Extreme operator-valued continuous maps

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Abstract. Let $\mathscr{L}(H)$ denote the space of operators on a Hilbert space H. We show that the extreme points of the unit ball of the space of continuous functions $C(K, \mathscr{L}(H))$ (K-compact Hausdorff) are precisely the functions with extremal values. We show also that these extreme points are (a) strongly exposed if and only if $\dim H < \infty$ and $\operatorname{card} K < \infty$, (b) exposed if and only if H is separable and K carries a strictly positive measure.

1. Introduction

Let $C(K, X)$ denote the Banach space of all continuous functions from a compact Hausdorff space K to a Banach space X equipped with the supremum norm $\|f\| = \sup_{k \in K} \|f(k)\|$. By $B(X)$ we denote the unit ball of the Banach space X.

There is a natural conjecture about extreme points:

$$(*) \qquad f \in \operatorname{ext} B(C(K, X)) \quad \text{if and only if} \quad f(k) \in \operatorname{ext} B(X) \quad \text{for all} \quad k \in K.$$

Obviously the "if part" in $(*)$ always holds. This conjecture has been proved to be true under various additional assumptions. We list some of the known results. The conjecture is true if:

1) X is strictly convex,
2) $B(X)$ is polytope,
3) $X = (C(K_1))^* = M(K_1)$, where K_1 is a compact Hausdorff space ([31], Theorem 4, 5). Note that in this case the conjecture $(*)$ is equivalent to the fact that extreme operators in the unit ball of the space of compact operators $\mathscr{L}(C(K_1), C(K))$ are nice in the sense of Morris and Phelps. For other similar results on the space of weak* continuous functions from K into $(C(K_1))^*$ (this corresponds to the case of extreme point of the unit ball of the space of all bounded linear operators $\mathscr{L}(C(K_1), C(K))$ (see [6], p. 490)) under various assumptions on K and K_1 (see [2], [1], [34], [9], [20], see also [35], [36] for negative examples).
4) X has 3.2.I.P. ([37]).

5) $B(X)$ is stable (to get it we use (iii')) in [4] and Michael's selection Theorem [29]).
6) $X=L^1(\mu)$ ([39]).
7) $X=L^\varphi(\mu)$ is an Orlicz space equipped with the Luxemburg norm ([18]).
8) $X=M(K_1, Z)$ is the space of Z-valued regular Borel measures of finite variation, Z is a Banach space, K_1 is a compact Hausdorff space ([40]).

The conjecture $(*)$ is not true for every Banach space X. A negative example was given by Blumental, Lindenstrauss and Phelps for $X=R^4$ equipped with a suitable norm ([2]). The class of finite dimensional spaces for which conjecture $(*)$ holds was described by Papadopoulou ([32]). She proved that $B(X)$ ($\dim X<\infty$) is stable if and only if all k-skeletons ($k=0, 1, \ldots, n$) of $B(X)$ are closed (a k-skeleton of $B(X)$ is a set of all $x\in B(X)$ such that the face generated by x has dimension less than or equal to k).

After dealing with the conjecture for Banach spaces the next natural step is to consider X as a space of linear operators. We denote by $\mathscr{L}(Y)$ the space of all bounded linear operators from a Banach space Y into itself equipped with the operator norm.

For $X=\mathscr{L}(l^p)$, $1<p<\infty$, $p\neq 2$, the conjecture does not holds. Indeed, we have the following example. Let $(a_n)\in l^p$ be such that $\|(a_n)\|_p=1$ and $a_n>0$ for all n. We define $f\in C([0, 1], \mathscr{L}(l^p))$ by $f(k)=T_k$, $k\in[0, 1]$ where

$$T_k((x_n)) = x_1\left(\sqrt[p]{(1-k)}, \sqrt[p]{a_1^p+a_2^p}, \sqrt[p]{k}\cdot\sqrt[p]{a_1^p+a_2^p}, a_3, a_4, \ldots\right),$$

$(x_n)\in l^p$. We have for $p>2$, $T_k\in \operatorname{ext} B(\mathscr{L}(l^p))$ for $k\in(0, 1)$ and $T_k\notin \operatorname{ext} B(\mathscr{L}(l^p))$ for $k=0$ or 1 (see [15], see also [14]). Using adjoint operators an analogous example can be written for $p\in(1, 2)$.

In this note we consider the case $p=2$. It turns out that the conjecture $(*)$ holds for the space of operators acting on a Hilbert space (Section 2).

In Section 3 we consider under what conditions elements of $\operatorname{ext} B(C(K, \mathscr{L}(H)))$ are exposed or strongly exposed. We show that extreme points of $B(C(K, \mathscr{L}(H)))$ are
(a) strongly exposed if and only if $\dim H<\infty$ and $\operatorname{card} K<\infty$,
(b) exposed if and only if H is separable and K carries a strictly positive measure.

2. Extreme points in $B(C(K, (H)))$

Let H be a (real or complex) Hilbert space. We denote by $P_E\in\mathscr{L}(H)$ the orthogonal projection onto a subspace E of H. The set of extreme points $\operatorname{ext} B(\mathscr{L}(H))$ coincides with the set of all isometries and coisometries (see [23] for the complex case, and [16] for the real case). Our aim is to characterize $\operatorname{ext} B(K, \mathscr{L}(H))$. Note that the below presented Theorem was proved in [14, p. 314, $(**)$] in the case when H is finite dimensional.

In the finite dimensional case the dual of $\mathscr{L}(H)$ was also considered, and it turns out that the unit ball of $\mathscr{L}(H)^*$ is stable (see [17]), so the conjecture $(*)$ also holds for $C(K, B(\mathscr{L}(H)^*))$ $(\dim H<\infty)$.

Now we recall some facts we will use in the proof of Theorem 1 below.

Let $T\in\mathscr{L}(H)$ be a contraction. We put

$$M(T)=\{h\in H\colon \|Th\|=\|h\|\},$$

We have $T^*Th=h$ for $h\in M(T)$. Moreover $T(M(T))=M(T^*)$ and $T(M^\perp(T))\subset M^\perp(T^*)$.

Obviously, if $T\notin \operatorname{ext} B(\mathscr{L}(H))$ then neither $TT^*=I$ nor $T^*T=I$ i.e.

$$M^\perp(T)\neq\{0\}\neq M^\perp(T^*).$$

Put $S=((I+T^*T)/2)^{1/2}$. We have $I/2\leqq S\leqq I$. Therefore S^{-1}, $S^{1/2}$ and $S^{-1/2}$ exist. We have $0\leqq S^{1/2}\leqq I$. Hence

$$0\leqq S^{1/2}(2S^{1/2}-I)=2S-S^{1/2}\leqq I.$$

Since $T^*T\leqq S^2$ we have

$$\|Tx\|^2=\langle T^*Tx, x\rangle\leqq\langle S^2x, x\rangle=\|Sx\|^2.$$

So $\|TS^{-1}\|\leqq 1$. We get

$$\|TS^{-1/2}\|\leqq\|TS^{-1}\|\,\|S^{1/2}\|\leqq 1$$

and

$$\|T(2I-S^{-1/2})\|\leqq\|TS^{-1}\|\,\|2S-S^{1/2}\|\leqq 1.$$

Theorem 1. *Let H be a Hilbert space and let K be a compact Hausdorff space. Then*

$$f\in\operatorname{ext} B(C(K,\mathscr{L}(H)))$$

if and only if

$$f(k)\in\operatorname{ext} B(\mathscr{L}(H))\quad \text{for all}\quad k\in K.$$

Proof. Let $f\in B(C(K,\mathscr{L}(H)))$. Assume that $f(k_0)$ fails to be extremal for some $k_0\in K$, i.e.

$$M^\perp(f(k_0))\neq\{0\}\neq M^\perp(f(k_0)).$$

We need to prove that then

$$f\notin\operatorname{ext} B(C(K,\mathscr{L}(H))).$$

Put $T_k=f(k)$. We consider two cases.

1° There exist $k_0\in K$ and $x\in M^\perp(T_{k_0})$ such that $T_{k_0}x\neq 0$. Put $f_1(k)=T_kS_k^{-1/2}$, and $f_2(k)=2T_k-T_kS_k^{-1}S_k^{1/2}$, where $S_k=((I+T_k^*T_k)/2)^{1/2}$. Obviously

$$f_1, f_2\in B(C(K,\mathscr{L}(H)))\quad\text{and}\quad f=(f_1+f_2)/2.$$

Since

$$M(T_{k_0})=M(T_{k_0}^*T_{k_0})=M(S_{k_0}^2)=M(S_{k_0})=M(S_{k_0}^{1/2})$$

and $T_{k_0}^* T_{k_0} x \in M^\perp(S_{k_0}^{1/2})$ we have

$$\|S_{k_0}^{1/2} T_{k_0}^* T_{k_0} x\| < \|T_{k_0}^* T_{k_0} x\|.$$

Since $S_k^{1/2}$ and $T_k^* T_k$ commute, we get

$$T_{k_0}^* T_{k_0} S_{k_0}^{1/2} x \neq T_{k_0}^* T_{k_0} x, \quad \text{so} \quad T_{k_0} S_{k_0}^{1/2} x \neq T_{k_0} x.$$

Hence $T_{k_0} \neq T_{k_0} S_{k_0}^{-1/2}$ and $f_1(k_0) \neq f_2(k_0)$ i.e. f is not extreme.

$$2^\circ \qquad T_k(M^\perp(T_k)) = \{0\} = T_k^*(M^\perp(T_k^*)) \quad \text{for all} \quad k \in K.$$

Then $T_k^* T_k = P_{M(T_k)}$. Therefore $k \to P_{M(T_k)}$ is a continuous function, so $k \to P_{M^\perp(T_k)}$ is a continuous function, too. Fix $k_0 \in K$ such that $f(k_0) \notin \operatorname{ext} B(\mathscr{L}(H))$. Choose $e \in M^\perp(f(k_0))$ and $g \in M^\perp(f(k_0)^*)$ with $\|e\| = \|g\| = 1$. We have

$$\|T_k \pm P_{M^\perp(T_k^*)} g \otimes P_{M^\perp(T_k)} e\| \leqq 1.$$

Hence f is not extreme.

3. Exposed and strongly exposed points in $B(C(K, (H)))$

A point q_0 in a convex set Q of a (real or complex) Banach space E is said to be exposed if there exists a bounded $\mathbf{R}$-linear functional $\xi \colon E \to \mathbf{R}$ such that $\xi(q_0) > \xi(q)$ for all $q \in Q \setminus \{q_0\}$. An exposed point $q_0 \in Q$ is called strongly exposed if for any sequence $q_n \in Q$ the condition $\xi(q_n) \to \xi(q_0)$ implies $\|q_n - q_0\| \to 0$. Obviously each exposed point is extreme.

Note that extreme points of $B(\mathscr{L}(H))$ are strongly exposed in and only if H is a finite dimensional Hilbert space, and exposed but not strongly exposed if and only if H is separable infinite dimensional. Moreover there are no exposed points in $B(\mathscr{L}(H))$ if and only if H is not separable ([19]).

Consider the Bochner L^p-space $(1 < p < \infty)$. For $f \in L^p(\mu, X)$ (X is a Banach space), if $\|f\| = 1$ and $f(t)/\|f(t)\| \in \operatorname{ext} B(X)$ for $t \in \operatorname{supp} f$ μ-a.e., then

$$f \in \operatorname{ext} B(L^p(\mu, X)).$$

Generally the converse does not holds. A negative example was given by Greim [12] for a nonseparable Banach space X. But for all separable Banach spaces X this property characterizes extreme functions (see [38], [21], [22], [13]). For the strongly exposed points the analogous natural condition on the values of f are sufficient, whenever X is smooth ([10], see also [11]). Recently W. Kurtz considered strongly exposed points in Bochner—Orlicz spaces [25] (see also [26]). A compact Hausdorff space K is said to carry a strictly positive measure, if there exists a strictly positive Radon measure μ on X (i.e. $\mu(\mathscr{U}) > 0$ for all non-empty open subsets $\mathscr{U}$ of X).

Several authors have workes on the problem of the characterization of spaces X which carry a strictly positive measure. Maharam [28] has given necessary and sufficient conditions for the existence of strictly positive measures. Kelley considers strictly positive measures on Boolean algebras. In his work [24] he introduced the notion of the intersection number of a collection of subsets to give a characterization of spaces which carry a strictly positive measure.

Next it turns out that the countable chain condition is not sufficient for the existence of such a measure (see Gaifman [8]). Note that in the case of a compact Hausdorff space, the problem mentioned above is equivalent to the problem of existence of a finitely additive strictly positive measure. Rosenthal ([33], Th. 4.5b) proved that $C(X)$ carries a strictly positive functional if and only if its dual $C(X)^*$ contains a weakly compact total subset. Other results can be found in [7], [30], [3]. We refer the reader to [5, Chapter 6] for a survey of known results about strictly positive measures. In fact, we can consider a strictly positive measure on X as a functional on $C(X)$ which exposes the function 1.

Theorem 2. *If a Hilbert space H is separable and a Hausdorff compact space K carries a strictly positive measure, then each extreme point of $B(C(K, \mathscr{L}(H)))$ is exposed.*

If H is not separable or K does not carry a strictly positive measure then $B(C(K, \mathscr{L}(H)))$ contains no exposed points.

If H is infinite dimensional then $B(C(K, \mathscr{L}(H)))$ contains no strongly exposed points.

Proof. Suppose that H is separable and K carries a strictly positive measure. Assume that $\mu(K)=1$. We fix an orthonormal basis $\{e_i\}_{i\in I}$ in H. Fix

$$f_0 \in \operatorname{ext} B(C(K, \mathscr{L}(H))).$$

Put $K_1=\{k\in K\colon f(k) \text{ is an isometry}\}$ and $K_2=K\setminus K_1$. The set K_1 is closed. Let $(a_i)_{i\in I}$ be a sequence of strictly positive reals such that $\sum_{i\in I} a_i=1$. We define a functional ξ on $C(K, \mathscr{L}(H))$ by

$$\xi(f) = \int_{K_1} \sum_{i\in I} a_i \operatorname{Re}\langle [f(k)](e_i), [f_0(k)](e_i)\rangle \, d\mu + \int_{K_{2_0}} \sum_{i\in I} a_i \operatorname{Re}\langle [f(k)]^*(e_i), [f_0(k)]^*(e_i)\rangle \, d\mu,$$

$f\in C(K, \mathscr{L}(H))$. The functional exposes f_0 in $B(C(K, \mathscr{L}(H)))$. Indeed $\xi(f)\leqq 1=\xi(f_0)$ for all $f\in B(C(K, \mathscr{L}(H)))$.

Suppose that $\xi(f_1)=1$ for some $f_1\in B(C(K, \mathscr{L}(H)))$. Because

$$\operatorname{Re}\langle [f_1(k)](e_i), [f_0(k)](e_i)\rangle \leqq 1,$$

the condition $\xi(f_1)=1$ implies that $[f_1(k)](e_i)=[f_0(k)](e_i)$ for all $i\in I$ and $k\in K_1$

μ-a.e. Analogously $f_1=f_0$ μ-a.e. on K_2. Hence by continuity $f_1=f_0$. So we finish the proof of the first part of theorem.

Now suppose that a functional ξ_0 exposes $f_0 \in \operatorname{ext} B(C(K, \mathscr{L}(H)))$ in $B(C(K, \mathscr{L}(H)))$. We define a functional m on $C(K)$ by

$$m(h) = \xi_0(hf_0), \quad h \in C(K).$$

We claim that m is strictly positive. Indeed, suppose to get a contradiction, that there exists $h_0 \in C(K)$ such that $0 \leqq h_0(k) \leqq 1$, $h_0 \neq 0$, and $m(h_0) < 0$.

Then

$$\xi_0((1-h_0)f_0) = m(1-h_0) \leqq m(1) = \xi_0(f_0)$$

and $h_0 f_0 \neq 0$, which is impossible. It follows that K carries a strictly positive measure.

Consider now a function n on all subsets of I defined by

$$n(L) = \xi_0(f_0 P_{\overline{\operatorname{lin}}\{e_i : i \in L\}})$$

$L \subset I$ (in the case when all $f_0(k)$ are coisometries we define n by $n(L) = \xi_0(P_{\overline{\operatorname{lin}}\{e_i : i \in L\}} f_0)$). The function n is finitely additive on the family of all subsets of I. Moreover $n(I)=1$ and $n(L) \geqq 0$ for $L \subset I$. Suppose that $n(L_0)=0$ for some non-empty $L_0 \subset I$. Then

$$\xi_0(f_0 P_{\overline{\operatorname{lim}}\{e_i : i \in L_0\}}) = 0,$$

so

$$\xi_0(f_0 P_{\overline{\operatorname{lim}}\{e_i : i \in L\}}) = \xi_0(f_0)$$

(i.e. ξ_0 does not expose f_0). This contradiction proves that $n(L_0) > 0$. Hence I is countable and H is separable, which yields the second part of the theorem.

Let $(L_j)_{j \in \mathbf{N}}$ be a sequence of non-empty disjoint subsets of I. Then $n(L_j) \xrightarrow{j} 0$. Hence

$$\xi_0(f_0 P_{\overline{\operatorname{lin}}\{e_i : i \in L_j^c\}}) = \xi_0(f_0) - n(L_j) \xrightarrow{j} \xi_0(f_0)$$

and

$$\|f_0 - f_0 P_{\overline{\operatorname{lin}}\{e_i : i \in L_j^c\}}\| = \|f_0 P_{\overline{\operatorname{lin}}\{e_i : i \in L_j\}}\| = 1.$$

Therefore f_0 is not strongly exposed in the case when I is infinite (i.e. H is infinite dimensional). This proves the third part of the theorem.

Theorem 3. *Let H be a finite dimensional Hilbert space and K be a Hausdorff compact space. Then each extreme point of $B(C(K, \mathscr{L}(H)))$ is strongly exposed, if and only if K is finite.*

Proof. Let $\dim H < \infty$ and $\operatorname{card} K < \infty$. Fix $f_0 \in B(C(K, \mathscr{L}(H)))$. Let η_k be a functional on $\mathscr{L}(H)$ which strongly exposes $f_0(k)$, $k \in K$. It is easy to see that a functional ξ defined by

$$\xi(f) = \sum_{k\in K} \eta_k(f(k)), \quad f\in C(K, \mathscr{L}(H)),$$

exposes f_0.

Now suppose that card $K=\infty$. Let a functional ξ_0 expose $f_0\in \operatorname{ext} B(C(K, \mathscr{L}(H)))$. Let $(k_n)_{n\in \mathbf{N}}$ be a sequence of distinct points of K such that $\lim k_n=k_0$. Let $h_n\in C(K)$, $n\in\mathbf{N}$, be such that $0\leqq h_n\leqq 1$, $h_n(k_n)=1$ and $\operatorname{supp} h_{n_1}\cap \operatorname{supp} h_{n_2}=\emptyset$ if $n_1\neq n_2$. Put $a_n=\xi_0(h_n f_0)$. Note that $a_n\geqq 0$ because

$$\xi_0(f_0)-a_n = \xi_0((1-h_n)f_0) \leqq \xi_0(f_0).$$

For every finite subset L of $\mathbf{N}$ we have

$$\sum_{n\in L} a_n = \xi_0((\sum_{n\in L} h_n)f_0) \leqq \xi_0(f_0).$$

Hence $a_n\to 0$. Therefore

$$\xi_0((1-h_n)f_0) = \xi_0(f_0)-a_n \to \xi_0(f_0)$$

and

$$\|(1-h_n)f_0-f_0\| = \|h_n f\| = 1.$$

Thus f_0 is not strongly exposed by ξ_0. This ends the proof.

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