Representations of bounded harmonic functions

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Summary

An open subset D of R^d, $d \geqq 2$, is called Poissonian iff every bounded harmonic function on the set is a Poisson integral of a bounded function on its boundary. We show that the intersection of two Poissonian open sets is itself Poissonian and give a sufficient condition for the union of two Poissonian open sets to be Poissonian. Some necessary and sufficient conditions for an open set to be Poissonian are also given. In particular, we give a necessary and sufficient condition for a Greenian D to be Poissonian in terms of its Martin boundary.

1. Introduction

Let D be an open subset of R^d, $d \geqq 2$. A problem of long interest has been to characterize the functions harmonic on D that are in some special collection of harmonic functions on D. One collection that has attracted much attention is the collection of bounded harmonic functions on D.

Suppose D is an open ball. The classical Poisson integral representation then solves this problem in a very satisfactory manner. Every bounded harmonic function on D is the Poisson integral of a bounded measurable function on its geometric boundary ∂D. If we identify functions on ∂D that differ only on sets of harmonic measure 0 then this representation is unique. It is natural to inquire to what extent this result on the ball carries over to other open sets.

Let D be an open subset of R^d, $d \geqq 2$, and let ∂D be the boundary of D. For each $x \in D$ let $\pi_D(x, dy)$ be the harmonic measure of D at x. If D is unbounded let $\psi(x)$ be the harmonic measure of D at x on the set $\{\infty\}$.

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Definition 1. A bounded harmonic function f on D is representable if there is a bounded Borel function φ on ∂D and a constant α such that

$$f(x) = \pi_D \varphi(x) + \alpha \psi(x). \tag{1.1}$$

An open set D is Poissonian iff every bounded harmonic function is representable.

Remark. If D is bounded then $\psi(x) \equiv 0$ in the above definition. (See Fact 1 below.) If f is representable then its representation is unique provided we identify functions on ∂D that differ only on sets of harmonic measure 0. (See Fact 3.)

Let $\bar{R}^d$ be the space R^d compactified by adding the point ∞. View D as a subset of $\bar{R}^d$, so if D is unbounded its boundary now contains ∞, and let $(\partial D)^*$ denote the boundary of D in $\bar{R}^d$. The harmonic measure π_D^* of $(\partial D)^*$ is $\pi_D^*(x, A) = \pi_D(x, A) + \psi(x) 1_A(\infty)$. Note that (1.1) is the same as $f(x) = \pi_D^* \varphi^*(x)$, where $\varphi^*(x) = \varphi(x)$, $x \in \partial D$ and $\varphi^*(\infty) = \alpha$.

Not every open set is Poissonian. For example, as shown in [4] the set D that is the open disk punctured by a line segment from the origin to a point on the boundary is not Poissonian. Observe that this set is star shaped about the origin. In [4] an open set D was called *strongly star shaped* about the origin iff $\overline{rD} \subseteqq D$ for all $0 < r < 1$. It was shown in [4] that a strongly star shaped open set is Poissonian.

Let D be an open set. If D^c is a polar set then every bounded harmonic function on D is a constant so D is Poissonian. Suppose that D^c is not polar. Then D is Greenian and therefore it has a Martin boundary Δ. (For D not connected we take the Martin boundary of D to be the union of the Martin boundaries of its components.) The Martin representation shows that every bounded harmonic function on D is a Poisson integral of a bounded measurable function on Δ. That is, if $\mu_D(x, \cdot)$ is the harmonic measure at x on Δ then every bounded harmonic function f on D is of the form $f(x) = \mu_D \varphi(x)$ for φ a bounded measurable function on Δ. This representation is unique provided we identify functions on Δ differing only on sets of harmonic measure 0.

If D is Poissonian we have the exact analog of the classical result on the ball. There is a 1-1 correspondence between bounded π_D^* integrable functions on $(\partial D)^*$ and the bounded harmonic functions on D. This provides a very simple understanding of what the bounded harmonic functions are like on D. Also, in view of Fact 3, if we are willing to go along Brownian motion paths to $(\partial D)^*$, every bounded harmonic function has boundary values along such paths with the boundary function being the representing function φ^*. If D is not Poissonian the structure of the bounded harmonic functions is more complicated and for an adequate understanding of these functions we must replace $(\partial D)^*$ with the Martin boundary Δ. Now, in general, the Martin boundary is rather complicated, and often not exactly known, so it is somewhat unclear what exactly the representation in terms of the Martin boundary tells us about the bounded harmonic functions on D. Additional nice properties valid for all bounded harmonic functions on a Poissonian D but not valid on an arbitrary D will be given in Theorems 5 and 6.

Suppose the Martin boundary Δ and the geometric boundary $(\partial D)^*$ are the same (i.e. are homeomorphic), then D is Poissonian. On the other hand it is certainly not necessary for Δ and $(\partial D)^*$ to be the same for D to be Poissonian. For example take D the region inside the unit disk but exterior to the disk of center $(1/2, 0)$ and radius $1/2$. Here the Martin and geometric boundaries are different but our Theorem 1 shows D is a Poissonian open set.

In this example the Martin and geometric boundaries are not very different. Our next example will be much more extreme.

Example 1. Let Γ be a Jordan curve in R^2 and let D_1 and D_2 be the simply connected domains complementary to Γ. Assume D_1 is conformally equivalent to the interior of the unit circle and D_2 is conformally equivalent to the exterior of this circle. Then D_1 and D_2 are Poissonian. The Martin boundary of $D=D_1\cup D_2$ consists exactly of two copies of Γ. Call these Γ_1 and Γ_2. Thus here the Martin boundary $\Delta=\Gamma_1\cup\Gamma_2$ and the geometric boundary are topologically very different. Every point on the geometric boundary corresponds to exactly two minimal Martin boundary points. However D may or may not be Poissonian.

In [2] it is shown that the harmonic measures H_{D_1} and H_{D_2} are mutually singular iff the set of tangent points of Γ has 0 linear measure. Thus, if Γ is twisty enough, these measures are singular. Clearly D is Poissonian iff these measures are singular so it is possible to have D Poissonian. This example can be modified to have D connected by replacing a Jordan curve with a Jordan arc.

The above shows that it is possible to have D Poissonian but yet the Martin and geometric boundary are topologically quite different. It also shows that such conditions as having the set of points on ∂D that correspond to multiple points on Δ having harmonic measure 0 are not necessary.

There is really not much of a connection between the topology of $(\partial D)^*$ and that of Δ that is implied by D being Poissonian. For the problem of characterizing a Poissonian open set D in terms of its Martin boundary is really a *measure theoretic one*.

Let the measurable subsets of $(\partial D)^*$ be its Borel sets completed by the sets of harmonic measure 0 and let the measurable subsets of Δ be its Borel sets completed by the sets of harmonic measure 0. In Theorem 9 we will show that D is Poissonian iff there is a measurable mapping φ of $(\partial D)^*\to\Delta$ such that $\mu_D(x, \Lambda)=\pi_D^*(x, \varphi^{-1}(\Lambda))$.

Now for any open Greenian D there is always a measurable mapping $\psi\colon \Delta\to(\partial D)^*$ such that $\pi_D^*(x, B)=\mu_D(x, \psi^{-1}(B))$. (See § 9.) Using this fact and the result of Theorem 9 we will show in Theorem 10 that D is Poissonian iff $(\partial D)^*$

and Δ are related as follows. There is a set $B\subseteqq(\partial D)^*$ having zero harmonic measure and there is a set $\Delta_1\subseteqq\Delta$ that contains the non-minimal points of Δ and has 0 harmonic measure and there is a mapping φ: $\partial D\setminus B\to\Delta\setminus\Delta_1$ such that φ is 1-1, onto, bimeasurable, and $\pi_D(x, B)=\mu_D(x, \varphi(B))$, $\mu_D(x, \Lambda)=\pi_D^*(x, \varphi^{-1}(\Lambda))$. In other words D is Poissonian iff $(\partial D)^*$ and Δ are equivalent as measure spaces.

Example 1. (Continued.) Assume H_{D_1} and H_{D_2} are singular. Let B be a set $\subseteqq\Gamma$ such that $H_{D_1}(x, B)\equiv 1$ on D_1 and $H_{D_2}(x, B^c)\equiv 1$ on D_2. Let B_1 and B_2 the copies of B on Γ_1 and Γ_2. Then $\mu_D(x, B_1)\equiv 1$ on D_1 and $\mu_D(x, B_2^c)\equiv 1$ on D_2. Here $\Delta_1=B_1^c\cup B_2$. Let $x\to(\xi_1, \xi_2)\xi_i\in\Gamma_i$. Then $\varphi(x)=\xi_1$, $x\in B$ and $\varphi(x)=\xi_2$, $x\in B^c$ yields the mapping in Theorem 10.

Except for Theorems 9 and 10 our main concern in this paper will be to give conditions for D to be Poissonian that do not involve the Martin boundary of D. Our results are purely analytical. However, the methods used are purely probabilistic and fully involve the connection between classical potential theory and the theory of Brownian motion. The full story of this connection can be found in [4]. The proofs of our results will constantly use certain facts of probabilistic potential theory. Some of these facts cannot be found in the standard references in exactly the form required (but are easy consequences of facts that are in [4]). For this reason we will gather together in § 2 those facts that we shall need. One of our concerns in this paper will be to determine when an open set D that is put together from other open sets that are known to be Poissonian is itself Poissonian, e.g. intersections and unions. In Theorem 1 we will show rather remarkably that the intersection of two Poissonian sets is always itself Poissonian. This in turn will show (see Theorem 4) that Poissonianess of D is actually a local property of ∂D, a fact which is not obvious from either the definition or the characterization in terms of the Martin boundary.

There is an interesting strengthening of the result of Theorem 10 in terms of Brownian motion. Let the Brownian motion start at $x\in D$ and let T_D be the first time it leaves D (with $T_D=\infty$ if it never leaves D). Let $P_x(\cdot)$ be the law of a Brownian motion starting at x. Let X_{T_D} be the place where the Brownian motion first hits $(\partial D)^*$ (so $X_{T_D}=\infty$ on $[T_D=\infty]$). Let Z be the place the Brownian motion first hits Δ. Then $\mu_D(x, \cdot)$ is the distribution of Z and $\pi_D^*(x, \cdot)$ is the distribution of X_{T_D}. In theorem 9 we actually show that if D is Poissonian then $Z=\varphi(X_{T_D})$ a.e. P_x for all $x\in D$, and in Theorem 10 we show that also $X_{T_D}=\varphi^{-1}(Z)$ a.e. P_x for all $x\in D$.

Suppose D is Poissonian. Then if we identify functions on $(\partial D)^*$ that only differ on sets of $\pi_D^*(x, \cdot)$ measure 0 there is a 1-1 correspondence between bounded measurable functions on $(\partial D)^*$ and bounded harmonic functions on D. One can recapture the function φ^* on the boundary by taking limits along Brownian motion paths. (See Fact 3.) That is, starting from any $x\in D$, $\lim_{t\uparrow T_D} f(X_t)=\varphi^*(X_{T_D})$ with probability one. This suggests that in some sense a Brownian motion can only hit $(\partial D)^*$ in a unique way. But that is exactly what $P_x(Z=\varphi(X_{T_D}))\equiv 1$ on D tells us. The various approaches to a point on $(\partial D)^*$ are represented by the Martin boundary points that correspond to that point. Thus starting from any $x\in D$ the Brownian motion hits $(\partial D)^*$ at X_{T_D} only along the route that corresponds to the minimal Martin boundary point $\varphi(X_{T_D})$.

Let D be an open set. The point ∞ as a boundary point in $(\partial D)^*$ plays a distinguished role. It is the only point that can have positive harmonic measure. Now a Brownian motion starting at $x\in D$ can only go to ∞ in a unique way. To see this intuitively suppose first that $D=R^d$. Then that is certainly the case; for we are just following the entire path. Now, for any D, as we follow this path either at some finite time the path hits ∂D, in which case the path does not go to infinity, or it never hits ∂D, in which case it ignores ∂D. Suppose that D is Greenian. In general, there can be many minimal Martin boundary points that correspond to ∞. The consideration of Brownian motion above suggests that on the set Δ_0 that corresponds to ∞, either $\mu_D(x, \Delta_0)\equiv 0$ or there is exactly one point $\xi\in\Delta_0$ such that $\mu_D(x, \{\xi\})>0$ for some x and $\mu_D(x, \Delta_0\setminus\{\xi\})\equiv 0$. We will show in Theorem 11 that that is indeed the case.

Statement of results

Theorem 1. *Suppose D_1 and D_2 are Poissonian open sets. Then $D_1\cap D_2$ is a Poissonian open set.*

Theorem 2. *Let D be an open subset of R^d and let $\{r_n\}$ be a sequence of positive integers such that $r_n\to\infty$ as $n\to\infty$. Let B_r be the open ball of center 0 and radius r and let $D_n=D\cap B_{r_n}$. If each of the D_n are Poissonian open sets, then so is D. Conversely if D is Poissonian then so are all the D_n.*

Unlike the intersection, the union of two Poissonian open sets need not be Poissonian. For example let A_1 be the rectangle with vertices (0, 0), (2, 0), (0, 1), (2, 1), A_2 the rectangle with vertices (0, 1), (0, 2), (1, 0), (1, 2) and A_3 the rectangle with vertices (1, 0), (1, 2), (2, 0), (2, 2). Let D_1 be the L-shaped domain that is $A_1\cup A_2$ and let $D_2=A_1\cup A_3$. Then D_1, D_2 are Poissonian domains but their union is the square with vertices (0, 0), (2, 0), (0, 2), (2, 2) with the line segment connecting (1, 2) to (1, 1) removed, which is not a Poissonian domain.

Theorem 3. *Let D_1 and D_2 be a Poissonian open set and let $D=D_1\cup D_2$. If $\pi_D(x, \partial D_1\cap\partial D_2)\equiv 0$ on D then D is Poissonian.*

Let D be open and let $B(a, r)$ be the open ball of center a and radius r. Theorem 1 shows that if D is Poissonian then so is $D\cap B(a, r)$ for all $a\in\partial D$ and all r. John Garnett proposed that a converse should also be true. That this is the case is our next result.

Theorem 4. *In order for D to be Poissonian it is necessary and sufficient that there exist a finite or coutably infinite family of open balls $\{B(a_i, r_i)\}$ such that* (a) $\partial D\subseteqq\bigcup_i B(a_i, r_i)$ *and* (b) $B(a_i, r_i)\cap D$ *is Poissonian for each i.*

Definition 1. Let f be defined on $\bar{D}$, the closure of D. We say f is essentially continuous on $\bar{D}$ iff it is continuous at all points of $\bar{D}$ except perhaps for those in a polar subset of ∂D.

Theorem 5. *Let D be a connected open subset of R^d. Let φ be bounded on ∂D and let α be a constant. If* (1.1) *holds, then there is a sequence $\{f_n\}$ of bounded harmonic functions on D such that* (a) *f_n is essentially continuous on $\bar{D}$,* (b) $\sup_n \|f_n\|_\infty\leqq M$, *and* (c) *$f_n\to f$ uniformly on compact subsets of D. Conversely, if f is a bounded harmonic function on D such that there is a sequence $\{f_n\}$ of bounded harmonic functions on D satisfying* (a)—(c) *then there is a bounded φ and a constant α such that* (1.1) *holds.*

An immediate consequence of Theorem 5 is the following.

Corollary 1. *Let D be a connected open set. Then D is Poissonian iff every bounded harmonic function on D is the limit of a sequence $\{f_n\}$ bounded harmonic functions satisfying* (a)—(c) *of Theorem* 5.

Using a recent result of Ancona [1] it is possible to improve the continuity part of Theorem 5.

Theorem 6. *Let D be a connected open set and let φ be bounded on ∂D. Set $f=\pi_D\varphi$. There is then a sequence $\{f_n\}$ of bounded harmonic functions on D that are continuous on $\bar{D}$ such that* (b) *and* (c) *of Theorem* 5 *hold.*

Theorem 7. *Let D be Greenian. In order for D to be Poissonian it is necessary and sufficient that there be a measurable mapping φ: $(\partial D)^*\to\Delta$ such that $\mu_D(x, \Lambda)=\pi_D^*(x, \varphi^{-1}(\Lambda))$. If D is Poissonian we can choose φ such that $Z=\varphi(X_{T_D})$ a.e. P_x, $x\in D$.*

Theorem 8. *Let D be Greenian. In order for D to be Poissonian it is necessary and sufficient that the following holds. There are measurable sets $B\subseteqq(\partial D)^*$ and $\Delta_1\subseteqq\Delta$ such that $\pi_D^*(x, B)\equiv 0$, Δ_1 contains all the non-minimal points of Λ and $\mu_D(x, \Lambda_1)\equiv 0$, and a mapping φ: $\partial D\setminus D\to\Delta\setminus\Delta_1$ such that φ is* 1-1, *onto, bimeasurable and measure preserving in both directions with measure π_D^* on $(\partial D)^*$ and μ_D*

on Δ. In that case φ can be chosen so that $Z=\varphi(X_{T_D})$ and $X_{T_D}=\varphi^{-1}(Z)$ a.e. P_x for all x.

Theorem 9. *Let D be a Greenian open set with Martin boundary Δ and harmonic measure $\mu_D(x,\cdot)$ on Δ. There is a measurable mapping $\psi\colon \Delta\to(\partial D)^*$ such that $\psi(Z)=X_T$ P_x a.e. for all $x\in D$. Let $\Delta_0=\psi^{-1}(\{\infty\})$. Then $\mu_D(x,\Delta_0)=P_x(T_D=\infty)$. Either $\mu_D(x,\Delta_0)\equiv 0$ or there is a unique point $\xi\in\Delta_0$ such that $\mu_D(x,\{\xi\})=\mu_D(x,\Delta_0)$.*

We view the set Δ_0 as the points in the Martin boundary that correspond to ∞. The mapping ψ is not unique but another such mapping ψ_1 must satisfy $P_x(\psi(Z)=\psi_1(Z))=1$ for all $x\in D$. Let $\Delta_1=\psi_1^{-1}(\{\infty\})$. Then it must be that

$$\mu_D(x,\Delta_1\Delta\Delta_0)\equiv 0.$$

Let ξ_1 be the point picked out by ψ_1 in Δ_1 having positive measure. Suppose $\xi_1\neq\xi$. Then ξ_1 and ξ cannot be in $\Delta_1\cap\Delta_0$ nor can they be in $\Delta_1\Delta\Delta_0$, which is impossible. Thus $\xi_1=\xi$. Hence all maps ψ pick out the same point in Δ as the only point having positive measure corresponding to ∞.

Example 2. Denjoy Domains. Let K be a closed subset of a hyperplane in R^d. Its complement D in R^d is called a Denjoy domain. Such a domain may or may not be Poissonian. For example in R^2 if K is the x-axis then D is not Poissonian while for K a polar set D is Poissonian. It is of some interest to determine necessary and sufficient conditions on K for D to be Poissonian and when D is not Poissonian to give necessary and sufficient conditions on a bounded harmonic function f to be representable.

It is not difficult to show by direct example that if K has positive hyperplane Lebesgue measure then D is not Poissonian and then for f to be representable it is necessary that it be symmetric with respect to reflection across the hyperplane. It turns out that K having positive hyperplane Lebesgue measure is in fact necessary and sufficient for D to be non-Poissonian and the symmetry of f is sufficient as well as necessary for it to be representable. Originally, we proved these facts some three years ago by purely probabilistic arguments which did however use some refined properties of Brownian motion. Since then a purely analytic proof has been produced by Bishop and will appear in his paper on Poisson Domains [3]. For that reason we will omit our proofs here.

Some Remarks. The only previous results on conditions for D be the Poissonian that we know are in [2] and [4]. The result in [2], discussed in Example 1, and the result in [4] that a strongly star shaped domain is Poissonian, are proved by classical (non-probabilistic) methods. The results here on the other hand are obtained by probabilistic arguments. The probabilistic approach seems to us to be the more natural method for the problems addressed in this paper.

After seeing the first version of this paper, C. J. Bishop [3] obtained by purely analytic methods a necessary and sufficient condition for D to be Poissonian. This condition can be used to give analytic proofs of some of our results. In turn Bishop's results can be proved by our methods. See section 10.

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2. Preliminary facts

In order to prove Theorems 1—10 we will need some preliminary facts. These all may be found in [4] or are simple consequences of facts in [4].

Let X_t be the Brownian motion process on R^d. Let $T_D=\inf\{t>0\colon X_t\notin D\}(=\infty$ if $X_t\in D$ for all $t>0)$ be the first exist time from D and let $H_D(x, dy)=P_x(X_{T_D}\in dy, T_D<\infty)$ where $P_x(\cdot)$ is the law of the process X_t when $X_0=x$. Let E_x be expectation with respect to P_x.

Fact 1. $H_D(x, dy)=\pi_D(x, dy)$, $x\in D$, and $\psi(x)=P_x(T_D=\infty)$. If D is bounded $P_x(T_D<\infty)\equiv 1$, so $\psi(x)\equiv 0$.

Fact 2. Let f be a bounded harmonic function on D. Then for $x\in D$, $P_x(\lim_{t\uparrow T_D} f(X_t)$ exists$)=1$, and calling the limit ξ, $f(x)=E_x\xi$, $x\in D$. If D is Greenian with Martin boundary Δ, and ψ represents f on Δ, then $P_x(\xi=\psi(Z))=1$ for all $x\in D$.

Fact 3. If $f(x)=H_D\varphi(x)+\alpha P_x(T_D=\infty)$ for α a constant and φ bounded, then f is a bounded harmonic function on D, and for $x\in D$

$$\lim_{t\uparrow T_D} f(X_t) = \varphi(X_{T_D})1_{[T_D<\infty]}+\alpha 1_{[T_D=\infty]}$$

a.s. P_x.

Fact 4. If φ_1 and φ_2 are bounded and $H_D\varphi_1=H_D\varphi_2$ on D, then a.s. P_x for $x\in D$

$$\varphi_1(X_{T_D}) = \varphi_2(X_{T_D}) \quad \text{on} \quad [T_D<\infty].$$

Fact 5. For any x

$$H_D\varphi(X_{T_D}) = \varphi(X_{T_D}) \quad \text{a.s.} \quad P_x \quad \text{on} \quad [T_D<\infty].$$

Fact 6. If D_1 and D_2 are open sets and $D_1\subseteqq D_2$ then $H_{D_1}H_{D_2}(x, dy)=H_{D_2}(x, dy)$ for $x\in D_1$.

Fact 7. If f is a bounded harmonic function on D then there is a bounded harmonic function h on D such that $|h(x)|\leqq KP_x(T_D<\infty)$ for $x\in D$, where K is a constant, and for some constant α,

$$f(x) = h(x)+\alpha P_x(T_D=\infty) \quad \text{on} \quad D.$$

Fact 8. The complement, D^c, of D is called recurrent iff $P_x(T_D<\infty)\equiv 1$. Otherwise it is called transient. If D is bounded then D^c is recurrent.

Fact 9. If f is a bounded harmonic function on D and

$$f(x) = H_D\varphi(x)+\alpha P_x(T_D=\infty)$$

then $\alpha=0$ if $|f(x)|\leqq KP_x(T_D<\infty)$ for some constant K.

Fact 10. Let $x\in D$. If D is transient then

$$\lim_{t\uparrow T_D} P_{X_t}(T_D=\infty) = 0, \quad \text{a.s.} \quad P_x, \quad \text{on} \quad [T_D<\infty].$$

Using Fact 1 the representation (1.1) of f can be written as

$$f(x) = H_D\varphi(x)+\alpha P_x(T_D=\infty), \quad x\in D. \tag{2.1}$$

It is this representation that we shall use throughout the proofs.

3. Proof of Theorem 1

Let D_1 and D_2 be Poissonian and let $D=D_1\cap D_2$. If $D=\emptyset$ there is nothing to prove. Assume that $D\neq\emptyset$ and let f be a bounded harmonic function on D. Using Fact 7 it suffices to prove the theorem for those f that satisfy the condition $|f(x)|\leqq KP_x(T_D<\infty)$ for some constant K. We henceforth assume f is of this type.

The main problem is to use the fact that all bounded harmonic functions on the D_i are representable to conclude that this is true for those on $D=D_1\cap D_2$. Though analytically it is not clear how to proceed the basic Fact 2 suggests an approach. Consider a Brownian path starting at $x\in D_i$ that first exits D_i by eventually first entering $D_1\cap D_2$ and then first exiting D_i via $\partial D\cap\partial D_i$. The limit of f (harmonic on D) along such a path till the exit time from D_i makes sense and calling the limit ζ should be that $E_x\zeta$ is a harmonic function on D_i whose boundary values on $\partial D\cap\partial D_i$ should be those for f. The actual details are a bit more complicated but this is the essential idea on how to obtain a harmonic function on D_i whose boundary values yield those for f on $\partial D\cap\partial D_i$.

The proof will proceed via seven lemmas. The first, of a technical nature, establishes needed measurability. Lemmas 2—5 show the boundary of D_i can be decomposed into a "good part" and a "bad part", and the bad parts are not relevant. The next two lemmas show the limits of f along Brownian paths that first exit from the good parts exist and the expectation of these limits yield harmonic functions on D_i. The final lemma shows that the boundary values of these harmonic functions yield those for f so f is representable.

For a Borel set B let $\mathcal{T}_B=\inf\{t>0: X_t\in B\}(=\infty$ if $X_t\notin B$ for all $t>0)$ be the hitting time of B. Observe that $T_D=\mathcal{T}_{D^c}$.

Lemma 3.1. *Let* $W_1\subseteq\overline{W}_1\subseteq W_2\subseteq\ldots$ *be an increasing sequence of open subsets of* D_1 *with union* D_1. *Let* $T_n=T_{W_n}$. *Then a.s.* P_x, $x\in D_1$, $T_n\uparrow T_{D_1}$. *Let*

$$\Gamma = \varlimsup_{t\uparrow T_{D_1}} [X_t\in D_2^c].$$

Let $\mathcal{F}_n=\mathcal{F}_{T_n}$ *and let* $\mathcal{F}$ *be the σ-field generated by the* $\{\mathcal{F}_n\}$. *Then* $\Gamma\in\mathcal{F}$.

Proof. The first assertion is obvious. As to the second note that

$$\Gamma = \bigcap_n \bigcup_{m>n} [\mathcal{T}_{D_2^c}\circ\theta_{T_n} < T_m\circ\theta_{T_n} = T_m]$$

and the event $[\mathcal{T}_{D_2^c}\circ\theta_{T_n}<T_m]\in\mathcal{F}_m\subseteq\mathcal{F}$.

Lemma 3.2. *Let*

$$\Gamma_1 = \big[\omega: \varlimsup_{t\uparrow T_{D_1}(\omega)} [X_t(\omega)\in D_2^c,\ T_{D_1}(\omega)<\infty]\big].$$

Then there is a partition K_1 *and* G_1 *of* ∂D_1 *such that for all* $x\in D_1$

$$P_x([X_{T_{D_1}}\in G_1]\cap\Gamma_1) = 0, \tag{3.1}$$

and

$$P_x([X_{T_D}\in K_1]\cap\Gamma_1) = P_x(X_{T_{D_1}}\in K_1,\ T_{D_1}<\infty). \tag{3.2}$$

Proof. Let $h(x)=P_x(\Gamma_1)$. The function h is harmonic on D_1. To see this let $B_r(x)$ be the open ball of radius r and center x and let $T_r=T_{B_r(x)}$. Then $H_{B_r(x)}(x, dy)=\sigma_r(dy)$, the uniform distribution on $\partial B_r(x)$. Choose $r>0$ so that $\overline{B_r(x)}\subseteq D_1$. Then the strong Markov property shows

$$h(x) = P_x(\Gamma_1) = \int \sigma_r(dy)\, P_y(\Gamma_1) = \int \sigma_r(dy)\, h(y).$$

Thus h is harmonic.

Let W_n, $\mathcal{F}_n$ and $\mathcal{F}$ be as in Lemma 3.1. That lemma shows Γ_1 is $\mathcal{F}$ measurable. Consequently a.s. P_x for $x\in D$,

$$\lim_{n\to\infty} h(X_{T_n}) = \lim_{n\to\infty} E_{x_{T_n}} 1_{\Gamma_1} = \lim_{n\to\infty} E[1_{\Gamma_1}|\mathcal{F}_n] = 1_{\Gamma_1}. \tag{3.3}$$

By assumption D_1 is Poissonian. Thus by Fact 9 there is a bounded function ψ_1

such that $h=H_{D_1}\psi$ on D_1. By Fact 3, a.s. P_x, $x\in D_1$,

$$\lim_{t\uparrow T_{D_1}} h(X_t) = \psi(X_{T_{D_1}})1_{[T_{D_1}<\infty]}.$$

But the LHS is a.s. P_x equal to $\lim_n h(X_{T_n})$ which by (3.3) is 1_{Γ_1}. Thus a.s. P_x, $x\in D_1$,

$$\psi(X_{T_{D_1}})1_{[T_{D_1}<\infty]} = 1_{\Gamma_1}. \tag{3.4}$$

Let $A=\{x\in\partial D_1: \psi(x)=0 \text{ or } \psi(x)=1\}$. It follows from (3.4) that $H_D(x, \partial D_1\setminus A)\equiv 0$ on D_1. Taking $K_1=\{y\in\partial D_1: \psi(y)=1\}$ now yields the desired partition.

Lemma 3.3. *Let*

$$\Gamma_2 = [\omega: \overline{\lim_{t\uparrow T_{D_2}}} [X_t\in D_1^c, T_{D_2}<\infty]].$$

Then there is a partition on K_2 and G_2 of ∂D_2 such that for all $x\in D_2$,

$$P_x(X_{T_{D_2}}\in K_2\cap\Gamma_2) = P_x(X_{T_{D_2}}\in K_2, T_{D_2}<\infty)$$

and

$$P_x(X_{T_{D_2}}\in G_2\cap\Gamma_2) = 0.$$

Proof. Interchange the roles of D_1 and D_2 in Lemma 4.2.

Lemma 3.4. *Let $D=D_1\cap D_2$. For all $x\in D$, $H_D(x, K_1\cup K_2)=0$.*

Proof. By Lemma 3.2 for any $x\in D_1$ (so in particular for $x\in D$),

$$P_x([X_{T_{D_1}}\in K_1]\cap\Gamma_1^c) = 0.$$

Thus for $x\in D$

$$P_x([X_{T_{D_1}}\in K_1]\cap[T_{D_1}\leqq T_{D_2}]) = 0.$$

Since $T_{D_2}\geqq T_D$ we see that

$$P_x([X_{T_{D_1}}\in K_1]\cap[T_{D_1}\leqq T_D]) = 0.$$

Hence for $x\in D$, $P_x(X_{T_D}\in K_1)=0$. Similarly using Lemma 3.3 we find $P_x(X_{T_D}\in K_2)=0$.

Lemma 3.5. *Let $D=D_1\cap D_2$ and let f be a bounded harmonic function on D such that $|f(x)|\leqq KP_x(T_D<\infty)$ for some constant K. Then for all $x\in D_i$ a.s. P_x*

$$\lim_{t\uparrow T_{D_i}} f(X_t) = \xi_i \quad on \quad \Lambda_i = [\omega: X_{T_{D_i}}\in G_i, T_{D_i}<\infty]. \tag{3.5}$$

Proof. By Fact 2 we know that for all $x\in D$ $\lim_{t\uparrow T_D} f(X_t)$ exists a.s. P_x. By definition of G_i, a.s. P_x, there is a rational $q(\omega)$ such that on Λ_i

$$X_{q(\omega)}\in D \quad\text{and}\quad T_{D_i} = q+T_{D_i}\circ\theta_q.$$

Thus

$$P_x([\lim_{t\uparrow T_{D_i}} f(X_t) \text{ does not exist}]\cap A_i)$$

$$\leq \sum_{t\in Q} P_x(X_t\in D \text{ and } \lim_{s\uparrow t+T_{D_i}\circ\theta_t} X_s \text{ does not exist}).$$

But

$$P_x(X_t\in D \text{ and } \lim_{s\uparrow t+T_{D_i}\circ\theta_t} f(X_s) \text{ does not exist})$$

$$= \int_D \left(\frac{1}{2\pi t}\right)^{-d/2} e^{-|y-x|^2/2t} P_y(\lim_{s\uparrow T_{D_i}} f(X_s) \text{ does not exist})\, dy = 0.$$

Thus (3.5) holds.

Lemma 3.6. *Let f be as in Lemma 3.5. Define*

$$g_i(x) = E_x[\lim_{t\uparrow T_{D_i}} f(X_t);\ X_{T_{D_i}}\in G_i, T_{D_i} < \infty].$$

Then g_i is a bounded harmonic function on D_i and there is a bounded function φ_i on ∂D_i such that

(3.6) $$g_i(x) = H_{D_i}\varphi_i(x), \quad x\in D_i.$$

Proof. That g is harmonic on D_i follows by same kind of argument used in the proof of Lemma 3.2. The representation 3.6 follows by Fact 9 and the fact that D_i is a Poissonian open set.

Lemma 3.7. *Let f, φ_1, φ_2 be as in Lemma 3.6. Fix $x\in D$. Then a.s. P_x*

(3.7) $$\lim_{t\uparrow T_D} f(X_t) = \varphi_i(X_{T_D}) \quad on \quad [X_{T_D}\in G_i].$$

In particular, a.s. P_x, $\varphi_1(X_{T_D})=\varphi_2(X_{T_D})$ on $[X_{T_D}\in G_1\cap G_2]$.

Proof. Fact 2 shows $\lim_{t\uparrow T_D} f(X_t)=\xi$ exists a.s. P_x and that $\xi=0$ a.s. P_x on $[T_D=\infty]$. From Lemma 4.6 and Fact 3 we find a.s. P_x that

(3.8) $$\lim_{t\uparrow T_{D_i}} f(X_t) = \varphi_i(X_{T_{D_i}}) \quad \text{on} \quad [X_{T_{D_i}}\in G_i].$$

By definition of G_i a.s. P_x, $[X_{T_D}\in G_i]\subseteq[X_{T_{D_i}}\in G_i, T_D=T_{D_i}]$. Thus (3.8) implies (3.7).

Proof of Theorem 1. Define $\varphi(x)=\varphi_1(x)$ on G_1 and $\varphi(x)=\varphi_2(x)$ on G_2. Since $H_D(x, K_1\cup K_2)\equiv 0$ it follows from Lemma 3.7 that $\lim_{t\uparrow T_D} f(X_t)=\varphi(X_{T_D})$ a.s. P_x on $[T_D<\infty]$. Thus

$$f(x) = H_D\varphi(x), \quad x\in D.$$

4. Proof of Theorem 2

Let D, D_n, B_n be as in the statement of Theorem 1. Assume each D_n is a bounded Poissonian open set. Thus there is a bounded function φ_n such that

$$f(x) = H_{D_n}\varphi_n(x), \quad x \in D_n.$$

Define φ on ∂D as follows.

(4.1a) $$\varphi(x) = \varphi_1(x) \quad \text{on} \quad \partial D \cap \partial D_1$$

and

(4.1b) $$\varphi(x) = \varphi_n(x) \quad \text{on} \quad [\partial D \cap \partial D_n] \backslash [\partial D \cap \partial D_{n-1}], \quad n > 1.$$

Lemma 4.1. *For* $x \in D_r$

$$\varphi(X_{T_{D_r}}) = \varphi_r(X_{T_{D_r}}) \quad \textit{a.s. } P_x \textit{ on } \quad [T_{D_r} \in \partial D \cap \partial D_r].$$

Proof. Let $s > r$. Then for $x \in D_r$

$$H_{D_r}\varphi_r(x) = f(x) = H_{D_s}\varphi_s(x).$$

By Fact 6, $H_{D_s}\varphi_s(x) = H_{D_r}H_{D_s}\varphi_s(x)$, $x \in D_r$ so by Fact 4, for $x \in D_r$

$$\varphi_r(X_{T_{D_r}}) = H_{D_s}\varphi_s(X_{T_{D_r}}) \quad \text{a.s. } P_x.$$

By Fact 5 for $x \in D_r$

$$H_{D_s}\varphi_s(X_{T_{D_r}}) = \varphi_s(X_{T_{D_r}}) \quad \text{a.s. } P_x \text{ on } \quad [X_{T_{D_r}} \in \partial D \cap \partial D_r].$$

Thus for $x \in D$,

$$\varphi_r(X_{T_{D_r}}) = \varphi_s(X_{T_{D_r}}) \quad \text{a.s. } P_x \text{ on } \quad [X_{T_{D_r}} \in \partial D \cap \partial D_r].$$

Hence for $x \in D$

$$\varphi_r(X_{T_{D_r}}) = \varphi(X_{T_{D_r}}) \quad \text{a.s. } P_x \text{ on } \quad [X_{T_{D_r}} \in \partial D \cap \partial D_r].$$

Lemma 4.2. *Suppose for some constant* K, $|f(x)| \leqq K P_x(T_D < \infty)$. *Then for* φ *given by* (4.1) $f(x) = H_D\varphi(x)$ *on* D.

Proof. Using Lemma 4.1 we find for $x \in D_r$

(4.2) $$f(x) = H_{D_n}\varphi_n(x) = E_x[\varphi(X_{T_D});\ T_D \leqq T_{B_{r_n}}] + E_x[\varphi_r(X_{T_{B_{r_n}}});\ T_{B_{r_n}} < T_D].$$

If $y \in \partial B_{r_n} \cap D$ then $\lim_{x \to y} f(x) = f(y)$. By Fact 2

$$\lim_{t \uparrow T_{D_n}} f(X_t) = \varphi(X_{T_{D_n}}) \quad \text{a.s. } P_x, \quad x \in D_n.$$

Thus on the event

$$[X_{T_{D_n}} \in \partial B_{r_n} \cap D] = [T_{B_{r_n}} < T_D] \quad \varphi_r(X_{T_{B_{r_n}}}) = f(X_{T_{B_{r_n}}}) \quad \text{a.s. } P_x.$$

Hence by (4.2)

$$f(x) = E_x[\varphi(X_{T_D});\ T_D < T_{B_{r_n}}] + E_x[f(X_{T_{B_{r_n}}});\ T_{B_{r_n}} < T_D]. \tag{4.3}$$

Since $P_x(T_{B_{r_n}} \uparrow \infty) = 1$ for $x \in D$, the first term on the RHS of (3.3) converges to $H_D\varphi(x)$ as $n \to \infty$. The second term on the RHS of (3.3) is dominated by

$$KE_x[P_{X_{T_{B_{r_n}}}}(T_D < \infty);\ T_{B_{r_n}} < T_D] = KP_x(T_{B_{r_n}} < T_D < \infty),$$

and therefore this term converges to 0 as $n \to \infty$. Thus $f(x) = H_D\varphi(x)$, $x \in D$.

Proof of Theorem 2. By Fact 7 we can find a bounded harmonic function h on D satisfying the requirements of Lemma 3.2, and a constant α such that $f = h + \alpha P_\cdot(T_D = \infty)$ on D. By Lemma 3.2 $h = H_D\varphi$ for some bounded function φ. By Fact 1 $P_\cdot(T_D = \infty) = \psi$ on D. The converse statement follows at once from Theorem 1.

5. Proof of Theorem 3

Let D_1, D_2, and D be as in Theorem 3 and let $x \in D$. By Fact 7 $f(x) = h(x) + \alpha P_x(T_D = \infty)$ where α is a constant and h is a bounded harmonic function on D such that $|h(x)| \leqq KP_x(T_D < \infty)$. By assumption, there are bounded functions φ_i such that $h(x) = H_{D_i}\varphi_i(x) + \alpha_i P_x(T_{D_i} = \infty)$ for constants α_i. By Fact 9 and the fact that $P_x(T_D < \infty) \leqq P_x(T_{D_i} < \infty)$ for $x \in D_i$ we find $\alpha_i = 0$. Fact 2 shows that $\lim_{t \uparrow T_D} h(X_t) = \xi$ exists a.s. P_x, $x \in D$ and as in the proof of Lemma 4.7, $\xi = 0$ a.s. P_x on $[T_D = \infty]$. Let Q be the rationals on $(0, \infty)$. Then

$$P_x[X_{T_D} \in \partial D_i \cap (\partial D_1 \cap \partial D_2)^c,\ T_D < \infty;\ \xi \neq \varphi_i(X_{T_D})]$$

$$\leqq P_x\big(\textstyle\bigcup_{t \in Q} [X_t \in D_i \text{ and } \lim_{s \uparrow t + T_{D_i} \circ \theta_t} h(X_s) \neq \varphi_i(X_{t + T_{D_i}(\theta_t \omega)}),\ T_{D_i} < \infty]\big)$$

$$\leqq \textstyle\sum_{t \in Q} \displaystyle\int_{D_i} (2\pi t)^{-\alpha/2} e^{-|y-x|^2/2t} P_y\big(\lim_{s \uparrow T_{D_i}} h(X_s) \neq \varphi_i(X_{T_{D_i}}),\ T_{D_i} < \infty\big)\, dy = 0.$$

Thus a.s. P_x, $\xi = \varphi_i(X_{T_D})$ on

$$[X_{T_D} \in \partial D_i \cap (\partial D_1 \cap \partial D_2)^c,\ T_D < \infty].$$

Let $\varphi = \varphi_i$ on $(\partial D_i \cap \partial D) \cap (\partial D_1 \cap \partial D_2)^c$. Since $H_D(x, \partial D_1, \cap \partial D_2) \equiv 0$ on D it follows that

$$h(x) = E_x\xi = E_x[\xi;\ T_D < \infty] = E_x\varphi(X_{T_D}).$$

Thus

$$f(x) = E_x\varphi\big(X_{T_D} + \alpha P_x(T_D = \infty)\big).$$

6. Proof of Theorem 4

Sufficiency. By Fact 7 it suffices to consider bounded harmonic functions f that satisfy $|f(x)| \leqq KP_x(T_D < \infty)$ for some constant K. We assume henceforth that f satisfies this condition. Pick a point x_j in the j-th component of D and set $H(dy) = \sum_j 2^{-j} H_D(x_j, dy)$. Let D_i and $B(a_i, r_i)$ be as in the statement of the theorem. Since H is a finite measure, only countably many $\partial B(a_i, r_i) \cap \partial D$ for $r \leqq r_i$ can have positive measure. Using Theorem 1 we can replace the r_i by slightly smaller values if necessary to obtain balls that satisfy (a) and (b) and also satisfy (c) $H(\partial B(a_i, r_i) \cap \partial D) = 0$. We henceforth assume that we originally choose the r_i to satisfy this last condition (c).

For each i there is a bounded function φ_i on ∂D_i such that $f(y) = H_{D_i} \varphi_i(y), y \in D_i$. Let $P(\cdot) = \sum_j 2^{-j} P_{x_j}(\cdot)$. By Facts 2 and 9 a.s. P $\xi = \lim_{t \uparrow T_D} f(X_t)$ exists and $\xi = 0$ on $[T_D = \infty]$. By the same kind of argument used in the proof of Theorem 3 we can conclude that $\xi = \varphi_i(X_{T_D})$ a.s. P on $[X_{T_D} \in \partial D \cap \partial D_i, T_D < \infty]$. In particular, $\varphi_{i_1}(X_{T_D} = \varphi_{i_2}(X_{T_D}) = \dots$ a.s. P on $[X_{T_D} \in \partial D \cap \partial D_{i_1} \cap \partial D_{i_2} \cap \dots, T_D < \infty]$. (There may be finitely many or countably infinitely many i_j.) Thus we can find a function φ on ∂D such that $\|\varphi\|_\infty \leqq \|f\|_\infty$ such that for $H(dy)$ a.e. y, $\varphi_j(y) = \varphi(y)$ on $\partial D \cap \partial D_j$ and $\xi = \varphi(X_{T_D})$ a.s. P on $[T_D < \infty]$. Now for any $x \in D$, $H_D(x, dy) \ll H(dy)$. Thus for $H_D(x, dy)$ a.e. y, $\varphi_j(y) = \varphi(y)$ on $\partial D \cap \partial D_j$. Arguing as before we can now conclude a.s. P_x that $\lim_{t \uparrow T_D} f(X_t) = \varphi(X_{T_D})$ on $[T_D < \infty]$. By Facts 2 and 9 $f(x) = H_D \varphi(x)$.

Necessity. This follows at once from Theorem 1.

7. Proofs of Theorems 5 and 6

Throughout this section we assume D is connected.

Lemma 7.1. *Suppose φ is bounded on ∂D and let $f = H_D \varphi + \alpha P.(T_D = \infty)$ for α a constant. Then there is a sequence $\{f_n\}$ of bounded harmonic functions on D satisfying* (a)—(c) *of Theorem* 5.

Proof. Since D is connected the maximum principle shows the measures $H_D(x, dy)$ are equivalent for all $x \in D$. Fix $x_0 \in D$. An easy argument using Harnack's inequality shows there is a version $K(x, y)$ of the Radon—Nykodym derivative of $H_D(x, dy)$ with respect to $H_D(x_0, dy)$ such that for any compact subset $B \subseteqq D$.

(7.1) $$\sup_{x \in B} \sup_{y \in \partial D} K(x, y) = F < \infty.$$

We can find a sequence $\{\varphi_n\}$ of continuous functions on ∂D such that $\|\varphi_n\|_\infty \leqq \|\varphi\|_\infty$ and $H_D(x_0, dy)|\varphi_n(y)-\varphi(y)| \to 0$ as $n\to\infty$. Let $f_n = H_D\varphi_n + \alpha P(T_D=\infty)$. Using (6.1) it follows that $\{H_D\varphi_n\}$ satisfy (a)—(c) of Theorem 5 with $f=H_D\varphi$. Since $P(T_D=\infty)$ is essentially continuous on ∂D the $\{f_n\}$ has the required properties.

Lemma 7.2. *Suppose f is a bounded and harmonic on D and there is a sequence* $\{f_n\}$ *of bounded harmonic functions on D such that* (a)—(c) *of Theorem 5 hold. Then there is a bounded* φ *on* ∂D *and a constant* α *such that*

$$f = H_D\varphi + \alpha P(T_D=\infty) \quad \text{on } D. \tag{7.2}$$

Proof. Since f_n is essentially continuous on $\bar{D}$ it is a solution to the modified Dirichlet problem on D_n with boundary function f_n. By Theorem 2.10 of [2] there are bounded functions φ_n on ∂D and constants α_n such that

$$f_n = H_D\varphi_n + \alpha_n P(T_D=\infty) \quad \text{on } D.$$

Fix $x_0 \in D$. By Fact 3, a.s. P_{x_0},

$$|\varphi_n(X_{T_D})1_{[T_D<\infty]} + \alpha_n 1_{[T_D=\infty]}| = \lim_{t\uparrow T_D} |f_n(X_t)| \leqq M.$$

Thus a.s. P_{x_0}, $|\varphi_n(X_{T_D})| \leqq M$ on $[T_D<\infty]$ and $|\alpha_n| \leqq M$. Thus $\operatorname{ess\,sup} \varphi_n \leqq M$ (with respect to the measure $H_D(x_0, dy)$). Consequently, we can find a subsequence $\{\varphi_{n_j}\}$ of $\{\varphi_n\}$ and $\{\alpha_{n_j}\}$ of $\{\alpha_n\}$ such that $\alpha_{n_j} \to \alpha$ and for every $H_D(x_0, dy)$ integrable function ψ

$$\int H_D(x_0, dy)\psi(y)\varphi_{n_j}(y) \to \int H_D(x_0, dy)\psi(y)\varphi(y).$$

Taking $\psi(y)=K(x, y)$ where K is as in (6.1) we find $H_D\varphi_{n_j}(x) \to H_D\varphi(x)$. Thus for $x\in D$

$$f_{n_j}(x) \to H_D\varphi(x) + \alpha P_x(T_D=\infty).$$

Hence (7.2) holds.

Proof of Theorem 5. Immediate from Lemmas 6.1 and 6.2.

Lemma 7.3. *Let D be a bounded connected open set. Suppose* $f=H_D\varphi$ *where* φ *in bounded. Then there is a sequence* $\{f_n\}$, $n\geqq 1$ *of bounded harmonic functions on D that are continuous on* $\bar{D}$ *such that* $\sup_n \|f_n\|_\infty \leqq \|\varphi\|_\infty$ *and* $f_n \to f$ *uniformly on compact subsets of D.*

Proof. Let B be an open ball that contains $\bar{D}$. Let $K=D^c\cap\bar{B}$. By a theorem of Ancona [1] we can find compact subsets K_n of K such that all points of K_n are regular for K_n and the capacity $C(K\setminus K_n) \leqq 1/n$. Let $D_n = K_n^c \cap B$. Let $D_{1n}, D_{2n}, \ldots$ be the components of D_n. Since $T_{D_{in}} \leqq T_{D_n}$, $P_x(T_{D_n}=0) \leqq P_x(T_{D_{ni}}=0)$. Since all points of D_n^c are regular and $x \in \partial D_{ni}$ iff $x \in \partial D_n$ it follows that all points of ∂D_{ni} are regular for D_{ni}^c. Now as D is connected D must be contained in some component of D_n. Consequently we can assume that the D_n are themselves connected and all points on ∂D_n are regular for D_n^c. We can also assume that $D_1 \supset D_2 \supset \dots$. Now $C(D_n \setminus D) \leqq 1/n$ so $D_n \setminus D$ decreases to a set of capacity 0 and thus a polar set. Hence for each $x \in D$

$$H_D(x, D_n \setminus D) \to 0.$$

Let $\varphi_n = \varphi$ on $\partial D \cap \partial D_n$ and let $\varphi_n = 0$ on $\partial D \cap D_n$. Then for $x \in D$

$$|H_{D_n}\varphi_n(x) - H_D\varphi(x)| \leqq \|\varphi\|_\infty H_D(x, \partial D \cap D_n). \tag{7.3}$$

Fix $x_0 \in D$ and let $\varepsilon > 0$. There is then a sequence $\{\psi_n\}$ of bounded continuous functions on ∂D such that $\sup_n \|\psi_n\|_\infty \leqq \|\varphi\|_\infty$ and

$$\int H_{D_n}(x_0, dz)|\psi_n(z) - \varphi_n(z)| \leqq \varepsilon. \tag{7.4}$$

Let $f_n = H_{D_n}\psi_n$. Then $\sup_n \|f_n\| \leqq \|\varphi\|_\infty$. Let $K(x, y)$ be the version of the Radon—Nikodym derivative of $H_{D_n}(x, dz)$ with respect to $H_D(x_0, dz)$ that satisfies (7.1). Then for C a compact subset of D

$$\sup_{x \in C} |f_n(x) - f(x)| \leqq F\varepsilon + \|\varphi\|_\infty H_D(x_0, \partial D \cap \partial D_n).$$

Thus $f_n \to f$ uniformly on compacts. Since all points of ∂D_n are regular for D_n^c, f_n is continuous on $\bar{D}_n$.

Proof of Theorem 6. Suppose $f = H_D\varphi$ with φ bounded. Let D_r be the intersection of D with the open ball of center 0 and radius r. Then by Fact 6

$$f(x) = H_{D_r}H_D\varphi, \quad x \in D_r.$$

By Lemma 7.3 there is a sequence $\{f_{nr}\}$, $n = 1, 2, \dots$ of bounded harmonic functions on D_n. The sequence $\{f_{nr}\}$, $n = 1, 2, \dots$, $r = 1, 2, \dots$ then has the required properties.

8. Proof of Theorems 7 and 8

The measurable sets of $(\partial D)^*$ are the Borel sets completed with sets of π_D^* measure 0. The measurable sets of Δ are the Borel sets completed with sets of μ_D measure 0. Statements such as for P_x a.e. will be understood to mean for all $x \in D$.

Let f be a bounded harmonic function on D. The limit random variable ξ in Fact 2 can be identified with $\psi(Z)$ where $f(x) = \mu_D\psi(x)$. Fact 3 shows that for Poissonian D $\xi = \varphi^*(X_{T_D})$ a.e. P_x, where $f(x) = \pi_D^*\varphi^*(x)$.

Lemma 8.1. *Suppose D is Poissonian. Then for each measurable $\Lambda \subseteq \Delta$ there is a measurable $B \subseteq (\partial D)^*$ such that $1_B(X_{T_D}) = 1_\Lambda(Z)$ a.e. P_x. This B is unique modulo a set of π_D^* measure 0.*

Proof. The function $\mu_D(x, \Lambda)$ is bounded and harmonic on D so there is φ^* such that $\mu_D(x, \Lambda) = \pi_D^* \varphi^*(x)$. Hence $1_\Lambda(Z) = \varphi^*(X_{T_D})$ a.e. P_x. Let $B = \{y: \varphi^*(y) = 1\}$. Then $1_B(X_{T_D}) = 1_\Lambda(Z)$ a.e. P_x. If also $1_{B'}(X_{T_D}) = 1_\Lambda(Z)$ a.e. P_x, then $1_{B'}(X_{T_D}) = 1_B(X_{T_D})$ a.e. P_x, so $\pi_D^*(x, B \Delta B') \equiv 0$.

Lemma 8.2. *Let $P = \sum \alpha_i P_{x_i}$ where each x_i is in a different component of D, $\alpha_i > 0$ and $\sum_i \alpha_i = 1$. For each measurable $\Lambda \subseteq \Delta$ there is a measurable $B \subseteq (\partial D)^*$ satisfying the condition in Lemma 9.1 such that*

$$P(Z \in \Lambda | X_{T_D} = y) = 1 \quad \text{for all} \quad y \in B.$$

Proof. By Lemma 8.1 we can find B' such that $1_{B'}(X_{T_D}) = 1_\Lambda(Z)$ a.e. P. Hence

$$P(Z \in \Lambda | X_{T_D}) = 1_{B'}(X_{T_D}) \quad \text{a.e. } P.$$

That is

$$P(Z \in \Lambda | X_{T_D} = y) = 1 \quad \text{a.e.} \quad y \in B'$$

where a.e. is with respect to the measure $\sum_i \alpha_i \pi_D^*(x_i, \cdot)$. Thus we can find $B \subseteq B'$ such that $\pi_D^*(B' \setminus B) = \sum \alpha_i \pi_D^*(x_i, B' \setminus B) = 0$, so that $P(Z \in \Lambda | X_{T_D} = y) = 1$ for all $y \in B$.

Proof of Theorem 7. Suppose D is Poissonian. We can find countable nested partitions $\Lambda_{1n}, \Lambda_{2n}, \ldots$ of Δ such that $\sup_i \operatorname{diam}(\Lambda_{in}) \to 0$. Using Lemma 8.2 we can the find a set F having π_D^* measure 0 at x and a partition $B_{1n}, B_{2n}, \ldots$ of $(\partial D)^* \setminus F$ such that $P(Z \in \Lambda_{in} | X_{T_D} = y) = 1$ on B_{in}. Pick $\xi_{in} \in \Lambda_{in}$ arbitrarily and let $\varphi_n(y) = \xi_{in}$ on B_{in}. Now $\varphi_{n+m}(y) \in \Lambda_{in}$ for $m = 0, 1, 2, \ldots$ so it must be that for each $y \in (\partial D)^* \setminus F$, $\varphi_n(y) \to \varphi(y)$. Define $\varphi(y) = \xi$ for $y \in F$ where ξ is any point on Δ. Then φ is measurable and $P_x(Z = \varphi(y) | X_{T_D} = y) = 1$ for all $y \in (\partial D)^* \setminus F$. Hence $P_x(Z = \varphi(X_{T_D})) = 1$. Now $P_x(Z = \varphi(X_{T_D}))$ is harmonic on D, so $\sum_i \alpha_i P_{x_i}(Z = \varphi(X_{T_D})) = 1$ implies $P_x(Z = \varphi(X_{T_D})) \equiv 1$. Certainly then $\mu_D(x, \Lambda) = \pi_D^*(x, \varphi^{-1}(\Lambda))$. On the other hand if there is a φ such that this last equality holds then Z and $\varphi(X_{T_D})$ have thesame distribution so D is Poissonian.

Proof of Theorem 8. Let D be any open set such that D^c is not polar. Then for any measurable B on $(\partial D)^* P_x(X_{T_D} \in B)$ is a bounded harmonic function on D. The same arguments used to establish Theorem 9 can now be applied in reverse to show that there is a measurable mapping of $\psi: \Delta \to (\partial D)^*$ such that $\psi(Z) = X_{T_D}$ a.e. P_x. But then, if D is Poissonian, a.e. P_x, $\psi(\varphi(X_{T_D})) = X_{T_D}$ and $\varphi(\psi(Z)) = Z$. We can therefore find a set $B \subseteq (\partial D)^*$ having harmonic measure 0 and a set $\Delta_1 \subseteq \Delta$ that contains all of the non-minimal points of Δ that also has harmonic measure 0 such that on $(\partial D)^* \setminus B$ φ is 1-1 and onto $\Delta \setminus \Delta_1$.

9. Proof of Theorem 9

The existence of a function ψ having the stated properties was shown in the proof of Theorem 8. Pick a point $x_i \in D_i$, the i-th component of D. Let $\alpha_i > 0$, $\sum \alpha_i = 1$ and set $\mu_D(\cdot) = \sum_i \alpha_i \mu_D(x_i, \cdot)$. Assume $\mu_D(\Delta_0) > 0$. Let φ^* be bounded and measurable on Δ. Using Facts 2, 3, and 7 we can conclude that $\varphi^*(Z) = \alpha$ a.e. P_x on $[T_D = \infty]$ so $\varphi^*(\xi) = \alpha$ a.e. (μ_D) on Δ_0. Suppose Λ_1 and Λ_2 are measurable, disjoint, and have union Δ_0. Then Λ_1 and Λ_2 cannot both have positive μ_D measure. For otherwise $\varphi^* = a1_{\Lambda_1} + b1_{\Lambda_2}$ with $a \neq b$ would be a non-constant function on Δ_0. Since $\mu_D(\Delta_0) = \gamma$ is positive, exactly one of these two sets has measure γ. Let $\{\Lambda_{in}\}$ be nested countable partitions of Δ_0 such that $\sup_i \operatorname{diam}(\bar{\Lambda}_{in}) \to 0$. For each n there is exactly one i, say i_n, such that $\gamma = \mu_D(\Lambda_{i_n, n})$. Then $\bigcap_n \Lambda_{i_n, n}$ has exactly one point ξ and $\mu_D(\{\xi\}) = \gamma$. Thus $\mu_D(\Delta_0 \setminus \{\xi\}) = 0$ so $\mu_D(x, \Delta_0 \setminus \{\xi\}) \equiv 0$.

10. Proof of Bishop's Theorem

In this section we prove using probabilistic methods

Theorem (Bishop). *An open set D is Poissonian if and only if every pair of disjoint subdomains D_1 and D_2 with $\partial D_1 \cap \partial D_2 \subset \partial D$ have mutually singular harmonic measures.*

The condition give above will be refered to as Bishops condition henceforth. Before starting the proof of the above theorem we will require a lemma.

Lemma 10.1. *Let D be a non-Poissonian open set. Then there exists a bounded harmonic function f on D taking values in $[0, 1]$ and a starting measure μ supported on D, such that for Brownian motion $\{X(t)\colon t \geqq 0\}$*

(i) $P_\mu[\lim_{t \to T_D} f(X(t)) \in [0, 1]] = 1$ *and both of the possible limits have positive probability of occuring.*

(ii) *For $H_D(\mu)$ almost all y,* $P_\mu[\lim_{t \to T_D} f(X(t)) = 1 | X(T_D) = y] > 0$.

Proof. Let δ_x be the unit mass at the point x and let μ be the probability measure $\sum_{j=1}^\infty 2^{-j} \delta_{q(j)}$ where $\{q(j)]$ is dense in D. By assumption there exists a bounded non-representable function g. By fact 2 we know that P_μ a.s. $\lim_{t \to T_D} g(X(t))$ exists. Let us denote this random variable by Y. Let $K(X_T, dy)$ be the conditional probability of Y given X_T. By assumption $K(x\ )$ is not a unit mass at some point for $H_D(\mu\ )$ almost all x. Therefore we can find a number a such that for $x \in F \subset \partial D(H_D(\mu, F) > 0)$

$$K(x, (-\infty, a]) > 0 \quad \text{and} \quad K(x, (a, \infty)) > 0.$$

We now take as our harmonic function f the function

$$P_x[\lim_{t \to T_D} X(t) F \text{ or } Y > a] = P_x[\Gamma].$$

As in Lemma 3.2, a.s. the limiting value of $f(X(t))$ as t tends to T_D is equal to l_D, so condition (i) is satisfied. By our choice of a and F for $x \in F$.

$Q(x, \Gamma) > 0$ and $Q(x, \Gamma) > 0$, where $Q(,)$ is a regular conditional probability on the space of paths given X_T.

Proof of Theorem. We first show that if Bishop's condition fails for an open set D, then D cannot be Poissonian.

Let D_1 and D_2 be two subdomains for which Bishop's condition fails. As before $H_{D_i}(x_i,)$ are the respective harmonic measures which by assumption are not mutually singular. Let B be the set $\partial D_1 \cap \partial D_2$. For a Brownian motion $\{X_t\colon t \geqq 0\}$, we define

$$h(x) = P^x[X_{T_D} \in B, \varliminf_{t \to T_D} X_t \in D_1].$$

Now (as with Lemma 3.2) the function h is harmonic in D. Also, by Fact 3, if h is representable by boundary function φ we must have

(i) $\varphi = 1$ on a subset of B with full H_{D_1} measure.

(ii) $\varphi = 0$ on a subset of B with full H_{D_2} measure.

But these requirements are incompatible with the assumptions that the two harmonic measures are not mutually singular. Hence h is not representable and D is non-Poissonian.

We now show that if an open set D is non-Poissonian then Bishop's condition must fail. Let μ be a measure on D of countable dense support and let f be the harmonic measure guaranteed by Lemma 10.1. Consider the open sets $O_1 = \{x\colon f(x) > 3/4\}$ and $O_2 = \{x\colon f(x) < 1/4\}$. For D_i any components of the two open sets we must clearly have $D_1 \cap D_2$ is empty and $\partial D_1 \cap \partial D_2 \subset \partial D$. Furthermore it is clear from Lemma 10.1 that we may find components with non-mutually singular harmonic measure and so Bishop's condition must fail.

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