Pointwise regularity of solutions to nonlinear double obstacle problems

Tero Kilpeläinen[1] and William P. Ziemer[2]

1. Introduction

In this paper we investigate the continuity at a given point x_0 of the solution u of a nonlinear elliptic variational inequality with a double obstacle constraint of the form $\psi_1 \leqq u \leqq \psi_2$. The partial differential operators associated with our obstacle problem are quasi-linear and include operators with only L^∞ coefficients. The obstacles in this context are to be regarded as quite general and irregular. In particular, they may be discontinuous. Since the partial differential operators associated with our problem may have only L^∞ coefficients, we can expect at most Hölder continuity for the regularity of our solution. Indeed, we show that if both obstacles are locally Hölder continuous, then the solution is also locally Hölder continuous. We also show that if the obstacles are not continuous, but satisfy a Wiener-type regularity condition, the solution is still continuous. This work extends that of [MZ1] in which a similar investigation was undertaken for the case of a single obstacle. This work also extends the recent paper of [DMV] which is devoted to the double obstacle problem for linear operators with bounded measurable coefficients. Because their work involves linear operators, they are able to employ potential theoretic techniques to obtain many of their estimates. These techniques are not available for us in our context of nonlinear operators.

Since the one-obstacle problem is a special case of the two-obstacle problem, one cannot expect better results in this latter case. On the other hand, the two-obstacle problem is so similar to the unilateral case, that one would anticipate virtually identical results. However, we have not been able to achieve this with the general structure we consider. One significant difference between the single and

[1] Research conducted in part while visiting Indiana University.
[2] Research supported in part by a grant from the National Science Foundation.

double-obstacle problem is that in the former case, the solution to the obstacle problem turns out to be a supersolution of the corresponding differential equation whereas this is no longer true in the double obstacle situation. This does not allow the use of the weak Harnack inequality for supersolutions, which was a critical tool in the analysis of the unilateral problem. If we impose more conditions on the structure of our nonlinear problem, then we show that the solution does become a supersolution of the corresponding equation and in this case we obtain results that run parallel to the single obstacle problem. The conditions imposed on the structure are general enough to include the case of linear operators.

Let A and B denote, respectively, vector and scalar valued Borel functions defined on $\Omega\times R^1\times R^n$ satisfying the following structure conditions for fixed $n>p>1$, $\mu>0$, and $\nu\geqq 0$:

$$(1.1)\qquad \begin{aligned} |A(x,z,h)| &\leqq \mu|h|^{p-1}+\mu|z|^{p-1}+\nu \\ h\cdot A(x,z,h) &\geqq |h|^p-\mu|z|^p-\nu \\ |B(x,z,h)| &\leqq \mu|h|^{p-1}+\mu|z|^{p-1}+\nu \end{aligned}$$

for a.e. $x\in\Omega$ and all $z\in R^1$, $h\in R^n$.

The obstacles ψ_1 and ψ_2 are defined to be real-valued functions defined on Ω with $\psi_1\leqq\psi_2$. In addition, we assume that ψ_1 is bounded above and that ψ_2 is bounded below. A bounded function $u\in W^{1,p}(\Omega)$ is said to be a solution of the *double obstacle problem* if $\psi_1\leqq u\leqq\psi_2$ and

$$(1.2)\qquad \int_\Omega A(x,u,\nabla u)\cdot\nabla\varphi\,dx+\int_\Omega B(x,u,\nabla u)\varphi\,dx\geqq 0$$

for all $\varphi\in W_0^{1,p}(\Omega)$ such that $\psi_1\leqq u+\varphi\leqq\psi_2$.

Although we have made the assumption that $1<p<n$, it can be shown without too much difficulty that virtually all of the results below are valid in case $p=n$.

2. Preliminaries

Throughout the paper, we will utilize properties of the following classes of functions introduced in [MZ1] which are similar but not quite as general as the De Giorgi classes discussed in [LU]. If $G\subset R^n$ is an open set, $C>0$ and $\lambda\geqq 0$, we let

$$S^\varepsilon(G,p,C,\lambda)$$

denote the set of all nonnegative functions $v\in W^{1,p}(G)$ such that

$$(2.1)\qquad \int_G|\nabla(v(x)-k)^\varepsilon|\eta(x)^p\,dx\leqq C\int_G[(v-k)^\varepsilon]^p[\eta(x)^p+|\nabla\eta(x)|^p]\,dx$$
$$+C(k^p+\lambda)\int_{\{(v(x)-k)^\varepsilon>0\}}\eta(x)^p\,dx$$

for all $k\geqq 0$ and all nonnegative Lipschitz functions η on R^n with compact support in G. Here ε stands for either $+$ or $-$.

The next two lemmas reveal the relationship between the solution u of the double obstacle problem and the classes $S^\varepsilon(G, p, C, \lambda)$.

Lemma 2.1. *Let u be a solution of (1.2) and let $d_0\geqq 0$. If $|d|\leqq d_0$, then there exist constants $C>0$ and $\lambda>0$ such that*

(i) *If $d\geqq\psi_1$ in $G\subset\Omega$, then $(u-d)^+\in S^+(G,p,C,\lambda)$*

(ii) *If $d\leqq\psi_2$ in $G\subset\Omega$, then $(u-d)^-\in S^+(G,p,C,\lambda)$.*

Proof. (i) Let $0\leqq\eta\leqq 1$ be a cut-off function and let $v=(u-d)^+$. For $k\geqq 0$, let $w=(v-k)^+$ and $\varphi=-w\eta^p$. Then

$$u+\varphi\leqq u\leqq\psi_2$$

and

$$u+\varphi\geqq u-(u-d)^+=\min(u,d)\geqq\varphi_1$$

in G and φ is thus admissible in (1.2). Since

$$\nabla\varphi=-\nabla w\eta^p-pw\eta^{p-1}\nabla\eta$$

it follows that

$$\begin{aligned}(2.2)\qquad&\int_{\{u>d+k\}}(|\nabla u|^p-\mu|u|^p-v)\eta^p\,dx\\&\leqq p\int_{\{u>d+k\}}(\mu|\nabla u|^{p-1}+\mu|u|^{p-1}+v)w\eta^{p-1}|\nabla\eta|\,dx\\&+\int_{\{u>d+k\}}(\mu|\nabla u|^{p-1}+\mu|u|^{p-1}+v)w\eta^p\,dx.\end{aligned}$$

Applying Young's inequality, we obtain

$$\begin{aligned}(2.3)\qquad&\int_{\{u>d+k\}}|\nabla u|^p\eta^p\,dx\leqq C\int_{\{u>d+k\}}(\mu|u|^p+v)\eta^p\,dx\\&+C\int_{\{u>d+k\}}w^p(\eta^p+|\nabla\eta|^p)\,dx+p\int_{\{u>d+k\}}(\mu|u|^{p-1}+v)\,w(\eta^{p-1}|\nabla\eta|+\eta^p)\,dx.\end{aligned}$$

Since $u>d+k$ implies $u=w+d+k$ we have

$$(\mu|u|^p+v)\eta^p\leqq\big((\mu w+\mu d_0+\mu k)^p+v\big)\eta^p$$

and thus the first integral on the right side of (2.3) is bounded by

$$(2.4)\qquad C_1\int_{\{w>0\}}(w^p+k^p+\lambda_1)\eta^p\,dx\quad\text{where}\quad\lambda_1=\lambda_1(d_0,v).$$

Also, the inequality

$$\begin{aligned}&(\mu|u|^{p-1}+v)w(\eta^p+\eta^{p-1}|\nabla\eta|)\\&=\mu|u|^{p-1}w\eta^{p-1}|\nabla\eta|+vw\eta^{p-1}|\nabla\eta|+\mu|u|^{p-1}w\eta^p+vw\eta^p\\&\leqq C(|u|^p+v^{p/(p-1)}+w^p)\eta^p+C(|u|^p\eta^p+w^p|\nabla\eta|^p)+C(\eta^p+w^p|\nabla\eta|^p,\end{aligned}$$

together with the fact

$$(\mu|u|^p+v^{p/(p-1)})\eta^p \leqq C_2(w^p+k^p+\lambda_2)\eta^p \quad \text{where} \quad \lambda_2=\lambda_2(d_0, v),$$

implies the estimate

$$C_2\int_{\{w>0\}} w^p(\eta^p+|\nabla\eta|^p)\,dx+C_2(k^p+\lambda)\int_{\{w>0\}}\eta^p\,dx \tag{2.5}$$

for the last integral of (2.3). The estimates (2.4) and (2.5) yield

$$\int|\nabla w|^p\eta^p\,dx \leqq C\int w^p(\eta^p+|\nabla\eta|^p)\,dx+C(k^p+\lambda)\int_{\{w>0\}}\eta^p\,dx$$

with $\lambda=\lambda(d_0, v)$. This proves (i).

The proof of (ii) is similar. Write $v=(u-d)^-$ and $w=(v-k)^+$. Then

$$\varphi = w\eta^p$$

is admissible for the (1.2) Proceeding as in (i) we obtain the desired result. □

Lemma 2.2. *With the same hypotheses as Lemma* 2.1 *and with* $v=(u-d)^\varepsilon$, *there exists a constant* $C=C(\mu, v, p, n, \|u\|_{\infty;G})$ *such that*

$$\int_{B(x_0,r)}|\nabla v|^p\eta^p\,dx \leqq C\left(\frac{1}{r}\int_{B(x_0,2r)}|\nabla v|^{p-1}\eta^{p-1}\,dx+r^{n-1}\right)$$

for every $B(x_0, 2r)\subset G$ *and every nonnegative cut-off function whose support is contained in* $B(x_0, 2r)$.

Proof. Consider the proof of the case $v=(u-d)^+$. Let η be a cut-off function such that $\eta=1$ on $B(x, r)$ and whose support is contained in $B(x, 2r)$. The desired result follows from a standard application of Young's inequality in (2.2). The proof in case $v=(u-d)^-$ is similar. □

Lemma 2.3. *Let u be a solution of* (1.2) *and let* $d_0\geqq 0$. *If* $d\leqq d_0$, *then there exist constants* $C>0$ *and* $\lambda\geqq 0$ *such that if* $M=\sup_G(u-d)^\varepsilon$, *then*

(i) $d\geqq\psi_1$ *in G implies* $M-(u-d)^+\in S^-(G, p, C, \lambda)$

and

(ii) $d\leqq\psi_2$ *in G implies* $M-(u-d)^-\in S^-(G, p, C, \lambda)$.

Proof. It will be sufficient to establish (i), the proof of (ii) being similar. Let $v=M-(u-d)^+$ and let η be a cut-off function. Fix $k\geqq 0$ with $k\leqq M$. With

$$w = (v-k)^-$$

and

$$\varphi = -w\eta^p$$

we have

$$\psi_1 \leqq \min(u, d) = u-(u-d)^+ \leqq u-(k-v)$$
$$\leqq u+\varphi \leqq u \leqq \psi_2$$

in G, whence φ is an admissible test function for (1.2). Thus, proceeding as in Lemma 2.1, we obtain

$$\int |\nabla w|^p \eta^p\,dx \leqq C\int w^p(\eta^p+|\nabla\eta|^p)\,dx+C(k^p+\lambda)\int_{\{w>0\}} \eta^p\,dx \tag{2.6}$$

where $\lambda=\lambda(d_0, v)$. This proves the necessary estimate for the case $k\leqq M$. If $k\geqq \sup_G v=M$, then

$$(v-k)^- = k-v = (k-M)+(v-M)^-$$

and (2.6) holds with $w=(v-M)^-$ and k replaced by M. This implies that (2.6) holds with $w=(v-k)^-$ since $k-M$ is nonnegative. □

The following results established in [MZ1] will be needed.

Theorem 2.4. *Let* $r\in(0, 1]$, $x_0\in R^n$, *and* $v\in S^\varepsilon(B(x_0, r), p, C, \lambda)$. *Define*

$$w(y) = r^{-1}v(ry+x_0)$$

for all $y\in B(0, 1)$. *Then*

$$w\in S^\varepsilon(B(0, 1), p, C, \lambda).$$

This is proved by a straightforward change of variables.

Theorem 2.5. *If* $v\in S^\varepsilon(B(0, 1), p, C, \lambda)$, *then*

$$v+2\lambda^{1/p}\in S^\varepsilon(B(0, 1), p, 2C, 0).$$

From these two results, one easily concludes the following.

Corollary 2.6. *If* $v\in S^\varepsilon(B(x_0, r), p, C, \lambda)$ *then*

$$v+2\lambda^{1/p}r\in S^\varepsilon(B(x_0, r), p, 2C, 0).$$

Finally we also recall the weak Harnack inequalities established in [MZ1] for functions in the classes S^+ and S^-. They play a central role in this paper.

Theorem 2.7. *Let* $p\in(1, n)$, $C>0$ *and* $\gamma\in(0, p]$. *There exists a constant* $C'=C'(n, p, C, \gamma)$ *such that for* $x_0\in R^n$, $\sigma\in(0, 1)$, $r\in(0, 1)$, $\lambda\geqq 0$, *and*

$$v\in S^+(B(x_0, r), p, C, \lambda)$$

we have

$$\sup_{B(x_0,\sigma r)} v \leqq C'(1-\sigma)^{-\xi}\left[\left\{\fint_{B(x_0,r)} v(x)^\gamma\,dx\right\}^{1/\gamma}+\lambda^{1/p}r\right]$$

where $\xi=n/\gamma$.

Theorem 2.8. *Let* $p\in(1,n)$, $\sigma\in(0,1)$ *and* $C>0$. *There exist positive constants* C', C'', and $\gamma\in(0,1)$ *depending only on* n, p, σ *and* C *such that for* $x_0\in R^n$, $r\in(0,1)$, $\lambda\geqq 0$, *and*

$$v\in S^-(B(x_0,r),p,C,\lambda)$$

with $v\geqq 0$, *we have*

$$\inf_{B(x_0,\sigma r)} v \geqq C'\left\{\fint_{B(x_0,r)} v(x)^\gamma\,dx\right\}^{1/\gamma} - C''\lambda^{1/p} r.$$

Corollary 2.9. *Let* u *be a bounded solution of* (1.2).

(i) *If* $d\geqq\psi_1$ *in* $G\subset\Omega$, *then* $(u-d)^+$ *is upper semicontinuous in* G.

(ii) *If* $d\leqq\psi_2$ *in* $G\subset\Omega$, *then* $(u-d)^-$ *is upper semicontinuous in* G.

Proof. This follows immediately from Lemmas 2.3 and 2.8. Indeed, for the first part of the corollary, we have $\mu(r)-(u-d)^+\in S^-(G,p,C,\lambda)$ where $\mu(r)=\sup_{B(x_0,r)}(u-d)^+$, $x_0\in G$, and r sufficiently small. Hence,

$$\mu(r)-\mu(r/2)\geqq C'\left\{\fint_{B(x_0,r)}[\mu(r)-(u-d)^+]^\gamma\,dx\right\}^{1/\gamma} - C''\lambda^{1/p} r.$$

Since u is bounded, this implies that

$$\fint_{B(x_0,r)}[\mu(r)-(u-d)^+]\,dx\to 0 \quad\text{as}\quad r\to 0.$$

Thus, it follows that $(u-d)^+$ has a Lebesgue point at each point of G and that the value of $(u-d)^+$ is equal to its upper limit at each point of G, which implies upper semicontinuity. A similar argument holds for the second part. □

Remark 2.10. Theorems 2.7 and 2.8 are counterparts to the well-known weak Harnack inequalities for sub and super solutions of equations in divergence form [GT, Chapter 8]. While sub and super solutions are elements of the classes S^+ and S^-, the greater generality of these classes do not yield results with the same precision as with sub and super solutions. For example, the exponent γ in Theorem 2.8 can be taken as any positive number less than $n(p-1)/(n-p)$ in the case of a weak supersolution (see [T]) whereas no such bound is available to functions in the class S^-. A similar phenomenon was encountered in [DT] where weak Harnack inequalities were established for quasiminima, but where the exponent in the weak Harnack inequality for super quasiminima is not quite as strong as for supersolution of elliptic equations. This slight difference will be a factor in some of our results below.

3. Regularity properties of the solution

In this section we will initiate a study of the regularity of the solution to (1.2). For this we will introduce the following notation. We let $|E|$ denote the Lebesgue measure of a set $E\subset R^n$. We will also need a more refined method of measuring sets.

Definition 3.1. For $1<p<n$, the p-capacity of a compact set $K\subset R^n$ is defined as

$$\gamma_p(K) = \inf\left\{\int_{R^n} |\nabla v|^p\,dx\right\}$$

where the infimum is taken over all functions $v\in C_0^\infty(R^n)$ such that $v\geqq 1$ on K. The definition of γ_p can be extended to all sets by standard methods.

The p-capacity is a capacity in the sense of Brelot—Choquet and enjoys the following properties that will be needed later, cf. [AH]. If $q<p$ then

$$\gamma_q(E) \leqq C[\gamma_p(E)]^{(n-q)/(n-p)} \quad \text{for every} \quad E\subset B(0,1) \tag{3.1}$$
$$\gamma_p[B(x,r)] = Cr^{n-p} \quad \text{for every ball} \quad B(x,r).$$

Since we are concerned with point-wise regularity, we will consider a fixed point $x_0\in\Omega$ throughout the remainder of the paper. Let

$$\bar\psi_1(r) = p\text{-}\sup_{B(x_0,r)}\psi_1 \qquad \underline{\psi}_1(r) = p\text{-}\inf_{B(x_0,r)}\psi_1$$
$$\bar\psi_1(x_0) = \lim_{r\to 0}\bar\psi_1(r) \qquad \underline{\psi}_1(x_0) = \lim_{r\to 0}\underline{\psi}_1(r) \tag{3.2}$$
$$\underline{\psi}_1(E_1,r) = \inf_{E_1\cap B(x_0,r)}\psi_1 \qquad \bar\psi_1(E_1,r) = \sup_{E_1\cap B(x_0,r)}\psi_1.$$

Here p-sup and p-inf are the essential supremum and infimum in the sense of p-capacity. Similar notation will be used for ψ_2 and the solution u.

Theorem 3.2. *Let u be a solution of* (1.2) *and assume $\bar\psi_1(x_0)\leqq\underline{\psi}_2(x_0)$ where $x_0\in\Omega$. Then u possesses a Lebesgue point at x_0.*

Proof. Let

$$\mu(r) = \sup_{B(x_0,r)}[u-\bar\psi_1(r)],$$
$$v(x) = \mu(r)-[u(x)-\bar\psi_1(r)]^+.$$

Lemma 2.3 implies that $v\in S^-(B(x_0,r),p,C,\lambda)$ and therefore we conclude from Theorem 2.8 that

$$\left\{\fint_{B(x_0,r)}[\mu(r)-(u(x)-\bar\psi_1(r))^+]^\gamma\,dx\right\}^{1/\gamma} - C'\lambda^{1/p}r \leqq C\inf_{B(x_0,r/2)} v$$
$$\leqq C[\mu(r)-\bar u(r/2)+\bar\psi_1(r)] = C[\bar u(r)-\bar u(r/2)].$$

Observe that the right-side tends to 0 as $r\to 0$. Thus,

$$\lim_{r\to 0}\left\{\fint_{B(x_0,r)}[\mu(r)-(u(x)-\bar{\psi}_1(r))^+]^\gamma\,dx\right\}^{1/\gamma}=0$$

and since u is bounded, we obtain the same conclusion with $\gamma=1$. Now let $\mu(x_0)=\lim_{r\to 0}\mu(r)$ and assume that $\mu(x_0)>0$. Then we have

$$\lim_{r\to 0}\fint_{B(x_0,r)}[\mu(x_0)-(u(x)-\bar{\psi}_1(x_0))^+]\,dx=0.$$

Now write

$$\fint_{B(x_0,r)}[\mu(x_0)-(u(x)-\bar{\psi}_1(x_0))^+]\,dx$$

$$=|B(x_0,r)|^{-1}\int_{B(x_0,r)\cap\{u\geqq\bar{\psi}_1(x_0)\}}|\mu(x_0)+\bar{\psi}_1(x_0)-u(x)|\,dx$$

$$+\mu(x_0)\frac{|B(x_0,r)\cap\{u<\bar{\psi}_1(x_0)\}|}{|B(x_0,r)|}.$$

Notice that both terms on the right tend to 0 as $r\to 0$ and since we are assuming $\mu(x_0)\neq 0$, we have

$$\lim_{r\to 0}\frac{|B(x_0,r)\cap\{u<\bar{\psi}_1(x_0)\}|}{|B(x_0,r)|}=0.$$

Since u is bounded, it now follows that

$$\lim_{r\to 0}\fint_{B(x_0,r)}|\mu(x_0)+\bar{\psi}_1(x_0)-u(x)|\,dx=0$$

if $\mu(x_0)\neq 0$, our desired conclusion. In case $\mu(x_0)=0$, then we have

$$\operatorname{ess}\limsup_{x\to x_0}u(x)=\bar{\psi}_1(x_0).$$

In this situation, let

$$v(x)=\lambda(r)-(u(x)-\underline{\psi}_2(r))^-$$

where $\lambda(r)=\sup_{B(x_0,r)}(u-\underline{\psi}_2(r))^-$. Then as above, we find that u possesses a Lebesgue point at x_0 if $\lim_{r\to 0}\lambda(r)=\lambda(x_0)>0$. If $\lambda(x_0)=0$, we have

$$\operatorname{ess}\liminf_{x\to x_0}u(x)=\underline{\psi}_2(x_0).$$

This, along with the case $\mu(x_0)=0$ and the assumption $\bar{\psi}_1(x_0)\leqq\underline{\psi}_2(x_0)$ implies that the essential limit of u at x_0 exists and therefore u possesses a Lebesgue point there. □

This result immediately leads to our first conclusion on regularity.

Corollary 3.3. *If u is a solution of* (1.2) *and* $\bar{\psi}_1(x_0)\leqq u(x_0)\leqq\underline{\psi}_2(x_0)$, *then u is continuous at x_0.*

Proof. By the previous result, note that u has a Lebesgue point at x_0 and therefore the theorem has meaning. If $d>u(x_0)$, then $d>\psi_1$ in a neighborhood of x_0. Moreover, by Corollary 2.9, $v=(u-d)^+$ is upper semicontinuous at x_0 and $v(x_0)=0$. It follows that

$$\limsup_{x\to x_0} u(x) \leqq \limsup_{x\to x_0} v(x)+d = d.$$

Now let $d\to u(x_0)$ to obtain that u is upper semicontinuous at x_0. A dual argument shows that u is also lower semicontinuous at x_0, thus establishing the desired conclusion. □

Now we have shown that u has a Lebesgue point under the assumption $\bar{\psi}_1(x_0)\leqq \underline{\psi}_2(x_0)$, we wish to use this result to prove more, namely, that u is finely continuous at x_0. For this purpose, we will need the following result.

Theorem 3.4. *Let $p>1$ and $C>0$. There exist constants $C'=C'(n,p,C)$ and $\alpha<p$ such that if $v\in S^-(G,p,C,0)$ with $v(x)>0$ for almost all $x\in G$, then*

$$\int_G v(x)^{-\beta}|\nabla v(x)|^p\eta(x)^p\,dx \leqq C'\int_G v(x)^{p-\beta}[\eta(x)^p+|\nabla\eta(x)|^p]\,dx$$

for every $\beta>\alpha$ and every nonnegative Lipschitz function η with compact support in G.

Proof. Since $v+t\in S^-(G,p,C,0)$ whenever $t>0$, we may assume that v is bounded away from 0 on G. Observe also that (2.1) holds whenever $\eta\in W_0^{1,p}(G)$. Let $T>0$, $\gamma>0$ and set

$$v_T(x) = \inf\{T, v(x)\}$$
$$\eta_T(x) = v_T(x)^\gamma\eta(x).$$

Now replace η by η_T in (2.1). This yields

$$\int_G |\nabla(v-k)^-|^p v_T^{p\gamma}\eta^p\,dx \leqq C_1\int_G [(v-k)^-]^p[v_T^{p\gamma}\eta^p+\gamma^p v_T^{p\gamma-p}|\nabla v_T|^p\eta^p$$
$$+v_T^{p\gamma}|\nabla\eta|^p]\,dx+C_1k^p\int_{\{(v-k)^->0\}} v_T^{p\gamma}\eta^p\,dx$$

where $C_1=C_1(C,p)$. Let $\beta\geqq p$ and multiply both sides of the previous inequality by $k^{-p\gamma-\beta-1}$ and integrate with respect to k over $(0,\infty)$ to obtain

$$\frac{1}{p\gamma+\beta}\int_G v^{-p\gamma-\beta}v_T^{p\gamma}|\nabla v|^p\eta^p\,dx$$

$$\leqq \frac{2C_1}{-p+p\gamma+\beta}\int_G v^{-p-p\gamma-\beta}[v_T^{p\gamma}\eta^p+\gamma^p v_T^{-p+p\gamma}|\nabla v_T|^p\eta^p+v_T^{p\gamma}|\nabla\eta|^p]\,dx.$$

By letting $T\to\infty$ and observing that

$$\frac{p\gamma+\beta}{-p+p\gamma+\beta} = 1+\frac{p}{-p+p\gamma+\beta} \leqq 1+\frac{1}{\gamma}$$

we obtain

$$\int_G v^{-\beta}|\nabla v|^p \eta^p\,dx \leqq 2C_1\left(1+\frac{1}{\gamma}\right)\int_G v^{p-\beta}[\eta^p+|\nabla\eta|^p]\,dx$$

$$+2C_1\left(1+\frac{1}{\gamma}\right)\gamma^p\int_G v^{-\beta}|\nabla v|^p\eta^p\,dx.$$

A suitable choice of γ now yields the required inequality under our assumption of $\beta\geqq p$. However, it is possible to make a suitable choice of γ under a slightly weaker constraint on β. For a certain range of β less than p, we need $\gamma>0$ such that

$$0<\frac{(\gamma)^p(p\gamma+\beta)}{p\gamma+\beta-p}<\varepsilon \tag{3.3}$$

for arbitrary $\varepsilon>0$. Clearly, such a γ exists, say γ_1, in case $\beta\geqq p$. Under the assumption that $\beta<p$, there exists γ_2 such that

$$\left(\frac{p\gamma_2+\beta}{p\gamma_2}\right)(\gamma_2)^p \leqq \left(\frac{p\gamma_2+p}{p\gamma_2}\right)(\gamma_2)^p \leqq \frac{\varepsilon}{2}. \tag{3.4}$$

Now let $\gamma_0=\min(\gamma_1,\gamma_2)$ and place an additional constraint on β by requiring

$$\frac{\beta-p}{p\gamma_0}\geqq-\frac{1}{2}.$$

Then, $p\gamma_0+\beta-p>0$ and because of (3.4), it is easy to see that (3.3) is satisfied. □

We now proceed to prove continuity of u at points x_0 where the obstacles ψ_1 and ψ_2 possess some regularity. The amount of regularity required is given by the following definition.

Definition 3.5. A point x_0 is said to be an *upper (p,η)-Wiener point for ψ_1* if there exists a set E_1 such that

$$\int_0^1\left(\frac{\gamma_p[E_1\cap B(x_0,r)]}{r^{n-p}}\right)^{1/\eta}\frac{dr}{r}=\infty \tag{3.5}$$

and

$$\liminf_{\substack{x\to x_0\\ x\in E_1}}\psi_1(x)\geqq\bar{\psi}_1(x_0).$$

Similarly, x_0 is said to be a *lower (p,η)-Wiener point for ψ_2* if there exists a set E_2 satisfying the same condition as (3.5) and such that

$$\limsup_{\substack{x\to x_0\\ x\in E_2}}\psi_2(x)\leqq\underline{\psi}_2(x_0).$$

A set E_1 satisfying (3.5) is said to be *not thin* at x_0.

Remark 3.6. In the usual definition of the Wiener integral for nonlinear problems, the exponent $1/\eta$ is replaced by $1/(p-1)$, cf. [MA], [GZ]. As suggested in Remark 2.10, we are forced to work with a slightly weaker Wiener condition because of the greater generality of the classes S^+ and S^-. This will manifest itself in our main result below where a crucial role is played by the weak Harnack inequality, Theorem 2.8.

Theorem 3.7. *Let $x_0\in\Omega$ be both an upper (p,γ)-Wiener point for ψ_1 and a lower (p,γ)-Wiener point for ψ_2 where $1/\gamma$ is the exponent that appears in Theorem 2.8. If $\bar{\psi}_1(x_0)\leqq\underline{\psi}_2(x_0)$, then a weak solution u of (1.2) is continuous at x_0.*

Proof. Recall Theorem 3.2 which states that u of (1.2) is defined at x_0. If

$$\bar{\psi}_1(x_0)\leqq u(x_0)\leqq \underline{\psi}_2(x_0) \tag{3.6}$$

then u is continuous at x_0 by Corollary 3.3. Thus, it will be sufficient to prove (3.6). In fact, it is sufficient to prove

$$\bar{\psi}_1(x_0)\leqq u(x_0). \tag{3.7}$$

Since the other inequality in (3.6) will follow by a dual argument. Let

$$v=(u-d)^- \quad \text{where} \quad d<\bar{\psi}_1(x_0)=u(x_0).$$

Since $d<\underline{\psi}_2(2r)$ for all small $r>0$, note that

$$v\in S^+(B(x_0,2r),p,C,\lambda).$$

Also, let

$$m(r)=\sup_{B(x_0,r)}(u-d)^-$$

$$w=m(2r)-v\in S^-(B(x_0,2r),p,C,\lambda)$$

$$A=\{u>d\}.$$

If E_1 is a set which is not thin at x_0 and has the property that

$$\liminf_{\substack{x\to x_0\\ x\in E_1}}\psi_1(x)\geqq\bar{\psi}_1(x_0),$$

then since $u\geqq\psi_1$ q.e. we have

$$E_1\cap B(x_0,r)\subset A\cap B(x_0,r) \tag{3.8}$$

for all small $r>0$. Let η be a cut-off function supported in $B(x_0,2r)$ such that $\eta=1$ on $B(x_0,r)$. Then,

$$m(2r)^p\gamma_p[A\cap B(x_0,r)]\leqq\int_{B(x_0,2r)}|\nabla(\eta w)|^p\,dx$$

$$\leqq C\Big(\int_{B(x_0,2r)}|\nabla w|^p\eta^p+w^p|\nabla\eta|^p\,dx\Big).$$

Now let $w'=w+2\lambda^{1/p}r$ and refer to Corollary 2.6 to find that

$$w'\in S^-(B(x_0,2r),p,C,0).$$

Thus, we obtain

$$\Big(\int_{B(x_0,2r)}|\nabla w|^p\eta^p+w^p|\nabla\eta|^p\,dx\Big)=\Big(\int_{B(x_0,2r)}|\nabla w'|^p\eta^p+w^p|\nabla\eta|^p\,dx\Big)$$

$$\leqq C\Big(\frac{1}{r}\int_{B(x_0,2r)}|\nabla w'|^{p-1}\eta^{p-1}\,dx+r^{n-p}\fint_{B(x_0,2r)}w^p\,dx+r^{n-1}\Big)$$

(by Lemma 2.2)

$$\leqq C\Big(\frac{1}{r}\int_{B(x_0,2r)}\big(\eta(w')^{(-\beta)/p}|\nabla w'|\big)^{p-1}\big((w')^{(\beta(p-1))/p}\big)\,dx+I(r)\Big)$$

where $\beta<p$ is obtained from Theorem 3.4 and

$$I(r)=r^{n-p}\fint_{B(x_0,2r)}w^p\,dx+r^{n-1}$$

$$\leqq C\Big(\frac{1}{r}\Big(\int_{B(x_0,2r)}(w')^{-\beta}|\nabla w'|^p\eta^p\,dx\Big)^{(p-1)/p}\Big(\int_{B(x_0,2r)}(w')^{\beta(p-1)}\,dx\Big)^{1/p}+I(r)\Big)$$

$$\leqq C\Big(\frac{1}{r}\Big(\int_{B(x_0,2r)}(w')^{p-\beta}[\eta^p+|\nabla\eta|^p]\,dx\Big)^{(p-1)/p}\Big(\int_{B(x_0,2r)}(w')^{\beta(p-1)}\,dx\Big)^{1/p}+I(r)\Big)$$

(by Theorem 3.4)

$$\leqq C\Big(\frac{1}{r}r^{n(p-1)/p}\Big(\fint_{B(x_0,2r)}(w')^{p-\beta}\,dx\Big)^{(p-1)/p}\cdot r^{n/p}\Big(\fint_{B(x_0,2r)}(w')^{\beta(p-1)}\,dx\Big)^{1/p}+I(r)\Big)$$

$$\leqq C\Big(r^{n-p}\Big(\fint_{B(x_0,2r)}(w')^{p-\beta}\,dx\Big)^{(p-1)/p}\Big(\fint_{B(x_0,2r)}(w')^{\beta(p-1)}\,dx\Big)^{1/p}+I(r)\Big)$$

$$\leqq Cr^{n-p}\Big(\Big(\fint_{B(x_0,2r)}(w')^{p-\beta}\,dx\Big)^{(p-1)/p}\Big(\fint_{B(x_0,2r)}(w')^{\gamma}\,dx\Big)^{1/p}+\fint_{B(x_0,2r)}w^p\,dx+r^{p-1}\Big)$$

(with $\beta(p-1)>\gamma$ and from the fact that w' is bounded)

$$\leqq Cr^{n-p}\Big(\fint_{B(x_0,2r)}(w')^{\gamma}\,dx+\fint_{B(x_0,2r)}w^p\,dx+r^{p-1}\Big)$$

$$\leqq Cr^{n-p}\big((\inf_{B(x_0,r)}w')^{\gamma}+(\inf_{B(x_0,r)}w)^{\gamma}+r^{p-1}\big)$$

(using the fact that w is bounded and from Theorem 2.8)

$$\leqq Cr^{n-p}\big[(m(2r)-m(r)+2\lambda^{1/p}r)^{\gamma}+r^{p-1}\big]$$

(since $\inf_{B(x_0,r)} w$ is <1 for small r).

That is,

(3.9) $$m(2r)^p\Big(\frac{\gamma_p[A\cap B(x_0,r)]}{r^{n-p}}\Big)\leqq C[m(2r)-m(r)+2\lambda^{1/p}r]^{\gamma}\quad\text{for}\quad 0<r<1,$$

which implies

$$\int_0^1 m(2r)^{p/\gamma}\left(\frac{\gamma_p[A\cap B(x_0, r)]}{r^{n-p}}\right)^{1/\gamma}\frac{dr}{r}<\infty.$$

Referring to (3.8), we obtain

$$\infty=\int_0^1\left(\frac{\gamma_p[E_1\cap B(x_0, r)]}{r^{n-p}}\right)^{1/\gamma}\frac{dr}{r}\leqq\int_0^1\left(\frac{\gamma_p[A\cap B(x_0, r)]}{r^{n-p}}\right)^{1/\gamma}\frac{dr}{r}$$

it follows that $m(r)\to 0$ as $r\to 0$. This implies that $\lim_{r\to 0}\underline{u}(r)\geqq d$; that is, $\underline{u}(x_0)\geqq d$.

Recall that this is proved for any $d<\bar{\psi}_1(x_0)$. Thus, we have that

$$u(x_0)=\lim_{r\to 0}\fint_{B(x_0, r)}u(x)\,dx\geqq\underline{u}(x_0)\geqq\bar{\psi}_1(x_0).\quad\square$$

Remark 3.8. It is easily seen that the hypothesis $\bar{\psi}_1(x_0)\leqq\underline{\psi}_2(x_0)$ is necessary for a solution to be continuous at x_0. Indeed, since $u\geqq\psi_1$ q.e. and $u\leqq\psi_2$ q.e., it follows

$$\begin{aligned}\bar{\psi}_1(x_0)&=\inf_{r>0}\ p\text{-}\sup_{B(x_0,r)}\psi_1\leqq\inf_{r>0}\ p\text{-}\sup_{B(x_0,r)}u=u(x_0)\\&=\sup_{r>0}\ p\text{-}\inf_{B(x_0,r)}u\leqq\sup_{r>0}\ p\text{-}\inf_{B(x_0,r)}\psi_2=\underline{\psi}_2(x_0).\end{aligned}$$

4. Modulus of continuity

Inequality (3.9) is fundamental in establishing the continuity of the solution. However, it is not strong enough to establish a meaningful bound on the modulus of continuity. For this purpose, one might employ a substitute for γ_p. The capacity needed is γ_q where

$$q=n-\frac{p(n-p)}{\gamma}$$

where γ is the number that appears in Theorem 2.8. Referring to (3.1) it is easily seen that (3.9) becomes

$$m(2r)^\gamma\left(\frac{\gamma_q[A\cap B(x_0, r)]}{r^{n-q}}\right)^{\gamma/p}\leqq C[m(2r)-m(r)+2\lambda^{1/p}r]^\gamma\quad\text{for}\quad 0<r<1. \tag{4.1}$$

That is,

$$m(2r)\left(\frac{\gamma_q[A\cap B(x_0, r)]}{r^{n-q}}\right)^{1/p}\leqq C[m(2r)-m(r)+2\lambda^{1/p}r]\quad\text{for}\quad 0<r<1. \tag{4.2}$$

Now that we have (4.2), it follows immediately from [GZ, Theorem 2.7] that $m(r)$

satisfies the following growth condition:

$$m(r) \leqq C_1 \exp\left(-C_2 \int_{2s}^{r} A(t)\frac{dt}{t}\right) \quad \text{for every} \quad s \leqq r/2$$

where

$$A(t) = \left(\frac{\gamma_q[E_1 \cap B(x_0, t)]}{t^{n-q}}\right)^{1/p}.$$

From this it would be possible to obtain a modulus of continuity. The difficulty with this argument is that in the definition of the number q, we have no estimate for the number γ and therefore q might be negative.

In order to obtain a modulus of continuity under the general structure (1.1), we require the following assumptions on the sets E_1 and E_2 that appear in Definition 3.5:

(4.3)
$$\liminf_{r\to 0} \frac{|E_1 \cap B(x_0, r)|}{|B(x_0, r)|} > 0$$

and

(4.4)
$$\liminf_{r\to 0} \frac{|E_2 \cap B(x_0, r)|}{|B(x_0, r)|} > 0.$$

Theorem 4.1. *If u is a solution of* (1.2), *E_1, E_2 respectively satisfy* (4.3) *and* (4.4), *and $\bar{\psi}_1(x_0) \leqq \underline{\psi}_2(x_0)$, then u is continuous at x_0 and its modulus of continuity is estimated by*

$$\omega(r) \leqq C[r^\alpha + M_1(r) + M_2(r)]$$

where

$$M_1(r) = \sup(\bar{\psi}_1(x_0) - \underline{\psi}_1(E_1, r), \underline{\psi}_2(x_0) - \underline{\psi}_2(2r))$$

$$M_2(r) = \sup(\bar{\psi}_2(E_2, r) - \underline{\psi}_2(x_0), \bar{\psi}_1(2r) - \bar{\psi}_1(x_0)).$$

Proof. We know from Theorem 3.2 that u is defined at x_0 and that (4.3) implies $\bar{\psi}_1(x_0) \leqq u(x_0)$. Likewise, (4.4) implies that $u(x_0) \leqq \underline{\psi}_2(x_0)$. Hence, by Theorem 3.7 it follows that u is continuous at x_0. If

$$\bar{\psi}_1(x_0) < u(x_0) < \underline{\psi}_2(x_0),$$

then u is a solution of the associated equation in a neighborhood of x_0 and is therefore Hölder continuous there. This establishes part of our conclusion.

Next we assume

(4.5)
$$\bar{\psi}_1(x_0) = u(x_0) < \underline{\psi}_2(x_0)$$

and consider the function w defined in the proof of Theorem 3.7 with $d = d(r) =$

$\underline{\psi}_1(E_1, r)-r$ where $\underline{\psi}_1(E_1, r)=\inf_{B(x_0,r)\cap E_1}\psi_1$. Because of assumption (4.5), note that $d(r)<\underline{\psi}_2(r)$ and that $E_1\cap B(x_0, r)\subset A\cap B(x_0, r)$ for all small $r>0$. Then, the functions v and w introduced in Theorem 3.7 belong to the classes S^+ and S^- respectively and by Theorem 2.8

$$m(r)\left(\frac{|E_1\cap B(x_0, r)|}{|B(x_0, r)|}\right)^{1/\gamma} \leqq \left(\fint_{B(x_0,r)} (w(x))^\gamma dx\right)^{1/\gamma}$$

$$\leqq C \inf_{B(x_0,r/2)} w+C'\lambda^{1/p}r \leqq C[m(r)-m(r/2)+C'\lambda^{1/p}r].$$

Because of (4.3), this implies

$$m(r) \leqq C''[m(r)-m(r/2)+C'\lambda^{1/p}r]$$

for all small $r>0$. This implies

$$m(r) \leqq Cr^\alpha$$

for some $0<\alpha<1$ and for all small $r>0$, cf. [GT, Lemma 8.23]. Now

$$u(x_0)-\underline{u}(r) = m(r)+\bar{\psi}_1(x_0)-\underline{\psi}_1(E_1, r)+r$$

so that

$$\underline{\omega}(r) \equiv u(x_0)-\underline{u}(r) \leqq \bar{\psi}_1(x_0)-\underline{\psi}_1(E_1, r)+Cr^\alpha. \tag{4.6}$$

This provides an estimate for the lower oscillation of u under assumption (4.5). We now proceed to obtain an estimate for the lower oscillation in case

$$\bar{\psi}_1(x_0) = u(x_0) = \psi_2(x_0).$$

The proof in this situation proceeds exactly as the one above except that now we take

$$d(r) = \inf\{\underline{\psi}_1(E_1, r), \underline{\psi}_2(r)\}-r < \underline{\psi}_2(r)$$

which will ensure that the functions v and w are in the appropriate classes and that $E_1\cap B(x_0, r)\subset A\cap B(x_0, r)$. We then obtain

$$u(x_0)-\underline{u}(r) \leqq \bar{\psi}_1(x_0)+m(r)-d(r) \leqq C[r^\alpha+M_1(r)].$$

This shows that under assumption (4.3)

$$\underline{\omega}(r) \leqq C[r^\alpha+M_1(r)]$$

for all small $r>0$. A dual argument establishes

$$\bar{\omega}(r) \equiv \bar{u}(r)-u(x_0) \leqq C[r^\alpha+M_2(r)]$$

under the assumption (4.4). □

Corollary 4.2. *If u is a solution of* (1.2) *and both ψ_1 and ψ_2 are locally Hölder continuous with $\psi_1 \leqq \psi_2$ on Ω, then u is locally Hölder continuous on Ω.*

In order to obtain stronger results, we modify the structure (1.1) of A and B as follows so that in addition, we require

$$A(x,u,h)\cdot h \geqq 0$$

$$A(x,u,0) = 0 \tag{4.7}$$

$$B(x,u,0) = 0$$

whenever $u \in R^1$ and $h \in R^n$.

Although this structure is more restrictive, it nevertheless includes a wide class of interesting equations. For example, the p-Laplacean

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0,$$

and the uniformly elliptic linear equations

$$Lu = -\sum_{i,j=1}^{n} (a^{ij}(x)u_{x_j})_{x_i}$$

considered in [DMV] where $a^{ij} \in L^\infty$. Also, the equations

$$-\operatorname{div} A(x,\nabla u) = 0$$

with $A(x,h)\cdot h \approx |h|^p$, associated with a nonlinear potential theory [HKM] satisfy these assumptions.

The key lemma which results from these extra assumptions is the following.

Lemma 4.3. *Suppose that A and B satisfy* (1.1) *and* (4.7). *Then*

(i) *If $d \geqq \psi_1$ in G, then $v = \max(u,d)$ is a subsolution of* (1.2) *in G; that is*

$$\int A(x,v,\nabla v)\cdot\nabla\varphi + B(x,v,\nabla v)\varphi\,dx \leqq 0.$$

for every nonnegative test function φ.

(ii) *If $d \leqq \psi_2$ in G, then $w = \min(u,d)$ is a supersolution of* (1.2) *in G.*

Proof. Let $\varphi \in C_0^\infty(G)$, $\varphi \geqq 0$. Fix $\varepsilon > 0$ and write

$$\eta = -\varepsilon \min\left(\varphi, \frac{(u-d)^+}{\varepsilon}\right).$$

Then $u+\eta \leqq u \leqq \psi_2$ and

$$u+\eta \geqq u-\varepsilon\frac{(u-d)^+}{\varepsilon} = \min(u,d) \geqq \psi_1.$$

Thus, η is admissible for (1.2) and it follows that

$$0 \geqq \varepsilon \int_G A(x, u, \nabla u) \cdot \nabla \min\left(\varphi, \frac{(u-d)^+}{\varepsilon}\right) dx$$

$$+\varepsilon \int_G B(x, u, \nabla u) \min\left(\varphi, \frac{(u-d)^+}{\varepsilon}\right) dx$$

$$= \varepsilon \int_{\{x \in G: (u-d)^+ \leqq \varepsilon\varphi\}} A(x, u, \nabla u) \cdot \nabla u \, dx + \varepsilon \int_{\{x \in G: (u-d)^+ > \varepsilon\varphi\}} A(x, u, \nabla u) \cdot \nabla \varphi \, dx$$

$$+\varepsilon \int_G B(x, u, \nabla u) \min\left(\varphi, \frac{(u-d)^+}{\varepsilon}\right) dx.$$

Thus, assumption (4.7) implies

$$\int_{G \cap \{(u-d)^+ > \varepsilon\varphi\}} A(x, v, \nabla v) \cdot \nabla \varphi \, dx$$

$$+\int_{G \cap \{(u-d)^+ > 0\}} B(x, v, \nabla v) \min\left(\varphi, \frac{(u-d)^+}{\varepsilon}\right) dx \leqq 0.$$

Now with $\varepsilon \to 0$ we obtain

$$\int_{G \cap \{(u-d)^+ > 0\}} A(x, v, \nabla v) \cdot \nabla \varphi \, dx + \int_{G \cap \{(u-d)^+ > 0\}} B(x, v, \nabla v) \varphi \, dx \leqq 0$$

and, since by assumption (4.7), we have $A(x, v, \nabla v) = 0$ and $B(x, v, \nabla v) = 0$ in the set

$$G \cap \{(u-d)^+ = 0\} = G \cap \{v = 0\}$$

we have

$$\int_G A(x, v, \nabla v) \cdot \nabla \varphi \, dx + \int_G B(x, v, \nabla v) \varphi \, dx \leqq 0$$

as desired.

The proof of (ii) is similar with η defined this time as

$$\eta = \varepsilon \min\left(\varphi, \frac{(u-d)^-}{\varepsilon}\right). \quad \square$$

Theorem 4.4. *If u is a solution* (1.2) *with structure* (1.1) *and* (4.7), *x_0 is a point that satisfies Definition* 3.5 *with $\eta = p-1$, and $\bar{\psi}_1(x_0) \leqq \underline{\psi}_2(x_0)$, then u is continuous at x_0 and its modulus of continuity for all small $r > 0$ and all $s < r/2$ is given by*

$$\omega(r) \leqq C\left[\exp\left(-C' \int_{2s}^r A_1(t) \frac{dt}{t}\right) + \exp\left(-C' \int_{2s}^r A_2(t) \frac{dt}{t}\right) + M_1(r) + M_2(r)\right]$$

where

$$A_i = \left(\frac{\gamma_p[E_i \cap B(x_0, t)]}{t^{n-p}}\right)^{1/(p-1)} \qquad i = 1, 2$$

and where $M_i(r)$, $i = 1, 2$ are as in Theorem 4.1.

Proof. We proceed as in the proof of Theorem 4.1 and thus first prove

$$\bar{\psi}_1(x_0) \leqq u(x_0) \leqq \underline{\psi}_2(x_0). \tag{4.8}$$

Let $d=d(r)=\underline{\psi}_1(E_1, 2r)-r$ and note that since $d<\bar{\psi}_1(x_0)$, we have $d<\underline{\psi}_2(2r)$ for all small $r>0$. Consequently, we infer from Theorem 4.3 that $\min(u, d)$ is a supersolution in $B(x_0, 2r)$ of an equation of type (1.1) and (1.2). Let $f=\min(u, d)$. Recall from [MZ1, Corollary 4.4] that f is finely continuous at all points of $B(x_0, 2r)$. Thus there is a set $E_0 \subset E_1$ such that $f \geqq \min(\psi_1, d)$ on E_0 and that

$$\lim_{\substack{x\to x_0\\ x\in E}} f(x) = f(x_0).$$

Then since f is approximately continuous

$$u(x_0) \geqq f(x_0) = \lim_{\substack{x\to x_0\\ x\in E_0}} f(x) \geqq \liminf_{\substack{x\to x_0\\ x\in E_0}} \min\left(\psi_1(x), d\right) = d$$

and by letting $r\to 0$ we have

$$u(x_0) \geqq \bar{\psi}_1(x_0).$$

Thus, the first inequality of (4.8) is established and the second is obtained by a dual argument. Now let

$$\begin{aligned} m(r) &= \sup_{B(x_0, r)} (u-d)^- \\ v &= (u-d)^- \\ w &= m(2r)-v \\ w' &= w+\lambda r \end{aligned} \tag{4.9}$$

where λ is a number that dominates the sum of all the coefficients that appear in (1.1).

We now collect some often used estimates that arise when dealing with sub and supersolutions of (1.2). Since v is a subsolution of an equation of type (1.2), one may use $\varphi=\eta^p v$ as a test function and employ standard estimates to obtain

$$C^{-1}\int_{B(x_0, 2r)} \eta^p |\nabla v|^p\,dx \leqq m(2r)\int_{B(x_0, 2r)} \eta^{p-1}|\nabla\eta|\,|\nabla v|^{p-1}\,dx + m(2r)\int_{B(x_0, 2r)} \eta^p+|\nabla\eta|^p\,dx. \tag{4.10}$$

Also, since w is a supersolution, so is w' and because of the definition of λ one obtains the following estimate, cf. [GZ, equation (23)]:

$$\int_{B(x_0, 2r)} w'(x)^{-\beta}|\nabla w'(x)|^p \eta(x)^p\,dx \leqq C'\int_{B(x_0, 2r)} w'(x)^{p-\beta}[\eta(x)^p+|\nabla\eta(x)|^p]\,dx \tag{4.11}$$

for every $\beta>1$ and every nonnegative Lipschitz function η with compact support in $B(x_0, 2r)$. Finally, we will need the following weak Harnack inequality for supersolutions:

$$\inf_{B(x_0, \sigma r)} v \geqq C'\left\{\fint_{B(x_0, r)} v(x)^\gamma\,dx\right\}^{1/\gamma} - C''\lambda^{1/p} r \tag{4.12}$$

for any $\gamma < n(p-1)/(n-p)$ and any $\sigma \in (0, 1)$, cf. [T]. Proceeding as in the proof of Theorem 3.7, we let $A=\{u>d\}$ and since $E_1 \cap B(x_0, r) \subset A \cap B(x_0, r)$ for small $r>0$ we obtain

$$\begin{aligned}
& m(2r)^p \gamma_p[E_1 \cap B(x_0, r)] \\
&\leqq m(2r)^p \gamma_p[A \cap B(x_0, r)] \\
&\leqq \int_{B(x_0, 2r)} |\nabla(\eta w)|^p \, dx \\
&\leqq C \Big(\int_{B(x_0, 2r)} |\nabla w|^p \eta^p + w^p |\nabla \eta|^p \, dx \Big) \\
&= \Big(\int_{B(x_0, 2r)} |\nabla w'|^p \eta^p + w^p |\nabla \eta|^p \, dx \Big) \\
&\leqq C \Big(\frac{m(2r)}{r} \int_{B(x_0, 2r)} |\nabla w'|^{p-1} \eta^{p-1} \, dx + r^{n-p} \int_{B(x_0, 2r)} w^p \, dx + m(2r) r^{n-p} \Big) \\
&\hspace{28em} \text{(by (4.10))} \\
&\leqq C \Big(\frac{m(2r)}{r} \int_{B(x_0, 2r)} \big(\eta (w')^{-(1-\vartheta)} |\nabla w'|\big)^{p-1} \big((w')^{(1-\vartheta)(p-1)}\big) \, dx + I(r) \Big)
\end{aligned}$$

where ϑ is such that $1 < (1-\vartheta)p < n/(n-p)$ and

$$I(r) = r^{n-p} \int_{B(x_0, 2r)} w^p \, dx + m(2r) r^{n-p}$$

$$\begin{aligned}
&\leqq C \Big(\frac{m(2r)}{r} \Big(\int_{B(x_0, 2r)} (w')^{-(1-\vartheta)p} |\nabla w'|^p \eta^p \, dx \Big)^{(p-1)/p} \\
&\times \Big(\int_{B(x_0, 2r)} (w')^{(1-\vartheta)(p-1)p} dx \Big)^{1/p} + I(r) \Big) \\
&\leqq C \Big(\frac{m(2r)}{r} \Big(\int_{B(x_0, 2r)} (w')^{\vartheta p} [\eta^p + |\nabla \eta|^p] \, dx \Big)^{(p-1)/p} \Big(\int_{B(x_0, 2r)} (w')^{(1-\vartheta)(p-1)p} dx \Big)^{1/p} + I(r) \Big) \\
&\hspace{28em} \text{(by (4.11))} \\
&\leqq C \Big(\frac{m(2r)}{r} r^{n(p-1)/p} \Big(\int_{B(x_0, 2r)} (w')^{\vartheta p} \, dx \Big)^{(p-1)/p} \\
&\times r^{n/p} \Big(\int_{B(x_0, 2r)} (w')^{(1-\vartheta)(p-1)p} \, dx \Big)^{1/p} + I(r) \Big) \\
&\leqq C \Big(m(2r) r^{n-p} \Big(\int_{B(x_0, 2r)} (w')^{\vartheta p} dx \Big)^{(p-1)/p} \Big(\int_{B(x_0, 2r)} (w')^{(1-\vartheta)(p-1)p} dx \Big)^{1/p} + I(r) \Big) \\
&\leqq C \big(m(2r) r^{n-p} \big(\inf_{B(x_0, r)} w'^{\vartheta(p-1)} \big) \big(\inf_{B(x_0, r)} w'^{(1-\vartheta)(p-1)} \big) + I(r) \big) \quad \text{by (4.12)} \\
&\leqq C m(2r) r^{n-p} \inf_{B(x_0, r)} (w')^{p-1} + C m(2r) r^{n-p} \big(\inf_{B(x_0, r)} w + C' r \big)^{p-1} \\
&\leqq C m(2r) (r^{n-p}) [m(2r) - m(r) + \lambda r]^{p-1} + r^{n-p} \quad \text{for} \quad 0 < r < 1.
\end{aligned}$$

That is,

$$m(2r)^{p-1}\left(\frac{\gamma_p[E_1\cap B(x_0,r)]}{r^{n-p}}\right) \leqq C[m(2r)-m(r)+\lambda r]^{p-1} \quad \text{for} \quad 0<r<1, \tag{4.13}$$

which implies

$$\int_0^1 m(2r)\left(\frac{\gamma_p[E_1\cap B(x_0,r)]}{r^{n-p}}\right)^{1/(p-1)}\frac{dr}{r} < \infty.$$

Now refer to [GZ, Theorem 2.7] to see that this implies the existence of constants C_1 and C_2 such that for sufficiently small $r>0$

$$m(r) \leqq C_1 \exp\left(-C_2\int_{2s}^r A_1(t)\frac{dt}{t}\right)$$

whenever $s \leqq r/2$ and where

$$A_1(t) = \left(\frac{\gamma_p[E_1\cap B(x_0,t)]}{t^{n-p}}\right)^{1/(p-1)}.$$

As in the proof of Theorem 4.1 we have

$$\begin{aligned}\underline{\omega}(r) = u(x_0)-\underline{u}(r) &\leqq m(r)+\underline{\psi}_1(x_0)-\underline{\psi}_2(E_1,r)+r \\ &\leqq C_1\exp\left(-C_2\int_{2s}^r A_1(t)\frac{dt}{t}\right)+M_1(r)\end{aligned}$$

for every $s\leqq r/2$. This is the desired conclusion under (4.8). In case

$$\bar{\psi}_1(x_0) = u(x_0) = \underline{\psi}_2(x_0)$$

an estimate for the lower oscillation is obtained in the same way except that $d(r)$ is taken as

$$d(r) = \inf\{\underline{\psi}_1(E_1,r), \underline{\psi}_2(2r)\}-r.$$

An estimate for the upper oscillation is obtained in a similar way. □

5. Sharpness of results

In this section we show that the Wiener conditions for the obstacles ψ_1 and ψ_2 assumed in Theorem 4.4 are necessary for a non-trivial class of equations, including the linear equations of [DMV]. For this we assume that $A\colon R^n\times R^n\to R^n$ is a Borel function satisfying the following assumptions for almost every x, all h, and for some $0<\gamma\leqq\mu<\infty$:

$$|A(x,h)| \leqq \mu |h|^{p-1} \tag{5.1}$$
$$A(x,h)\cdot h \geqq \gamma |h|^p$$
$$\big(A(x,h_1)-A(x,h_2)\big)\cdot(h_1-h_2) > 0 \quad \text{for} \quad h_1 \neq h_2$$
$$A(x,\lambda h) = |\lambda|^{p-2}\lambda A(x,h) \quad \lambda\in R^1.$$

Lemma 5.1. *Suppose that $p>n-1$ or that $p=2$ and the equation*

$$-\operatorname{div} A(x,\nabla u) = 0$$

is linear. If x_0 is not a p-Wiener point of ψ_1, then there is a solution v of the unilateral obstacle problem in a neighborhood U of x_0 such that v is not continuous at x_0. Moreover,

$$\operatorname{ess}\liminf_{x\to x_0} v(x) < \bar{\psi}_1(x_0).$$

Proof. The cases $p>n-1$ are proved in [HK, Theorem 1.16]. The linear case is done by Mosco [M, Thm. 5.2]. This linear case can be easily treated without using Green's function by using arguments of [HK, Theorem 1.16] and [HKM, Theorem 3.2]. □

Theorem 5.2. *Suppose A satisfies* (5.1) *and that $p>n-1$ or that $-\operatorname{div} A(x,\nabla u)=0$ is linear $(p=2)$. Suppose, further, that there is a (quasicontinuous) Sobolev function u_0 such that $\psi_1\leqq u_0\leqq\psi_2$ in a neighborhood U of x_0. If x_0 is neither an upper $(p,p-1)$-Wiener point for ψ_1 nor a lower $(p,p-1)$-Wiener point for ψ_2, then there is a solution u which is discontinuous at x_0.*

Proof. Let us assume that x_0 is not a regular point for ψ_1. If u is a continuous solution of the double obstacle problem, then $u(x_0)\geqq\bar{\psi}_1(x_0)$, (see Remark 3.8). Thus, it is enough to show that for some solution u

$$\operatorname{ess}\liminf_{x\to x_0} u(x) < \bar{\psi}_1(x_0).$$

To this end, let v be the discontinuous solution to the single obstacle problem given by Lemma 5.1 with

$$\operatorname{ess}\liminf_{x\to x_0} v(x) < \bar{\psi}_1(x_0).$$

Let u be a solution to the double obstacle problem with

$$u-\min(v,u_0)\in W_0^{1,p}(U).$$

To conclude the proof, we show that $u\leqq v$ in U. For this, let

$$\eta = \min(v-u,0)\in W_0^{1,p}(U).$$

Then, $u+\eta \leqq u \leqq \psi_2$ and $u+\eta = \min(v, u) \geqq \psi_1$. Thus,

$$0 \leqq \int_U \big(A(x, \nabla u) - A(x, \nabla u)\big) \cdot \nabla \eta \, dx$$
$$= -\int_{\{v<u\}} \big(A(x, \nabla u) - A(x, \nabla v)\big) \cdot (\nabla u - \nabla v) \, dx = 0.$$

Thus, $\eta=0$ by (5.1), thus proving that $u \leqq v$ a.e. as desired. □

Remark 5.3. By refining the argument of [HK] we could dispense with the homogeneity assumption in (5.1).

6. Monotone operators

In this section we study operators that satisfy (1.1) as well as the following:

$$\text{(6.1)} \qquad \begin{gathered} \big(A(x, \eta, h_1) - A(x, \eta, h_2)\big) \cdot (h_1 - h_2) > 0 \quad \text{for} \quad h_1 \neq h_2 \\ |B(x, \eta, h)| \leqq \mu |\eta|^{p-1} + \nu. \end{gathered}$$

These operators are not covered by the structure imposed by (4.7). The object of this section is to prove the following result.

Theorem 6.1. *Suppose x_0 is a $(p, p-1)$-Wiener point for both obstacles ψ_1 and ψ_2. Then, if $\bar{\psi}_1(x_0) \leqq \underline{\psi}_2(x_0)$ and* (6.1) *holds, the solution u to the double obstacle problem is continuous at x_0.*

Before proving this result, we will need the following two Lemmas. Note that since u is bounded, the operator defined by

$$\tilde{A}(x, h) = A(x, u(x), h)$$
$$\tilde{B}(x) = B(x, u(x), \nabla u(x))$$

satisfy the same structural assumptions.

The following is a special case of a result proved in [MZ2].

Lemma 6.2. *Let $w \in W^{1,p}(\Omega)$, $w \geqq \psi_1$ (or $w \leqq \psi_2$). Then there exists $v \in W^{1,p}(\Omega)$ such that $v-w \in W_0^{1,p}(\Omega)$, $v \geqq \psi_1$ (or $v \leqq \psi_2$) and*

$$\int_\Omega \tilde{A}(x, \nabla v) \cdot \nabla \varphi \, dx + \int_\Omega \tilde{B}(x) \varphi \, dx \geqq 0$$

whenever $\varphi \in W_0^{1,p}(\Omega)$, $\varphi \geqq \psi_1 - u$ (or $\varphi \leqq \psi_2 - u$).

The following lemma is crucial in establishing Theorem 6.1.

Lemma 6.3. *Suppose that x_0 is a $(p, p-1)$-Wiener point for ψ_1 and ψ_2 and that*

$\bar{\psi}_1(x_0) \leqq \underline{\psi}_2(x_0)$. *Then for each* $d > \underline{\psi}_2(x_0)$,

$$\lim_{r\to 0} \fint_{B(x_0,r)} (u-d)^+ \, dx = 0$$

and for each $d < \bar{\psi}_1(x_0)$,

$$\lim_{r\to 0} \fint_{B(x_0,r)} (u-d)^- \, dx = 0.$$

Proof. We will prove only the first assertion, the proof of the second being similar.

Let $d > \underline{\psi}_2(x_0)$. Then, for some $\varepsilon > 0$, $\psi_1 < d - \varepsilon$ q.e. in some neighborhood U of x_0. Furthermore, the set

$$E = U \cap \{\psi_2(x) \leqq d\}$$

is not $(p, p-1)$-thin at x_0. Choose a compact set $K \subset E \cup \{x_0\}$ such that K is not $(p, p-1)$-thin at x_0 and let $G = U - K$. Let w be a quasicontinuous function in $W^{1,p}(\Omega)$ such that $w - u \in W_0^{1,p}(U)$ and $w = d$ in K. Let v be the solution to the single obstacle problem in G with obstacle ψ_1 and with $v - w \in W_0^{1,p}(G)$; the existence of v being assured by Lemma 6.2. We claim that $v \geqq u$ in G. Indeed, let

$$\eta = \min(v - u, 0).$$

Then $\eta \in W_0^{1,p}(G)$ is nonpositive and admissible for (1.2) since $\psi_1 \leqq u + \eta \leqq \psi_2$ in G. Thus,

$$\begin{aligned} 0 &\leqq \int_G \left(A(x, u(x), \nabla u(x)) - \tilde{A}(x, \nabla v)\right) \cdot \nabla \eta \, dx \\ &\quad + \int_G \left(B(x, u(x), \nabla u(x)) - \tilde{B}(x)\eta \, dx\right. \\ &= -\int_{\{u<v\}} \left(A(x, u(x), \nabla u(x)) - A(x, u(x), \nabla v(x))\right) \cdot (\nabla u - \nabla v) \, dx \\ &\leqq 0. \end{aligned}$$

Thus, $\eta = 0$ by (6.1) and hence $v \geqq u$ which establishes our claim.

Now v is the solution to a single obstacle problem and is therefore a supersolution in G of the associated equation. Since $v - w \in W_0^{1,p}(G)$ and $w = d$ in K, it follows from [GZ, Theorem 2.2] that

$$\liminf_{x\to x_0} v(x) \geqq d,$$

whence $v \geqq \psi_1 + \varepsilon$ q.e. in $B(x_0, r) \cap G$ for some $r > 0$. This implies that v in fact is a solution of the equation

$$-\operatorname{div} \tilde{A}(x, \nabla v) + \tilde{B}(x) = 0$$

in $B(x_0, r) \cap G$. Now since v is a solution, we may employ [GZ, Theorem 2.2] again to obtain

$$\lim_{x\to x_0} v(x) = d.$$

Thus,

$$\limsup_{r\to 0} \fint_{B(x_0,r)} (u-d)^+ dx \leqq \lim_{r\to 0} \fint_{B(x_0,r)} (v-d)^+ dx = 0$$

which completes the proof. □

We now return to the proof of Theorem 6.1. Under the hypotheses of this theorem, we know from Theorem 3.2 that u has a Lebesgue point at x_0. The result above implies that $\bar{\psi}_1(x_0) \leqq u(x_0) \leqq \underline{\psi}_2(x_0)$. Now appeal to Corollary 3.3. □

References

AH. ADAMS, D. R. and HEDBERG, L. I., Inclusion relations among fine topologies in non-linear potential theory, *Indiana Univ. Math. J.* **33** (1984), 117—126.

DMV. DAL MASO, G., MOSCO, U. and VIVALDI, M. A., A pointwise regularity theory for the two-obstacle problem, *Acta Math.* **163** (1989), 57—107.

DBT. DIBENEDETTO, E. and TRUDINGER, N. S., Harnack inequalities for quasi-minima of variational integrals, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **4** (1984), 295—307.

GZ. GARIEPY, R. and ZIEMER, W. P., A regularity condition at the boundary for solutions of quasilinear elliptic equations, *Arch. Rational Mech. Anal.* **67** (1977), 25—39.

GT. GILBARG, D. and TRUDINGER, N. S., *Elliptic partial differential equations of second order*, Second Ed. Springer-Verlag, New York, 1983.

HK. HEINONEN, J. and KILPELÄINEN, T., On the Wiener criterion and quasilinear obstacle problems, *Trans. Amer. Math. Soc.* **310** (1988), 239—255.

HKM. HEINONEN, J., KILPELÄINEN, T. and MARTIO, O., Fine topology and quasilinear elliptic equations, *Ann. Inst. Fourier (Grenoble)* **39** 2 (1989), 293—318.

LU. LADYZHENSKAYA, O. A. and URAL'TSEVA, N. N., *Linear and quasilinear elliptic equations*, Academic Press, New York, 1968.

MA. MAZ'YA, V. G., On the continuity at a boundary point of the solution of quasi-linear elliptic equations, *Vestnik Leningrad. Univ. Mat. Mekh. Astronom.* **25**:13 (1970), 42—55 (in Russian). English translation: *Vestnik Leningrad Univ. Math.* **3** (1976), 225—242.

MZ1. MICHAEL, J. H. and ZIEMER, W. P., Interior regularity for solutions to obstacle problems, *Nonlinear Analysis, TMA* **10** (1986), 1427—1448.

MZ2. MICHAEL, J. H. and ZIEMER, W. P., Existence of solutions to obstacle problems, Preprint, 1988.

M. MOSCO, U., Wiener criterion and potential estimates for the obstacle problem, *Indiana Univ. Math. J.* **36** (1987), 455—494.

T. TRUDINGER, N. S., On Harnack type inequalities and their applications to quasilinear elliptic equations, *Comm. Pure Appl. Math.* **20** (1967), 721—747.

Received November 20, 1989

Tero Kilpeläinen
Department of Mathematics
University of Jyväskylä
SF-40100 Jyväskylä, Finland

William P. Ziemer
Department of Mathematics
Indiana University
Bloomington, Indiana 47405
U.S.A.