Characteristic Cauchy problems and analytic continuation of holomorphic solutions

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Introduction

This paper is concerned with the analytic continuation of holomorphic solutions of a partial differential equation in a complex domain.

Let Ω, Ω' be open connected sets in $\mathbf{C}^{n+1}$ satisfying $\Omega\subset\Omega'$ and $\partial\Omega\cap\Omega'\neq\emptyset$, where $\partial\Omega$ denotes the boundary of Ω. Let $P(z, D)$ be a linear partial differential operator of order m with holomorphic coefficients defined in Ω' and let $z^0\in\partial\Omega\cap\Omega'$. We let (Q) denote the assertion

(Q): Every solution $u\in H(\Omega)$ to $Pu=0$ has a holomorphic extension to a neighborhood of z^0.

In the following we suppose 1) $z^0=0$ and 2) $x_0=0$ is the real tangent hyperplane of $\partial\Omega$ at 0 whenever $\partial\Omega$ is assumed to be in $\mathbf{C}^1$. We note that, on the real hyperplane $x_0=0$, there is a unique complex hyperplane passing the origin, which is clearly $z_0=0$. We call it the complex tangent hyperplane of $\partial\Omega$ at 0.

M. Zerner [13] has proved that (Q) is true if $\partial\Omega$ is in $\mathbf{C}^1$ and $z_0=0$ is non-characteristic. Y. Tsuno [9], Pallu de la Barrière [5] and J. Persson [7] in the simply characteristic case, J. Persson [6] and Y. Tsuno [10] in the case when $z_0=0$ is characteristic with constant multiplicity have given sufficient conditions for (Q) to be true. These results are all based on a precise form of the Cauchy—Kowalewsky theorem. By the way, if $\Omega\cap\{z_0=0\}=\emptyset$ and there is a solution to $Pu=0$ with singularities on and only on $z_0=0$, then (Q) is not true. When $z_0=0$ is characteristic with constant multiplicity, the existence of such singular solutions has been studied by J. Persson [6], [8], S. Ouchi [4] and so on.

The purpose of this article is to give sufficient conditions for (Q) to be true when $z_0=0$ is a characteristic hyperplane with varying multiplicity around $z=0$. The argument is based on a Cauchy—Kowalewsky type theorem for a Cauchy problem with $m-1$ data given on a characteristic hyperplane of the type mentioned above. It is theorem 1 in § 1, and it gives sharp estimates of the existence domain of the solutions. The analytic continuation theorems, theorem 2 and corollaries 1 and 2 are stated and proved in § 2.

Theorem 2 is an analogue of the results obtained by cones of analytic continuation in [6] and [7], which are based on the precised Cauchy—Kowalewsky theorem. It seems however to be impossible to prove theorem 2 by using them, because our result depends essentially on lower order terms, whereas cones of analytic continuation are decided only by the principal part of the equation. We should also remark it is not assumed in theorem 2 that $\Omega\cap\{z_0=0\}\neq\emptyset$. It is important, for it means the non-existence of such singular solutions as mentioned above for the equations which we treat in this paper. Corollary 2 states this fact as an analytic continuation theorem.

We use the following notations in this paper:

$$z=(z_0,\hat{z})=(z_0,z_1,\dots,z_n);\ z_i=x_i+\sqrt{-1}y_i,\quad i=0,1,\dots,n;$$

$$\zeta=(\zeta_0,\dots,\zeta_n);\quad D=(D_0,\dots,D_n);$$

$$D_i=\partial/\partial z_i=(\partial/\partial x_i-\sqrt{-1}\partial/\partial y_i)/2;$$

$$\bar{D}_i=\partial/\partial\bar{z}_i=(\partial/\partial x_i+\sqrt{-1}\partial/\partial y_i)/2;$$

$$\mathrm{grad}_z F=(D_0F,\dots,D_nF);\quad F_{z_i}=D_iF;$$

$$F_{\bar{z}_i}=\bar{D}_iF;\quad \bar{\mathbf{N}}=\{0,1,2,\dots\};$$

$H(\Omega)$ the space of all holomorphic functions in Ω;
$p=O(q)$ means p/q is bounded as $q\to 0$;
$p=o(q)$ means p/q tends to 0 as $q\to 0$.

1. Cauchy—Kowalewsky type theorem

1.1. Denote by $p_m(z,\zeta)$ the principal symbol of $P(z,D)$ and presume

Assumption I. It holds that

$$p_m(z,N)\equiv 0\quad\text{in }\Omega',\quad\text{and}\quad(\partial p_m/\partial\zeta_i)(0,N)=0$$

for every i, where $N=(1,0,\dots,0)$.

Let $\lambda_1,\dots,\lambda_n$ be the eigen-values of the matrix

$$M=((\partial^2 p_m/\partial\zeta_i\partial z_j)(0,N);\quad i,j=1,\dots,n). \tag{1.1}$$

Suppose $\lambda_1, \ldots, \lambda_k \neq 0$ and $\lambda_{k+1} = \ldots = \lambda_n = 0$. Besides, put

$$\mu = p_{m-1}(0, N), \tag{1.2}$$

where $p_{m-1}(z, \zeta)$ denotes the homogeneous part of degree $m-1$, in the ζ variable, of the symbol of $P(z, D)$. We assume

Assumption II. The following i), ii) and iii) hold:

i) The convex hull of λ_i $(1 \leqq i \leqq k)$ in the complex number plane does not contain the origin.

ii) $-\mu \notin \{\sum_{i=1}^k \lambda_i \beta_i;\ \beta_i \in \overline{\mathrm{N}}\}$.

iii) There is an integer h with $0 \leqq h \leqq k$ such that

$$p_m(z, \zeta) = O\left(\sum_{i=1}^h |\zeta_i|^2 + \sum_{i=1}^h |z_i|^2 + \sum_{i=h+1}^k |z_i|\right)$$

as $(z_1, \ldots, z_k, \zeta_1, \ldots, \zeta_h) \to 0$.

1.2. Let τ be a complex parameter. The hyperplane $z_0 = \tau$ is characteristic because of Assumption I. We consider the Cauchy problem with $m-1$ data on this hyperplane: Given $f(z)$ and $v(z)$ holomorphic in a neighborhood of $(\tau, 0)$, obtain a holomorphic solution $u(z)$ to

$$P(z, D)u = f, \quad u - v = O\left((z_0 - \tau)^{m-1}\right). \tag{1.3}$$

Denote $z = (z_0, z', z'')$, $z' = (z_1, \ldots, z_h)$, $z'' = (z_{h+1}, \ldots, z_n)$, $|z'| = \max_{1 \leqq i \leqq h} |z_i|$ and $|z''| = \max_{h+1 \leqq i \leqq n} |z_i|$ with h from Assumption II—iii). Then we have

Theorem 1. *Suppose Assumptions I and II hold. Then there are three positive constants* $\delta > 0$, $r_1 > 0$ *and* $0 < c < 1$ *such that for every* $|\tau| < \delta$, *every* $0 < r < r_1$ *and every* $f(z)$, $v(z)$ *holomorphic in* $\{|z_0 - \tau| < r^2,\ |z'| < r,\ |z''| < r^2\}$, *the characteristic Cauchy problem* (1.3) *has a unique holomorphic solution in*

$$\{|z_0 - \tau| < (cr)^2,\ |z'| < cr,\ |z''| < (cr)^2\}.$$

This is an improvement on theorem 7.1 in our previous paper [3]. We will give the proof in § 3.

2. Analytic continuation theorem

2.1. From theorem 1 we have

Theorem 2. *Suppose Assumptions I and II hold. Then there exists a positive constant* $\gamma = \gamma(P)$ *depending only on the operator* $P(z, D)$, *such that the following prop-*

erty holds: If there is a sequence $\tau^{(q)}\to 0$, $q\to\infty$, $\tau^{(q)}\neq 0$, *such that*

$$\{z;\ z_0=\tau^{(q)},\ |z'|\leqq|\gamma\tau^{(q)}|^{1/2},\ |z''|\leqq|\gamma\tau^{(q)}|\}\subset\Omega, \tag{2.1}$$

then (Q) *is true.*

Proof. We apply theorem 1 by taking $\tau=\tau^{(q)}$ and $r=|2\tau^{(q)}|^{1/2}/c$, where c is the constant appearing there. Put $\gamma=2/c^2$, then $|\gamma\tau^{(q)}|=r^2$. Hence, by the assumption, u is holomorphic in some neighborhood of

$$S_{q,r}=\{z;\ z_0=\tau^{(q)},\ |z'|\leqq r,\ |z''|\leqq r^2\}.$$

Set

$$v=\sum_{i=0}^{m-2}(D_0^i u)(\tau^{(q)},\hat{z})(z_0-\tau^{(q)})^i/i!\quad\text{with}\quad \hat{z}=(z',z'').$$

Then v is naturally holomorphic in

$$\{z;\ |z_0-\tau^{(q)}|<r^2,\ |z'|<r,\ |z''|<r^2\}.$$

Therefore it follows from theorem 1 that there exists a unique holomorphic solution $\tilde{u}(z)$ to $P\tilde{u}=0$ in

$$V_{q,r}=\{z;\ |z_0-\tau^{(q)}|<(cr)^2,\ |z'|<cr,\ |z''|<(cr)^2\}$$

satisfying $\tilde{u}-v=O((z_0-\tau^{(q)})^{m-1})$. By the uniqueness, $\tilde{u}=u$ in a neighborhood of $(\tau^{(q)},0)$. On the other hand, since $(cr)^2=|2\tau^{(q)}|$, the set $V_{q,r}$ contains $z=0$. This completes the proof.

We say $\partial\Omega$ is in C^k, $k\geqq 1$, in a neighborhood of 0, if there is a real-valued C^k function $F(z)$ defined in some neighborhood V of 0 such that $F(0)=0$, $\mathrm{grad}_z F(0)=N$ with $N=(1,0,\dots,0)$ and $\Omega\cap V=\{z\in\Omega'\cap V;\ F(z)<0\}$. We then have

Corollary 1. *Let assumptions I and II be fulfilled and assume* $\partial\Omega$ *is in* $\mathbf{C}^2$ *in a neighborhood of* 0. *Then there exists a positive constant* $\alpha=\alpha(P)$ *depending only on* $P(z,D)$ *such that the following property holds: If*

$$\max_{|z'|\leqq 1}\left\{\sum_{i,j=1}^{h}F_{z_i\bar{z}_j}(0)z_i\bar{z}_j+\mathrm{Re}\sum_{i,j=1}^{h}F_{z_iz_j}(0)z_iz_j\right\}<\alpha, \tag{2.2}$$

then (Q) *is true.*

Proof. At the section $z_0=-r$ with $r>0$,

$$F(-r,\hat{z})=F(-r,0)+\sum_{i=1}^{n}\{F_{z_i}(-r,0)z_i+F_{\bar{z}_i}(-r,0)\bar{z}_i\}$$

$$+\frac{1}{2}\sum_{i,j=1}^{n}\{F_{z_iz_j}(-r,0)z_iz_j+2F_{z_i\bar{z}_j}(-r,0)z_i\bar{z}_j+F_{\bar{z}_i\bar{z}_j}(-r,0)\bar{z}_i\bar{z}_j\}+o(|\hat{z}|^2).$$

Note that $F(-r,0)=-2r+o(r)$, $F_{z_i}(-r,0)=O(r)$ for $1\leqq i\leqq n$, $F_{z_iz_j}(-r,0)=F_{z_iz_j}(0,0)+O(r)$ for $1\leqq i,j\leqq n$ and so on. Let γ be the same constant as in theorem 2. Then we have

$$F(z)=-2r+\sum_{i,j=1}^{h}F_{z_i\bar{z}_j}(0)z_i\bar{z}_j+\mathrm{Re}\sum_{i,j=1}^{h}F_{z_iz_j}(0)z_iz_j+o(r)$$

on $S_r=\{z;\ z_0=-r,\ |z'|\leqq(\gamma r)^{1/2},\ |z''|\leqq\gamma r\}$.

On the other hand, it follows from (2.2) that

$$\sum_{i,j=1}^{h}F_{z_i\bar{z}_j}(0)z_i\bar{z}_j+\mathrm{Re}\sum_{i,j=1}^{h}F_{z_iz_j}(0)z_iz_j<\alpha\gamma r,\quad \text{on}\quad S_r.$$

Therefore, if we take $\alpha=2/\gamma$, then $F<0$ on S_r for sufficiently small r, and so the condition (2.1) in Theorem 2 is satisfied. This completes the proof.

Corollary 2. *Suppose Assumptions I and II hold. Let* $U=\{z;\ |z_i|<r$ *for every* $i\}$ *with* $r>0$, *and assume* $U\subset\Omega'$. *If* u *is a holomorphic solution to* $Pu=0$ *on the universal covering space of* $U\setminus\{z_0=0\}$, *then* u *has a unique holomorphic extension to* U.

Proof. We can apply corollary 1 by taking $F=2x_0$. We see u becomes holomorphic in $U'=\{z;\ |z_i|<r'$ for every $i\}$ with some $0<r'<r$. Since $U'\cup\{U\setminus\{z_0=0\}\}$ is simply connected, u is single-valued there. As well-known in the theory of several complex variables (cf. [1], chapter II, theorem 1.5), u has a holomorphic extension to U. This completes the proof.

Remark 1. 1) In theorem 2, we need not assume any smoothness of $\partial\Omega$.

2) In corollary 1, the condition (2.2) is imposed only to the $(z_1,\dots,z_h)$ direction; and in particular when $h=0$, (Q) is true without (2.2).

2.2. We give an example. Let $1\leqq m\leqq n$ and

$$P=D_0^2-z_0^2(D_1^2+\dots+D_m^2)+b_0D_0+\dots+b_nD_n \tag{2.3}$$

with $b_i(z)$ holomorphic in a neighborhood of the origin in $\mathbf{C}^{n+1}$. Change the variables by $w_0=z_1+z_0^2/2$, $w_1=z_0$, $w_j=z_j$ for $j\geqq2$ and denote w by z again. Then P is transformed into

$$\tilde{P}=2z_1D_1D_0+D_1^2-z_1^2\sum_{j=2}^{m}D_j^2+(b_0z_1+b_1+1)D_0+b_0D_1+\sum_{j=2}^{n}b_jD_j.$$

Its principal symbol is $\tilde{P}_2=2z_1\zeta_1\zeta_0+\zeta_1^2-z_1^2(\zeta_2^2+\dots+\zeta_m^2)$; $\lambda_1=2$; $\lambda_j=0$ for $j=2,\dots,n$; $\mu=b_1(0)+1$. Assumptions I and II are clearly satisfied if $b_1(0)\neq-1,-3,-5,\dots$. In the present case, the condition (2.2) in corollary 1 is equal to

$$F_{z_1\bar{z}_1}(0)+|F_{z_1z_1}(0)|<\alpha. \tag{2.4}$$

By the way, when b_j are constants and $b_0=0$, the equation $\tilde{P}u=0$ has solutions

$$u_1 = z_0^{(b_1-1)/4}(z_0-z_1^2)^{-(b_1+1)/4}, \tag{2.5}$$

$$u_2 = z_1 z_0^{(b_1-3)/4}(z_0-z_1^2)^{-(b_1+3)/4}.$$

In particular, when $b_1=-1, -3, -5, \dots$, one of u_1, u_2 is singular only on $z_0=0$. This means that theorem 2 and corollaries 1 and 2 are not valid without assumption II—ii).

See [3] for other examples which satisfy assumptions I and II.

Remark 2. Let P be the operator in (2.3). J. Urabe [11] and C. Wagschal [12] considered the non-characteristic Cauchy problem

$$Pu = 0, \quad (D_0^i u)(0, \hat{z}) = v_i(\hat{z}), \quad i = 0, 1 \tag{2.6}$$

where $v_i(\hat{z})$ are singular on $z_1=0$ (meromorphic data are considered in [11] for more general operators, and so are ramified data in [12] when $m=1$). They proved there exists a neighborhood $U=\{z; |z_i|<r \text{ for every } i\}$ with $0<r<1$ and a holomorphic solution $u(z)$ to (2.6) on the universal covering space of $U\setminus(K_1\cup K_2)$, where $K_1=\{z_1=\frac{1}{2}z_0^2\}$ and $K_2=\{z_1=-\frac{1}{2}z_1^2\}$. By corollary 2 we can see, if $b_1(0)\neq \pm1, \pm3, \pm5, \dots$, the solution $u(z)$ is nowhere holomorphic on $K_1\cup K_2$, that is, the singularities of $v_i(\hat{z})$ surely propagate onto both K_1 and K_2.

3. Proof of Theorem 1

Modifying some points, we follow the proof of Theorem 7.1 in [3]. In parallel with $z=(z_0, \hat{z})=(z_0, z', z'')$, we denote $\alpha=(\alpha_0, \hat{\alpha})=(\alpha_0, \alpha', \alpha'')$ for multi-indices, too. We write

$$P(z, D) = \sum_{j=0}^{m} \sum_{|\hat{\alpha}|\leqq j} a_{m-j,\hat{\alpha}}(z) D_0^{m-j} D^{\hat{\alpha}}, \tag{3.1}$$

where $D^{\hat{\alpha}}=D_1^{\alpha_1}\dots D_n^{\alpha_n}$, and $|\hat{\alpha}|=\alpha_1+\dots+\alpha_n$. We assume all the coefficients are holomorphic in $\{z; |z|<r_0\}$, $r_0>0$, where $|z|=\max_{0\leqq i\leqq n}|z_i|$.

3.1. It follows from i) and ii) of assumption II that there is a positive constant δ such that

$$\left|\sum_{i=1}^{k} \lambda_i \beta_i+\mu\right| \geqq \delta\left(\sum_{i=1}^{k} \beta_i+1\right) \quad \text{for every} \quad \beta\in\mathbf{N}^{n+1}. \tag{3.2}$$

3.2. It follows from assumption I and iii) of assumption II that

$$a_{m,0} \equiv 0, \tag{3.3}$$

$$a_{m-j,\hat{\alpha}} = \begin{cases} O\left(\sum_{i=1}^{h} |z_i|^2 + \sum_{i=h+1}^{k} |z_i|\right), & \text{when } \alpha' = 0 \\ O\left(\sum_{i=1}^{k} |z_i|\right), & \text{when } |\alpha'| = 1 \end{cases}$$

as $(z_1, \ldots, z_k) \to 0$, for $j \geqq 1$, $|\hat{\alpha}| = j$.

3.3. By a suitable linear transform of the $\hat{z}$ variable, we may suppose the matrix M defined in (1.1) has the form

$$M = \begin{pmatrix} \lambda_1 & m_{ij} & & 0 & & \\ & \ddots & & \vdots & & \\ & & & 0 & & \\ & & \lambda_k & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}, \tag{3.4}$$

where the unspecified components are all zero; besides we may suppose

$$|m_{ij}| \leqq \delta_1^{j-i}, \quad \text{for} \quad 1 \leqq i < j \leqq k \tag{3.5}$$

with $\delta_1 = \min\{1/2, \delta/8n\}$, where δ is the constant appearing in (3.2). See [3], proposition 4.1.

3.4. We define

$$T = \sum_{i=1}^{k} \lambda_i z_i D_i + \mu, \quad t(\beta) = \sum_{i=1}^{k} \lambda_i \beta_i + \mu. \tag{3.6}$$

Let τ be a constant, $v = \Sigma v_{p\beta}(z_0-\tau)^p z^\beta / p!\,\beta!$, and $w = \Sigma w_{p\beta}(z_0-\tau)^p z^\beta / p!\,\beta!$, where $z^\beta = z_1^{\beta_1} \ldots z_n^{\beta_n}$ and $\beta! = \beta_1! \ldots \beta_n!$. Then, if $w = Tv$, we have $w_{n\beta} = t(\beta) v_{p\beta}$. Because of (3.2), we can define $w = T^{-1}v$ by $w_{p\beta} = v_{p\beta}/t(\beta)$. If v is holomorphic at $(\tau, 0)$, so are Tv and $T^{-1}v$.

3.5. We reduce the Cauchy problem (1.3) to a system of integro-differential equations. Let T be as in (3.6). We define

$$\begin{aligned} a_1 &= T - \textstyle\sum_{|\hat{\alpha}| \leqq 1} a_{m-1,\hat{\alpha}}(z) D^{\hat{\alpha}}, \\ a_j &= -\textstyle\sum_{|\hat{\alpha}| \leqq j} a_{m-j,\hat{\alpha}}(z) D^{\hat{\alpha}}, \quad j = 2, \ldots, m, \\ u_j &= D_0^j u, \quad j = 0, 1, \ldots, m-1, \\ D_0^{-1} u &= \int_\tau^{z_0} u(t, \hat{z})\, dt. \end{aligned} \tag{3.7}$$

We suppose $v=0$ in the Cauchy problem (1.3). Then $u = O((z_0-\tau)^{m-1})$ as $z_0 \to \tau$. It follows from this that

$$u_j = D_0^{-1} u_{j+1}, \quad j = 0, 1, \ldots, m-2. \tag{3.8}$$

Put

(3.9)
$$\mathbf{u} = {}^t(u_0, \ldots, u_{m-1}),$$
$$\mathbf{f} = {}^t(0, \ldots, 0, T^{-1}f),$$
$$A = \begin{pmatrix} 0 & D_0^{-1} & & & \\ & 0 & D_0^{-1} & & \\ & & \ddots & \ddots & \\ & & & 0 & D_0^{-1} \\ T^{-1}a_m & \cdots & \cdots & \cdots & T^{-1}a_1 \end{pmatrix}.$$

Then the Cauchy problem (1.3) with $v=0$ is reduced to the equation

(3.10)
$$\mathbf{u} = A\mathbf{u} + \mathbf{f}.$$

Conversely, if $\mathbf{u}={}^t(u_0, \ldots, u_{m-1})$ is a solution to (3.10) with $\mathbf{f}={}^t(0, \ldots, 0, T^{-1}f)$, then $u=u_0$ gives a solution to the Cauchy problem (1.3) with $v=0$.

3.6. We introduce a Banach space composed of holomorphic functions, in which we consider the equation (3.10). Let $\tau\in\mathbf{C}$, $1\leqq R_1\leqq R_2\leqq R_3$ and $s=0, 1, 2, \ldots$. Denote by $H_s^{(\tau)}(R_1, R_2, R_3)$ or simply by $H_s^{(\tau)}$ the set of power series

$$v = \Sigma v_{p\beta}^{\tau}(z_0-\tau)^p z^\beta/p!\,\hat{\beta}!$$

which satisfy

(3.11)
$$\sup_{p,\beta} |v_{p\beta}^{\tau}|/[|\beta'|, |\beta''|+p+s]!\, R_1^{|\beta'|} R_2^{|\beta''|} R_3^{p+s} < \infty,$$

where we use the notation

(3.12)
$$[k, n]! = k!(k+2)(k+4)\ldots(k+2n).$$

Define the norm $\|v\|_s^{(\tau)}=\|v\|_{s;R_1,R_2,R_3}^{(\tau)}$ by the left-hand side of (3.11). Then the space $H_s^{(\tau)}(R_1, R_2, R_3)$ is a Banach space.

Next, let $0<\sigma<1$ and denote by $\mathbf{H}^{(\tau)}(R_1, R_2, R_3, \sigma)$ or simply by $\mathbf{H}^{(\tau)}$ the set of all $\mathbf{u}={}^t(u_0, \ldots, u_{m-1})$ with $u_s\in H_s^{(\tau)}(R_1, R_2, R_3)$, defining the norm

(3.13)
$$|||\mathbf{u}|||^{(\tau)} = |||\mathbf{u}|||_{R_1,R_2,R_3,\sigma}^{(\tau)} = \max_{0\leqq s\leqq m-1} \|u_s\|_{s;R_1,R_2,R_3}^{(\tau)}\sigma^{-s}.$$

This space is a Banach space as well.

We consider the equation (3.10) in the space $\mathbf{H}^{(\tau)}(R_1, R_2, R_3, \sigma)$. If the operator norm of A is less than 1, we see, by the contraction principle, that the equation (3.10) has one and only one solution. So, in what follows, we will try to estimate the operator norm of A.

3.7. The following 1)～12) hold:

1) $\|T^{-1}D^{\alpha'}v\|_s^{(\tau)} \leqq (R_1^2/\delta R_3)\|v\|_{s-1}^{(\tau)}$, for $|\alpha'| = 2$

2) $\|T^{-1}z^{\gamma'}v\|_s^{(\tau)} \leqq (R_3/\delta R_1^2)\|v\|_{s+1}^{(\tau)}$, for $|\gamma'|=2$

3) $\|T^{-1}z_jD_iv\|_s^{(\tau)} \leqq (1/\delta)\|v\|_s^{(\tau)}$, for $i, j=1, \dots, h$

4) $\|T^{-1}z_jv\|_s^{(\tau)} \leqq (R_3/\delta R_2)\|v\|_{s+1}^{(\tau)}$, for $j=h+1, \dots, k$

5) $\|T^{-1}v\|_s^{(\tau)} \leqq (1/\delta)\|v\|_s^{(\tau)}$

6) $\|D^{\hat\alpha}v\|_s^{(\tau)} \leqq (R_1^{|\alpha'|}R_2^{|\alpha''|}/R_3^{|\hat\alpha|})\|v\|_{s-|\hat\alpha|}^{(\tau)}$, for every $\hat\alpha$

7) $\|z^{\hat\gamma}v\|_s^{(\tau)} \leqq (1/R_1^{|\gamma'|}R_2^{|\gamma''|})\|v\|_s^{(\tau)}$, for every $\hat\gamma$

8) $\|D_0v\|_s^{(\tau)} \leqq \|v\|_{s-1}^{(\tau)}$

9) $\|D_0^{-1}v\|_s^{(\tau)} = \|v\|_{s+1}^{(\tau)}$

10) $\|(z_0-\tau)v\|_s^{(\tau)} \leqq (1/R_3)\|v\|_s^{(\tau)}$

11) $\|v\|_s^{(\tau)} \leqq (1/2sR_3)\|v\|_{s-1}^{(\tau)}$

12) Let $a(z)=\sum a_{p\beta}^\tau(z_0-\tau)^pz^\beta/p!\hat\beta!$,

and suppose that with some positive constants A_0 and R_0,

$$|a_{p\beta}^\tau| \leqq A_0R_0^{p+|\beta|}p!\hat\beta! \quad \text{for every} \quad p, \hat\beta. \tag{3.14}$$

Then, if $R_0<R_1\leqq R_2\leqq R_3$, it holds that

$$\|av\|_s^{(\tau)} \leqq A_0\theta\|v\|_s^{(\tau)}, \quad \text{with} \quad \theta=\prod_{i=1}^3 R_i/(R_i-R_0).$$

We can prove these properties in an elementary way by consulting the proofs of Propositions 5.1, 5.2 and 5.3 in [3]. So we leave it to the reader.

3.8. Now we estimate the operator norm of A. First, put

$$m_{ij}^\tau = D_j a_{m-1,e_i}(\tau, 0), \quad i, j=1, 2, \dots, n, \tag{3.15}$$

where e_i is the i-th unit multi-index. It follows from assumption II—iii) that $m_{ij}^\tau=0$ for $k+1\leqq j\leqq n$ and for $h+1\leqq i\leqq n$, $1\leqq j\leqq h$. Besides, by the same assumption, we can write

$$\begin{aligned} a_1u_{m-1} &= \sum\nolimits^{(1)}(m_{ij}^0-m_{ij}^\tau)z_jD_iu_{m-1}-\sum\nolimits_{1\leqq i<j\leqq k}m_{ij}^0z_jD_iu_{m-1} \\ &\quad+\sum\nolimits^{(1)}z_jD_i\sum\nolimits_{0\leqq l\leqq n}(z_l-\tau_l)f_{ijl}^\tau u_{m-1} \\ &\quad+\sum\nolimits^{(2)}z^{\gamma'}D_i(f_{i\gamma'}^\tau u_{m-1})+\sum\nolimits_{0\leqq l\leqq n}(z_l-\tau_l)g_l^\tau u_{m-1} \\ &= \{a_{11}+a_{12}+a_{13}+a_{14}+a_{15}\}u_{m-1}, \end{aligned} \tag{3.16}$$

where $\tau_0=\tau$, $\tau_l=0$ for $1\leqq l\leqq n$; $\Sigma^{(1)}$ denotes the sum for $1\leqq i, j\leqq h$ and for $1\leqq i\leqq n$, $h+1\leqq j\leqq k$ and so does $\Sigma^{(2)}$ for $|\gamma'|=2$, $h+1\leqq i\leqq n$; taking $t_0>0$ sufficiently small and A_0, R_0 sufficiently large if necessary, we may suppose that the Taylor expansions of f_{ijl}^τ, $f_{i\gamma'}^\tau$ and g_l^τ about $(\tau, 0)$ all satisfy (3.14) for every $|\tau|<t_0$.

We estimate each term by making use of 1)$\sim$12). There is a positive constant M_1 such that $|m_{ij}^{\tau}-m_{ij}^{0}|\leqq M_1|\tau|$ for every $|\tau|\leqq t_0$, and so, by 3), 4) and 6), we have

$$\|T^{-1}a_{11}u_{m-1}\|_{m-1}^{(\tau)} \leqq (M_2|\tau|/\delta)\|u_{m-1}\|_{m-1}^{(\tau)},$$

where $M_2=n^2M_1$. By (3.5), 3), 4) and 6), we have

$$\|T^{-1}a_{12}u_{m-1}\|_{m-1}^{(\tau)} \leqq (1/4)\|u_{m-1}\|_{m-1}^{(\tau)}.$$

By 3), 4), 6), 7), 10) and 12), we have

$$\|T^{-1}a_{13}u_{m-1}\|_{m-1}^{(\tau)} \leqq (A_0\theta C_1/\delta R_1)\|u_{m-1}\|_{m-1}^{(\tau)}.$$

Here and hereafter C_i denotes a positive constant depending only on m and n. In the same way, we have

$$\|T^{-1}a_{14}u_{m-1}\|_{m-1}^{(\tau)} \leqq (A_0\theta C_2R_2/\delta R_1^2)\|u_{m-1}\|_{m-1}^{(\tau)},$$

$$\|T^{-1}a_{15}u_{m-1}\|_{m-1}^{(\tau)} \leqq (A_0\theta C_3/\delta R_1)\|u_{m-1}\|_{m-1}^{(\tau)}.$$

We thus obtain

$$(3.17)\qquad \|T^{-1}a_1u_{m-1}\|_{m-1}^{(\tau)} \leqq \left\{\frac{M_2t_0}{\delta}+\frac{1}{4}+\frac{A_0\theta C_2R_2}{\delta R_1^2}+\frac{A_0\theta C_4}{\delta R_1}\right\}\|u_{m-1}\|_{m-1}^{(\tau)}$$

for every $|\tau|<t_0$ and every $u_{m-1}\in H_{m-1}^{(\tau)}(R_1, R_2, R_3)$.

3.9. By assumption II—iii), we can write, for $j\geqq 2$,

$$(3.18)\qquad \begin{aligned} a_ju_{m-j} &= \sum\nolimits^{(3)} z^{\hat\gamma}D^{\hat\alpha}(f_{j\hat\alpha\hat\gamma}u_{m-j})+\sum\nolimits^{(4)} z_lD^{\hat\alpha}(f_{j\hat\alpha l}u_{m-j}) \\ &\quad +\sum\nolimits^{(5)}D^{\hat\alpha}(f_{j\hat\alpha}u_{m-j})+\sum\nolimits_{|\hat\alpha|<j}g_{j\hat\alpha}D^{\hat\alpha}u_{m-j} \\ &= \{a_{j1}+a_{j2}+a_{j3}+a_{j4}\}u_{m-j}, \end{aligned}$$

where $\Sigma^{(3)}$ denotes the sum for $|\hat\alpha|=j$ with $\alpha'=0$, $\hat\gamma=(\gamma', 0)$ with $|\gamma'|=2$ or $\gamma=e_j$ with $h+1\leqq j\leqq k$; so does $\Sigma^{(4)}$ for $|\hat\alpha|=j$ with $|\alpha'|=1$, $1\leqq l\leqq k$; and $\Sigma^{(5)}$ for $|\hat\alpha|=j$ with $|\alpha'|\geqq 2$. Taking t_0 smaller and A_0 and R_0 larger if necessary, we may suppose the Taylor expansions of $f_{j\hat\alpha\hat\gamma}$, $f_{j\hat\alpha l}$, $f_{j\hat\alpha}$ and $g_{j\hat\alpha}$ all satisfy (3.14) for every $|\tau|<t_0$.

If $R_2\leqq R_1^2$, then by 2), 4), 6) and 12) we have

$$\|T^{-1}a_{j1}u_{m-j}\|_{m-1}^{(\tau)} \leqq (A_0\theta C_5R_2^{j-1}/\delta R_3^{j-1})\|u_{m-j}\|_{m-j}^{(\tau)}.$$

In the same way, we have

$$\|T^{-1}a_{j2}u_{m-j}\|_{m-1}^{(\tau)} \leqq (A_0\theta C_6R_2^{j-1}/\delta R_3^{j-1})\|u_{m-j}\|_{m-j}^{(\tau)},$$

$$\|T^{-1}a_{j3}u_{m-j}\|_{m-1}^{(\tau)} \leqq (A_0\theta C_7R_1^2R_2^{j-2}/\delta R_3^{j-1})\|u_{m-j}\|_{m-j}^{(\tau)},$$

$$\|T^{-1}a_{j4}u_{m-j}\|_{m-1}^{(\tau)} \leqq (A_0\theta C_8R_2^{j-1}/\delta R_3^{j-1})\|u_{m-j}\|_{m-j}^{(\tau)}.$$

Hence for $j\geqq 2$ we obtain

(3.19) $$\|T^{-1}a_j u_{m-j}\|_{m-1}^{(\tau)}\sigma^{-m+1}$$
$$\leqq \frac{A_0\theta(C_7R_1^2+C_9R_2)}{\sigma\delta R_3}(R_2/\sigma R_3)^{j-2}\|u_{m-j}\|_{m-j}^{(\tau)}\sigma^{-m+j}.$$

3.10. By 9), we have

(3.20) $$\|D_0^{-1}u_{j+1}\|_j^{(\tau)}\sigma^{-j}=\sigma\|u_{j+1}\|_{j+1}^{(\tau)}\sigma^{-j-1}.$$

Putting (3.17), (3.19) and (3.20) together, we can get an estimate of the operator A: Set

(3.21) $$R_1=R,\quad R_2=\varrho_2R^2,\quad R_3=\varrho_3R^2.$$

If σ, R_0', ϱ_2 and ϱ_3 are constants satisfying

(3.22) $$0<\sigma<1,\quad R_0'>R_0,\quad R_0'^{-1}<\varrho_2<1<\varrho_3,\quad \sigma\varrho_3>\varrho_2,$$

then for every $R>R_0'$ and $|\tau|<t_0$ it holds that

(3.23) $$|||A\mathbf{u}|||^{(\tau)}\leqq\varepsilon|||\mathbf{u}|||^{(\tau)}\quad\text{for}\quad \mathbf{u}\in\mathbf{H}^{(\tau)}(R,\varrho_2R^2,\varrho_3R^2,\sigma),$$

with $\varepsilon=\max\left\{\sigma,\frac{M_2t_0}{\delta}+\frac{1}{4}+\frac{A_0\theta}{\delta}\left(C_2\varrho_2+\frac{C_4}{R_0'}+\frac{C_7+C_9\varrho_2}{\sigma\varrho_3-\varrho_2}\right)\right\}$.

We can take t_0, R_0', ϱ_2, ϱ_3 and σ so that $\frac{1}{4}<\varepsilon<1$. Hence, by the contraction principle, we see that equation (3.10) has a unique solution in the space $\mathbf{H}^{(\tau)}(R,\varrho_2R^2,\varrho_3R^2,\sigma)$ for every $|\tau|<t_0$ and every $R>R_0'$.

3.11. Theorem 1 is proved as follows. Let t_0, R_0', ϱ_2, ϱ_3 and σ be such as stated above. Suppose $f(z)$ is holomorphic in $\{z;|z_0-\tau|<r^2,|z'|<r,|z''|<r^2\}$ with $r>0$. Taking c_1 so that $0<c_1<\sqrt{\varrho_2}$, put $R=(c_1r)^{-1}$; then $r^{-1}<\sqrt{\varrho_2}R<R<\sqrt{\varrho_3}R$. We easily see that $f\in H_{m-1}^{(\tau)}(R,\varrho_2R^2,\varrho_3R^2)$, and besides

$$\mathbf{f}={}^t(0,\dots,0,T^{-1}f)\in\mathbf{H}^{(\tau)}(R,\varrho_2R^2,\varrho_3R^2,\sigma).$$

If $R>R_0'$ and $|\tau|<t_0$, then the equation (3.10) has a unique solution

$$\mathbf{u}={}^t(u_0,\dots,u_{m-1})\in\mathbf{H}^{(\tau)}(R,\varrho_2R^2,\varrho_3R^2,\sigma).$$

As noted in the paragraph 3.5, $u=u_0$ gives a solution to the Cauchy problem (1.3) with $v=0$. From the definition, the Taylor expansion $u=\Sigma u_{p\beta}^\tau(z_0-\tau)^pz^\beta/p!\beta!$ satisfies

$$|u_{p\beta}^\tau|\leqq\|u\|_0^{(\tau)}[|\beta'|,|\beta''|+p]!R^{|\beta'|}(\varrho_2R^2)^{|\beta''|}(\varrho_3R^2)^p$$
$$\leqq\|u\|_0^{(\tau)}(p+|\beta|)!(1/c_1r)^{|\beta'|}(2\varrho_2/c_1^2r^2)^{|\beta''|}(2\varrho_3/c_1^2r^2)^p.$$

Hence u is holomorphic in

$$\frac{|z_1|+\ldots+|z_h|}{c_1 r}+\frac{2\varrho_2(|z_{h+1}|+\ldots+|z_n|)}{c_1^2 r^2}+\frac{2\varrho_3|z_0-\tau|}{c_1^2 r^2}<1.$$

Take $c>0$ so that $(hc/c_1)+(2\varrho_2(n-h)c^2/c_1^2)+(2\varrho_3 c^2/c_1^2)\leqq 1$. Then u is holomorphic in

$$\{z;\ |z_0-\tau|<(cr)^2,\ |z'|<cr,\ |z''|<(cr)^2\}.$$

This completes the proof of Theorem 1.

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