of $f^{-1}(P)$, $\Sigma=\coprod_{i=1}^{d}\Sigma_i$ and $Q\in\Sigma_j$ for some j. Now by the projection formula,

$f_*(\Sigma_j \cdot W)_X = f_*(\Sigma_j \cdot f|_W^*(\Gamma))_X = (f_* \Sigma_j \cdot \Gamma)_S = (C \cdot \Gamma)_S = P$ since $\Sigma_j \to C$ has degree one. Thus $(\Sigma_j \cdot W)_X = 1$ so since $Q \in \Sigma_j \cap W$ and $\Sigma_j \subset V$, Q is a non-singular point on W (Fulton [6], p. 126) and W meets V normally. Finally, note that $f^{-1}(P)$ is connected so if $f^{-1}(P) \not\subset W$, then there are irreducible curves in $f^{-1}(P)$, K_1 and K_2, $K_1 \subset W$, $K_2 \not\subset W$, such that $K_1 \cap K_2 \neq \emptyset$. Since $W \cap V_i$ has pure dimension one (if not the empty set) for all $V_i \subset X_0$ and $W \cap K_2 \neq \emptyset$, we may assume that K_1 and K_2 are both contained in V_j for some $V_j \subset X_0$. In that case, though, since $K_1 \cap K_2 \neq \emptyset$, the fiber of $f|_{V_j}$: $V_j \to C$ over P is singular contrary to generic smoothness. Thus the fiber of $f|_W$: $W \to \Gamma$ over P is all of $f^{-1}(P)$.

2.3. Proposition (Step One). *Given* $\pi = \pi_S \circ f$ *a degeneration of minimal ruled surfaces, there is a birational map* α': $X \dashrightarrow X'$, *which is an isomorphism off* $D(f) \subset X_0$ and X_0', *composed of (smooth) blow-ups and blow-downs, such that for* $f' = f \circ (\alpha')^{-1}$, $\pi' = \pi_S \circ f'$ *is a degeneration of minimal ruled surfaces such that* $\dim(D(f')) = 0$.

Proof. All blow-ups and blow-downs will have centers over components of $D(f)$ and so are isomorphisms off X_0. Let $n(C)$ be the number of components of $f^{-1}(P)$ for P a general point of C, C a component of $(S_0)_{\text{red}}$. By 2.2.3, $n(C)$ is also the number of components of $f^{-1}(P) \cap W$, for W as defined in 2.1.

If $n(C) = 1$, then $f^{-1}(C) = V$ for V an irreducible surface; moreover, the general fiber of $f|_V$: $V \to C$ is irreducible. Thus (2.2.2) $\hat{C} = C$ and V is ruled over C. In particular, $f^{-1}(P)$ is an irreducible non-singular rational curve and so $C \not\subset D(f)$.

If $n(C) \geqq 2$, then $C \subset D(f)$ and so we want to birationally modify X over S to strictly decrease $n(C)$ without increasing $n(C')$ for any other component C' of S_0. Once $n(C)$ is one, then C is no longer a subset of $D(f)$. Since, in fact, the birational modifications of X over S will all be isomorphisms off C and its preimage, $n(C')$ will be unchanged for any C' other than C. Thus we may apply this procedure in turn to each component of $(S_0)_{\text{red}}$ contained in $D(f)$ until no component of $(S_0)_{\text{red}}$ is contained in $D(f')$ (where f' is the new map), in which case $\dim(D(f')) = 0$.

We now assume that $n(C) \geqq 2$. Using the notation of 2.1, there is (2.2.3) an irreducible non-singular ruled surface W on which $f^{-1}(P)$ is a fiber, and so, since $n(C) \geqq 2$, is a reducible fiber. Thus (0.4.2) there is an irreducible non-singular rational curve E, $E \subset f^{-1}(P)$ such that E is an exceptional curve on W. Now $E \subset V \cap W$ for some component V of X_0, so (2.2.2) E is an irreducible fiber of a ruling $V \to \hat{C}$. Now $(E \cdot V)_X = (E \cdot E)_W = -1$ since W meets V normally (2.2.3) and E is an exceptional curve on W. Thus V is generically contractible (0.2). Now using the minimizing lemma (0.3), blow-ups and blow-downs can be used to make the proper transform of V minimal ruled and still non-singular and generically contractible. Thus (0.2) the proper transform of V is smoothly contractible. Contracting the proper transform now strictly reduces $n(C)$ (in fact, by the degree of $\hat{C}\to C$) while not effecting $n(C')$ for any other C', components of $(S_0)_{\mathrm{red}}$. Finally, blow-ups with centers over the points in S corresponding to reducible fiber of V will return the strict transform of X_0 to normal crossings without effecting any $n(C')$ or the new $n(C)$.

3. STEP TWO: Reducing to a $\mathbf{P}^1$-bundle

3.1. Proposition (Step Two). *Given* $\pi=\pi_S\circ f$ *a degeneration of minimal ruled surfaces with* $\dim(D(f))=0$, *there is a birational map* $\beta\colon Y \dashrightarrow X$, *which is an isomorphism off* Y_0 *and* X_0, *composed of (smooth) blow-ups and blow-downs and blow-ups of fibers of f over points of S, and a birational morphism of non-singular surfaces* $\eta\colon Z\to S$:

$$\begin{array}{ccc} X & \overset{\beta}{\dashleftarrow} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ S & \underset{\eta}{\longleftarrow} & Z \end{array}$$

where $g=\eta^{-1}\circ f\circ\beta$, *all defined over* Δ, *such that for* $\pi_Z=\pi_S\circ\eta$, $\pi_Z\circ g$ *is a degeneration of minimal ruled surfaces and* $g\colon Y\to Z$ *is a* $\mathbf{P}^1$*-bundle.*

Proof. As noted in 1.2, $f\colon X\to S$ is a generalized conic bundle and so we may apply Sarkisov's theorem (0.11), yielding a standard conic bundle $g\colon Y\to Z$. By 0.11.3, β is an isomorphism off $D(f)$, which is contained in $(S_0)_{\mathrm{red}}$ (1.2), so all maps are isomorphisms off X_0. Thus it suffices to show that $D(g)=\emptyset$, since then g is a proper holomorphic map between non-singular complex manifolds and all fibers of g are isomorphic; thus g is in fact locally trivial (Fischer and Grauert [5], p. 89).

By 0.11.3, $D(g)\subset\eta^{-1}(D(f))$. Now $\dim D(f)=0$ and so $D(g)$ is contained in a finite union of trees of non-singular rational curves on Z (0.4.1). Since g is a conic bundle, either $D(g)=\emptyset$ or a curve (0.8.2). If $D(g)\neq\emptyset$, then, $D(g)$ is itself a union of trees of non-singular rational curves; in particular, there is an irreducible rational curve D, contained in $D(g)$, whose intersection with the other components of $D(g)$ is $\emptyset$ or one point. Now the natural double cover $\tilde{D}\to D$ (see 0.8.2) is therefore either unramified or ramified at exactly one point- this, however, is impossible since g is a standard conic bundle and so (0.10.1) $\tilde{D}$ is an irreducible non-singular double cover of $\mathbf{P}^1$ which is either unramified or has exactly one ramification point, both contrary to the Hurwitz formula. Thus $D(g)=\emptyset$, as claimed.

4. STEP THREE: Blowing back down

4.1. Definition. If $g\colon Y\to Z$ is a holomorphic map from a non-singular three-fold to a non-singular surface and E is a non-singular curve in Z, g is a $\mathbf{P}^1$*-bundle over* E provided, for $T=g^{-1}(E)$, $g|_T\colon T\to E$ is a $\mathbf{P}^1$-bundle. Note that this is equivalent to $g|_T$ being a $\mathbf{P}^1$-fibration by Fischer and Grauert [5].

4.2. Lemma. *Using the notation of* 4.1, *for g a $\mathbf{P}^1$-bundle over E, let Σ be a section on T (with respect to the ruling $g|_T$). If $\sigma\colon \tilde{Y}\to Y$ is the blow-up of Σ in Y, then $\tilde{T}$, the proper transform of T in $\tilde{Y}$, is a smoothly contractible ruled surface, under the ruling induced by $(g\circ\sigma)|_{\tilde{T}}$. In addition, if τ contracts $\tilde{T}$, then $g\cdot\sigma\cdot\tau^{-1}$ is again a $\mathbf{P}^1$-bundle over E.*

Proof. By assumption, $g|_T$ exhibits T as a non-singular minimal ruled surface in Y. Since $\sigma^{-1}|_T$ blows-up Σ, a divisor on T, $\tilde{T}\cong T$ and so $\tilde{T}$ is also a non-singular minimal ruled surface. By 0.2, to show the contractibility of $\tilde{T}$, it remains to show that $\tilde{F}\cdot\tilde{T}=-1$, where $\tilde{F}$ is a fiber on $\tilde{T}$. Now $F\cdot T=0$ since F is a fiber of g and so algebraically equivalent to a fiber of g not over E. In addition, note that $\sigma^*(T)\equiv \tilde{T}+\bar{T}$ where $\bar{T}$ is the preimage of Σ and so is also a non-singular minimal ruled surface over E; moreover, for $\bar{F}$ a fiber on $\bar{T}$, $\bar{F}\cdot\tilde{T}=1$. Thus by the projection formula, $F\cdot\tilde{T}=F\cdot\sigma^*T-F\cdot\bar{T}=\sigma_*F\cdot T-1=F\cdot T-1=-1$, as claimed. Clearly $\tau(\bar{T})$ is still a non-singular minimal ruled surface and so the fibers of $g\cdot\sigma\cdot\tau^{-1}$ over E are now irreducible non-singular rational curves.

4.3. Definition. If g is an $\mathbf{P}^1$-bundle over E as in 4.2, then the birational map $\tau\circ\sigma^{-1}$ is an *elementary modification of Y over E (centered at Σ).*

4.4. Note. An elementary modification of Y over E replaces T with a surface isomorphic to the blow-up of Σ; thus T is replaced by $\mathbf{P}(N_{\Sigma/Y})$.

4.5. Lemma. *If $g\colon Y\to Z$ is a $\mathbf{P}^1$-bundle over E and E is an exceptional curve on Z, then after a finite number of elementary modifications over E, g is the pull-back of a holomorphic map $\bar{Y}\to\bar{Z}$, where $\bar{Y}$ is non-singular and $Z\to\bar{Z}$ is the blow-down of E on Z.*

Note. This procedure gives a rational map which is an isomorphism off $g^{-1}(E)$ in Y and off a smooth rational curve in $\bar{Y}$.

Proof. This proof represents joint work with R. Miranda. We first use elementary modifications to make T into $\mathbf{P}^1\times\mathbf{P}^1$ and then show that it can be contracted "sideways".

Since E is an exceptional curve, E is rational and so T is a minimal rational ruled surface, i.e. $T\cong\mathbf{F}_N$ for some N. If $N=0$, then $T\cong\mathbf{P}^1\times\mathbf{P}^1$, as desired.

For $N \geqq 0$, let Σ be the negative section on T, so $(\Sigma^2)_T = -N$. Now $\Sigma \subset T \subset Y$, so there is an exact sequence

$$0 \to N_{\Sigma/T} \to N_{\Sigma/Y} \to N_{T/Y}|_\Sigma \to 0$$

Now $\Sigma \cong \mathbf{P}^1$ and so $N_{\Sigma/T} \cong \mathcal{O}_{\mathbf{P}^1}((\Sigma^2)_T) = \mathcal{O}_{\mathbf{P}^1}(-N)$. Also since E is non-singular, $E \subset Z$ is a local complete intersection and so (Fulton [6], chapter 6) $N_{T/Y} \cong g^*(N_{E/Z}) \cong g^* \mathcal{O}_E(-1)$ since $(E^2)_Z = -1$. Now $g^*(\mathcal{O}_E) \cong \mathcal{O}_T$, so

$$N_{T/Y}|_\Sigma \cong \mathcal{O}_T(-\text{one fiber})|_\Sigma \cong \mathcal{O}_{\mathbf{P}^1}(-1)$$

since $(\Sigma \cdot \text{one fiber})_T = 1$. Therefore, by the exact sequence above, $N_{\Sigma/Y}$ is an extension of $\mathcal{O}_{\mathbf{P}^1}(-1)$ by $\mathcal{O}_{\mathbf{P}^1}(-N)$ and so, if $N \geqq 1$, $N_{\Sigma/Y} \cong \mathcal{O}_{\mathbf{P}^1}(-a) \oplus \mathcal{O}_{\mathbf{P}^1}(-b)$ where $1 \leqq a \leqq b \leqq N$ and $a+b=N+1$. As noted in 4.4, an elementary modification of Y centered at Σ replaces T with $\mathbf{P}(N_{\Sigma/Y})$ and so, if $N \geqq 1$, $\mathbf{F}_N$ is replaced with $\mathbf{F}_n$ where $n = a-b \leqq N-1$. Thus after at most N elementary modifications, T can be made into $\mathbf{P}^1 \times \mathbf{P}^1$.

We now assume that $T \cong \mathbf{P}^1 \times \mathbf{P}^1$. As was shown in the paragraph above, $N_{T/Y} \cong \mathcal{O}_T(-\text{a fiber of } g|_T)$. Thus for F a fiber on T with respect to the other ruling (i.e. F is a section with respect to $g|_T$), $(F \cdot T)_Y = \deg_F(N_{T/Y}) = (-(\text{a fiber of } g|_T) \cdot F)_T = -1$ and so T is smoothly contractible in Y; moreover there is a commutative diagram

$$\begin{array}{ccc} Y & \to & \bar{Y} \\ \downarrow & & \downarrow \\ Z & \to & \bar{Z} \end{array}$$

where $Y \to \bar{Y}$ is the contraction of T above. Thus there is a holomorphic map $Y \to \bar{Y} \times_{\bar{Z}} Z$ which is a birational morphism contracting no divisors. By Zariski's main theorem, that map is an isomorphism and so g is a pull-back, as claimed.

5. Proof of Theorem 1.3

To prove theorem 1.3, we begin with $\pi = \pi_S \circ f \colon X \to \Delta$, a degeneration of minimal ruled surfaces. By Step One (proposition 2.3), there is a birational modification of X to X', where $X' \to S$ is again a degeneration of minimal ruled surfaces and the birational modification is an isomorphism off X_0 and X'_0, such that $\dim(D(f')) = 0$. Thus $\pi' = \pi_S \circ f'$ satisfies the hypotheses for Step Two (proposition 3.1) and so there is a birational modification, again an isomorphism off X'_0, blowing up points on S, yielding a $\mathbf{P}^1$-bundle $g \colon Y \to Z$ such that $\pi_Z \circ g$ is a degeneration of minimal ruled surfaces for $\pi_Z = \pi_S \circ \eta$. Note that η is a composition of (smooth) blow-ups of points $\eta = \sigma_2 \circ \ldots \circ \sigma_n$.

$$\begin{array}{ccccc} X & \overset{\alpha'}{\dashrightarrow} & X' & \overset{\beta}{\dashleftarrow} & Y \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & & & \downarrow{\scriptstyle g} \\ & S & & \underset{\eta}{\longleftarrow} & Z \end{array}$$

To complete the proof of theorem 1.3, it remains to blow Z back down to S while preserving the $\mathbf{P}^1$-bundle structure. Step Three (more precisely, lemma 4.5) accomplishes this as follows: let $Y_n=Y$, $Z_n=Z$ and $g_n=g$. Now since $\eta=\sigma_2\circ\ldots\circ\sigma_n$, there is an exceptional curve $E_n\subset Z_n$ which is blown down by σ_n. By theorem 4.5, there is a birational map γ_n: $Y_n \dashrightarrow Y_{n-1}$ such that $g_{n-1}=\sigma_n\circ g_n\circ\gamma_n^{-1}$ is a $\mathbf{P}^1$-bundle over $Z_{n-1}=\sigma_n(Z_n)$. Now Z_{n-1} is a non-singular surface containing an exceptional curve E_{n-1} which is blown down by σ_{n-1} and so by the above argument, there is a γ_{n-1} and g_{n-2} as above. Repeating this procedure for all exceptional curves contained in the exceptional divisor of η, we obtain a $\mathbf{P}^1$-bundle g_1: $Y_1\to Z_1=S$:

$$\begin{array}{ccccccccc} Y=Y_n & \overset{\gamma_n}{\dashrightarrow} & Y_{n-1} & \overset{\gamma_{n-1}}{\dashrightarrow} & \ldots & \overset{\gamma_3}{\dashrightarrow} & Y_2 & \overset{\gamma_2}{\dashrightarrow} & Y_1 \\ \downarrow{\scriptstyle g_n} & & \downarrow{\scriptstyle g_{n-1}} & & & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_1} \\ Z=Z_n & \xrightarrow{\sigma_n} & Z_{n-1} & \xrightarrow{\sigma_{n-1}} & \ldots & \xrightarrow{\sigma_3} & Z_3 & \xrightarrow{\sigma_2} & Z_1=S \end{array}$$

Thus $\alpha=\gamma_2\circ\ldots\circ\gamma_n\circ\beta^{-1}\circ\alpha'$: $X\dashrightarrow Y_1:=\tilde{X}$ is in fact a birational map over S, which is an isomorphism off X_0 and $\tilde{X}_0$, and g_0 is a $\mathbf{P}^1$-bundle, as claimed in the statement of 1.3. Note that in general g_1 is not f' since f' is not flat over S.

Finally, to pass from S to $\bar{S}$, a non-singular minimal model of S over Δ, we simply continue the procedure above:

$$\begin{array}{ccc} Y_i & \overset{\gamma_i}{\dashrightarrow} & Y_{i-1} \\ \downarrow{\scriptstyle g_i} & & \downarrow{\scriptstyle g_{i-1}} \\ Z_i & \xrightarrow{\sigma_i} & Z_{i-1} \end{array}$$

on past $Z_1=S$, contracting each exceptional curve in S_0 until there are no more, since $S\to\bar{S}$ is a composition of smooth blow-downs.

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