A remark on a theorem by F. Forstneriĉ

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Let B_n be the unit ball in $\mathbf{C}^n$ and bB_n be the unit sphere. Let γ be a simple closed curve of class $\mathscr{C}^2$ in bB_n. Assume that γ is complex tangential at one point at least (this means $\langle\dot{\gamma}, \gamma\rangle = 0$, where $\langle\ ,\ \rangle$ is the *hermitian* scalar product). F. Forstnerič has proved that then γ is polynomially convex [2]. This result has been used recently in [1].

The aim of this note is to present an extremely simple proof of this result (part I). Then in part II, we give a counterexample to show that the result fails to be true if the curve is assumed to be only of class $\mathscr{C}^1$ (and fails to be $\mathscr{C}^2$ just at one point, where a complex tangency occurs).

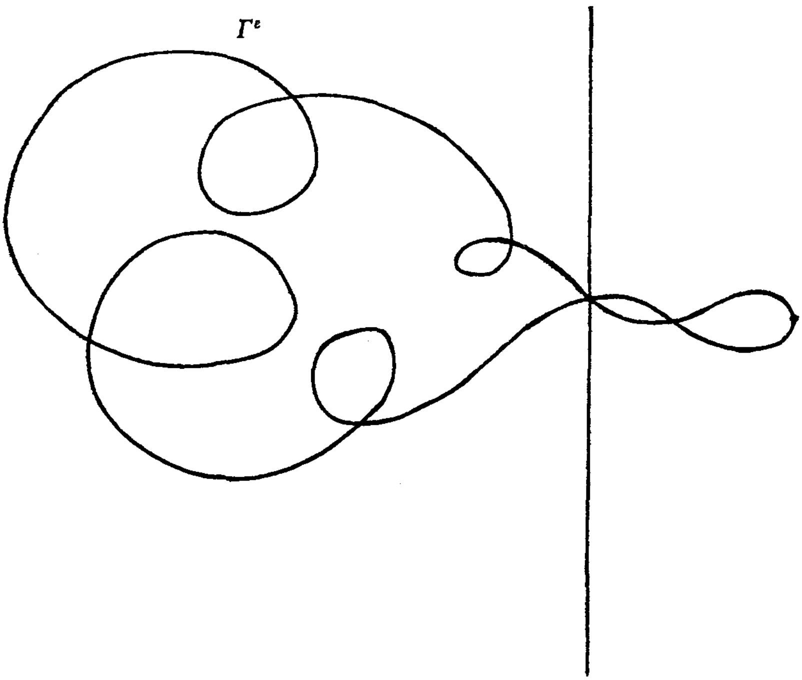
I. Proof of Forstnerič's result

We can assume that the curve γ is parametrized by $[-1, +1]$; $\gamma(1) = \gamma(-1)$, $\gamma(t) = (\gamma_1(t), \dots, \gamma_n(t))$, $\gamma(0) = (1, 0, \dots, 0)$ and $\dot{\gamma}(0) = (0, 1, 0, \dots, 0)$. In fact, we need only to assume that γ is $\mathscr{C}^1$, and $\mathscr{C}^2$ in a neighborhood of $t=0$. The result of Forstnerič is an easy consequence of the following totally elementary geometric fact:

Proposition. *For* $\varepsilon \in \mathbf{R}$, *and* $-1 \leqq t \leqq +1$, *set* $\Gamma^\varepsilon(t) = \gamma_1(t) + i\varepsilon\gamma_2(t)$ *(abusively* Γ^ε *will also denote the "geometric" image, which is the projection of* γ *under the map* $(z_1, z_2) \to z_1 + i\varepsilon z_2$*). There exists* $\varepsilon \in \mathbf{R}$, $|\varepsilon|$ *arbitrarily small,* $t_0 \in (0, 1)$ *and* $h > 0$ *such that:*

$$\begin{cases} \text{for } t_0 \leqq |t| \leqq 1 & \operatorname{Re}\Gamma^\varepsilon(t) \leqq 1-h \\ \text{for } -t_0 \leqq t < 0 & \frac{d}{dt}(\operatorname{Re}\Gamma^\varepsilon(t)) > 0 \\ \text{for } 0 < t \leqq t_0 & \frac{d}{dt}(\operatorname{Re}\Gamma^\varepsilon(t)) < 0 \\ \text{the intersection of the line } \operatorname{Re} z = 1-h \text{ and } \Gamma^\varepsilon \text{ consists of only one point.} \end{cases}$$

* Research supported in part by NSF grant DMS 8800610.



The last assertion is the important one. Using arguments from the theory of function algebras (see [3] or [5]), the result of Forstnerič can be deduced from the proposition in the following way. The curve γ can be decomposed into the union of two arcs γ' and γ'', which we take to be the intersection of γ with respectively the half spaces $\{\operatorname{Re}(z_1+i\varepsilon z_2)\geqq 1-h\}$ and $\{\operatorname{Re}(z_1+i\varepsilon z_2)\leqq 1-h\}$. It is immediate (using polynomials in $z_1+i\varepsilon z_2$) that γ' and γ'' are peak sets for $P(\gamma)$ (the closed subalgebra of $C(\gamma)$ generated by polynomials). Since γ' and γ'' are smooth arcs, one has $P(\gamma')=C(\gamma')$, and $P(\gamma'')=C(\gamma'')$ (see remark below). Then, given any $f\in C(\gamma)$ and $\varepsilon>0$, there exist polynomials P_1 and P_2 such that $|P_1-f|\leqq\frac{\varepsilon}{4}$ on γ' and $|P_2-f|\leqq\frac{\varepsilon}{4}$ on γ''. If $\chi\in P(\gamma)$ is a function which peaks on γ' (i.e. $\chi=1$ on γ', $|\chi|<1$ on $\gamma-\gamma'$), then for n large enough $\left|(\chi^n P_1+(1-\chi^n)P_2)-f\right|\leqq\varepsilon$ on γ. This shows that $P(\gamma)=C(\gamma)$, and therefore that γ is polynomially convex. In the last step, we have just rewritten a proof that the union of two peak interpolation sets is a (peak) interpolation set.

Proof of the proposition. One has

$$\gamma_2(t) = t+O(t^2) \quad\text{and}\quad \gamma_1(t) = 1+\left(-\frac{1}{2}+i\frac{a}{2}\right)t^2+o(t^2),$$

where $a=\frac{d^2}{dt^2}(\arg\gamma_1)|_{t=0}$. Therefore $\operatorname{Re}\Gamma^\varepsilon(t)=1-\frac{t^2}{2}+\varepsilon O(t^2)+o(t^2)$ and $\operatorname{Im}\Gamma^\varepsilon(t)=\varepsilon t+\dots$.

There exist $\varepsilon_0>0$ and $t_0\in(0, 1)$ so that for $|\varepsilon|\leqq\varepsilon_0$ and $|t|\leqq t_0$: $\left|t+\frac{d}{dt}(\operatorname{Re}\Gamma^\varepsilon)\right|\leqq\frac{|t|}{2}$. This already shows that for $|\varepsilon|\leqq\varepsilon_0$, $\operatorname{Re}\Gamma^\varepsilon$ in increasing on $[-t_0,0)$ and decreasing on $(0,t_0]$. Fix $h_0>0$ so that $\operatorname{Re}\gamma_1(t)<1-2h_0$ if $|t|\geqq t_0$. If $|\varepsilon|<h_0$ then $\operatorname{Re}\gamma_1(t)+i\varepsilon\gamma_2(t)<1-h_0$, if $|t|\geqq t_0$.

Set $\varepsilon_1=\min(\varepsilon_0, h_0)$. For any $h\in(0, h_0)$, and $|\varepsilon|\leqq\varepsilon_1$, the arc $\Gamma^\varepsilon([-t_0, 0))$ (resp. $\Gamma^\varepsilon(0, t_0])$ intersects the line $\operatorname{Re} z=1-h$ at only one point which we denote by $p^-(\varepsilon)$ (resp. $p^+(\varepsilon)$). Fix h small enough in order that $p^-(\varepsilon_1)<0<p^+(\varepsilon_1)$ and $p^-(-\varepsilon_1)>0>p^+(-\varepsilon_1)$, which is possible due to the fact that $\frac{d}{dt}(\operatorname{Im}\Gamma^\varepsilon)|_{t=0}=\varepsilon$.

By continuity, for some $\varepsilon\in[-\varepsilon_1+\varepsilon_1]$, $p^+(\varepsilon)=p^-(\varepsilon)$. Q.E.D.

Remark. Since γ', and γ'' are arcs *in the sphere,* one can prove in an elementary way that polynomials are dense in the space of continuous functions on γ', and γ'', without resorting to Stolzenberg's theorem. I do not know how much this has been noticed. F. Forstnerič and I came to notice it in a discussion, which is partly the origin of this paper. Here are some indications. Let Λ be a $\mathscr{C}^2$ arc in the unit sphere in $\mathbf{C}^n$, with $(1, 0, \dots, 0)$ as an end point. For $h>0$, let Λ' and Λ'' be respectively the intersection of Λ with the half spaces $\operatorname{Re} z_1\geqq 1-h$ and, $\operatorname{Re} z_1\leqq 1-h$. If h is small enough, the projection of Λ' in the z_1 plane is a simple arc. By using polynomials in z_1, one sees that Λ' is a peak interpolation set for $P(\Lambda)$. So, to show that $P(\Lambda)=C(\Lambda)$, it is enough to show that $P(\Lambda'')=C(\Lambda'')$ (the proof has essentially been given above). The problem has thus been reduced to the smaller arc Λ''. And the proof can be completed after a finite number of such steps. If the curve is only $\mathscr{C}^1$, the same proof works, if one replaces the region $\operatorname{Re} z_1>1-h$ by the region defined by: $|z_1-1|<1$ or $|z_1|>1$.

II. An example

1) Set $E=\left\{z\in\mathbf{C},\ |z|\leqq\frac{1}{2e},\ \operatorname{Im} z\geqq 0\right\}$. For $z\in E$, set $f(z)=z\log\frac{1}{z}$ where we take $\log\frac{1}{z}=\log\frac{1}{|z|}+i\theta$ with $-\pi\leqq\theta\leqq 0$, and $f(0)=0$. The function f is $(1-1)$ on E. Indeed $f'(z)=\log\frac{1}{z}-1$, so $\operatorname{Re} f'>0$ and we can apply the result in [4], page 294, exercise 12. The function f is $\mathscr{C}^1$ except at 0. However f maps $\left(-\frac{1}{2e}, \frac{1}{2e}\right)$ to a $\mathscr{C}^1$

curve. The positive real axis is mapped into itself, while the negative real axis is mapped into the curve $x=\frac{y}{\pi}\log\left(\frac{y}{\pi}\right)$. The inverse map f^{-1} is $\mathscr{C}^1$ on $f(E)$ and at 0: $(f^{-1})'(0)=0$. Finally notice that although f is not $\mathscr{C}^1$ on E, $|f|^2$ is $\mathscr{C}^1$ and even $\mathscr{C}^{1+\varepsilon}$ for every $\varepsilon\in(0, 1)$.

Composing f with a conformal mapping from Δ, the open unit disc in $\mathbf{C}$, onto a smooth domain in E whose boundary contains $\left[-\frac{1}{4e}, \frac{1}{4e}\right]$, we get therefore the following lemma.

Lemma. *There exists F, a $1-1$ continuous map from $\bar{\Delta}$ into* $\mathbf{C}$, *with the following properties:*

1) *F is holomorphic on Δ, $F(1)=0$, $|F|<1$;*
2) *F is smooth on $\bar{\Delta}-\{1\}$;*
3) *$|F|^2$ is of class $\mathscr{C}^{1+\varepsilon}$ on $\bar{\Delta}$ (for any $\varepsilon\in(0, 1)$);*
4) *$F(b\Delta)$ is a $\mathscr{C}^1$ curve ($\mathscr{C}^\infty$ except at 0);*
5) *F^{-1} is C^1 on $F(\bar{\Delta})$ and $(F^{-1})'(0)=0$.*

There is no claim that this lemma is original.

2) *Construction of the curve.*

Take F as in the lemma. Set $\Omega=F(\Delta)$. There exists G, a $\mathscr{C}^1$ function on $\bar{\Delta}$, holomorphic on Δ, such that on the unit circle $b\Delta$: $|G|=\sqrt{1-|F|^2}$, since $\sqrt{1-|F|^2}$ is of class $\mathscr{C}^{1+\varepsilon}$. We can impose $G(0)=1$. The map $z\mapsto\psi(z)=(z, G_0F^{-1}(z))$ is a $\mathscr{C}^1$ map from $\bar{\Omega}$ into the unit ball which maps $b\Omega$ to a $\mathscr{C}^1$ curve γ in the unit sphere in $\mathbf{C}^2$. This curve is not polynomially convex since its polynomial hull contains $\psi(\Omega)$. However $\psi(0)=(0, 1)$, and since $\psi'(0)=(1, 0)$ (due to $(F^{-1})'(0)=0$), the curve is complex tangential at the point $(0, 1)$.

References

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Received Oct. 11, 1989

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