Spaces of Lorentz type and complex interpolation

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1. Introduction

Lately there has been a great deal of interest in spaces of Lorentz type, which some authors have started to call spaces of Lorentz—Marcinkiewicz. These spaces are defined in the following way: given a non-negative function w on $(0, \infty)$, the space $\Lambda^q(w)$, $0<q\leqq\infty$, consists of those functions defined on a σ-finite measure space M, fixed from now on, for which

$$\|f\|_{\Lambda^q(w)} = \Big(\int_0^\infty [w(s)f^*(s)]^q\, ds\Big)^{1/q} \tag{1.1}$$

is finite; here $f^*(s)$ denotes the non-increasing rearrangement of f. Notice that this definition is a little different from the one used by other authors, since we use the measure ds instead of ds/s; we have done this for the sake of aesthetic beauty in our final results.

When $w\equiv 1$, $\Lambda^q(w)$ coincides with the Lebesgue space L^q. When $w(s)=s^{(1/p)-(1/q)}$, $1\leqq p<\infty$, we find that $\Lambda^q(w)$ is the familiar Lorentz space $L^{p,q}$. With other weight functions, such as, for example, $w(s)=s^{(1/p)-(1/q)}(1+|\log(s)|)^\alpha$, $\alpha\in\mathbf{R}$, $\Lambda^q(w)$ becomes a space of "Orlicz" type, which is sometimes a good substitute for end point results for the boundedness of operators.

The spaces defined by (1.1) have been extensively studied recently because of their connection to the method of real interpolation with a function parameter. In this method a pair (A_0, A_1) of compatible quasi-Banach spaces are given together with a function $\varphi\in B_\psi$ and intermediate spaces, denoted by $(A_0, A_1)_{\varphi,q}$, $0<q\leqq\infty$, are defined. (For the precise definitions see Section 2.) This theory has been developed in [6], [11], [12] and [13], and in particular if $\varphi(t)=t^\theta$ one obtains the classical real method of interpolation developed by J. L. Lions and J. Peetre (see [1], Chapter 3).

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The usual properties of the intermediate spaces are proved in this context. For now we shall only mention the result

$$(1.2)\qquad (L^1, L^\infty)_{\varphi,q} = \Lambda^q(w), \quad 1 \leqq q \leqq \infty,$$

where $w(s)=s^{(1-(1/q))}/\varphi(s)$, which can be found in [6], Lemma 3.1. Using the reiteration theorem and (1.2), it is easy to identify the intermediate spaces of the spaces $\Lambda^q(w)$ when $q \geqq 1$. (See [12] for details.)

When we consider interpolation by the complex method of A. P. Calderón [2], this question has not been treated. This is the problem we are going to address in this paper.

There are two ways to attack this problem. The first one is to find a relation between the real interpolation method with a function parameter and the complex interpolation method of A. P. Calderón. (See [1], page 102, for a relation of this type when we consider the classical real interpolation method.) This relation is as follows: given a pair of compatible Banach spaces A_0, A_1 and a pair of functions φ_0, φ_1 in β_ψ, we prove the equality

$$(1.3)\qquad [(A_0, A_1)_{\varphi_0,q_0}, (A_0, A_1)_{\varphi_1,q_1}]_\theta = (A_0, A_1)_{\varphi,q}$$

with equivalent norms, where $\varphi(s)=[\varphi_0(s)]^{1-\theta}[\varphi_1(s)]^\theta$, $\dfrac{1}{q}=\dfrac{1-\theta}{q_0}+\dfrac{\theta}{q_1}$ and $[\ ,\ ]_\theta$ denotes the intermediate space at level θ in the complex method of interpolation.

Using (1.3) and (1.2) we obtain, for $q_0 \geqq 1$, $q_1 \geqq 1$,

$$(1.4)\qquad [\Lambda^{q_0}(w_0), \Lambda^{q_1}(w_1)]_\theta = \Lambda^q(w)$$

where $w(s)=[w_0(s)]^{1-\theta}[w_1(s)]^\theta$, $\dfrac{1}{q}=\dfrac{1-\theta}{q_0}+\dfrac{\theta}{q_1}$ and w_0, w_1 satisfy $s^{(1-(1/q_j))}/w_j(s)\in B_\psi$, $j=0, 1$. In fact

$$[\Lambda^{q_0}(w_0), \Lambda^{q_1}(w_1)]_\theta = [(L^1, L^\infty)_{\varphi_0,q_0}, (L^1, L^\infty)_{\varphi_1,q_1}]_\theta = (L^1, L^\infty)_{\varphi,q} = \Lambda^q(w).$$

The other approach is based on an idea of Calderón [3]: to interpolate Lorentz spaces by the complex method of interpolation one can use a pair of inequalities due to Hardy, namely

$$(1.5)\qquad \left\{\int_0^\infty \left(\int_0^t f(s)\,ds\right)^q t^{-r-1}\,dt\right\}^{1/q} \leqq \frac{q}{r}\left\{\int_0^\infty [sf(s)]^q s^{-r-1}\,ds\right\}^{1/q}$$

and

$$(1.6)\qquad \left\{\int_0^\infty \left(\int_t^\infty f(s)\,ds\right)^q t^{r-1}\,dt\right\}^{1/q} \leqq \frac{q}{r}\left\{\int_0^\infty [sf(s)]^q s^{r-1}\,ds\right\}^{1/q}$$

for $q \geqq 1$, $r>0$ and $f \geqq 0$ (see [7], Chapter IX). If we define the operator $Sf(t)=\dfrac{1}{t}\displaystyle\int_0^t f(s)ds$, inequality (1.5) can be restated as saying that S is bounded on

$\mathscr{L}^q(s^{-(r/q)-(1/q)+1})$, $q\geqq 1$, $r>0$; here $\mathscr{L}^q(w(s))$ denotes the space of all f defined on $(0,\infty)$ such that $\|fw\|_{L^q}<\infty$ for a non-negative function w on $(0,\infty)$. Similarly (1.6) says that the dual of S, $S^*f(t)=\int_t^\infty \frac{f(s)}{s}\,ds$ is bounded on $\mathscr{L}^q(s^{(r/q)-(1/q)})$, $q\geqq 1$, $r>0$. Hence (1.5) and (1.6) are weighted inequalities for the operators S and S^* respectively. All pairs of weights (u,v) for which S is bounded from $\mathscr{L}^q(u)$ to $\mathscr{L}^q(v)$, $1\leqq q\leqq\infty$, have been characterized by B. Muckenhoupt in [14]. The result is that

$$\left\{\int_0^\infty (|Sf(s)|v(s))^q\,ds\right\}^{1/q}\leqq C\left\{\int_0^\infty (|f(s)|\,u(s))^q\,ds\right\}^{1/q} \tag{1.7}$$

if and only if

$$\sup_{r>0}\left(\int_r^\infty s^{-q}[v(s)]^q\,ds\right)^{1/q}\left(\int_0^r [u(s)]^{-q'}\,ds\right)^{1/q'}=K<\infty. \tag{1.8}$$

Similarly there is a result for S^*:

$$\left\{\int_0^\infty (|S^*f(s)|v(s))^q\,ds\right\}^{1/q}\leqq C\left\{\int_0^\infty (|f(s)|\,u(s))^q\,ds\right\}^{1/q} \tag{1.9}$$

if and only if

$$\sup_{r>0}\left(\int_0^r [v(s)]^q\,ds\right)^{1/q}\left(\int_r^\infty s^{-q'}[u(s)]^{-q'}\,ds\right)^{1/q'}=K<\infty. \tag{1.10}$$

If $u=v\equiv w$ and (1.7), or equivalently (1.8), holds, we write $w\in W_q(S)$ and similarly $w\in W_q(S^*)$ when (1.9), or equivalently (1.10), holds. We will denote by $C(S;q,w)$ the infimum of the constants C that appear on (1.7), and we shall use $C(S^*;q,w)$ for the smallest constant that could be used in (1.9) (here $u=v\equiv w$).

Using non-negative functions $w\in W_q(S)\cap W_q(S^*)$, we can identify the intermediate spaces of $\Lambda^q(w)$ in the complex method of interpolation. The result is

$$[\Lambda^{q_0}(w_0),\Lambda^{q_1}(w_1)]_\theta=\Lambda^q(w)\quad 1\leqq q_0,q_1<\infty \tag{1.11}$$

where $w(s)=[w_0(s)]^{1-\theta}[w_1(s)]^\theta$ and $\frac{1}{q}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}$ when $w_j\in W_{q_j}(S)\cap W_{q_j}(S^*)$, $j=0,1$. Details can be found in Section 3. The idea of using generalized Hardy inequalities in connection with interpolation has appeared also in [8] and [16].

There is a discrepancy between the parameters used in (1.4) and (1.11). The result in (1.11) is more general than that of (1.4), since we will prove in Section 4 that if $\varphi\in B_\psi$, $w(s)=s^{(1-(1/q))}/\varphi(s)\in W_q(S)\cap W_q(S^*)$, $1\leqq q<\infty$. Of course the important result in Section 2 is (1.3) and not (1.4).

All of the above mentioned results are proved in the context of the "St. Louis intermediate spaces", that is, the complex method of interpolation for families of

Banach spaces developed in [4]. A summary of this method is given in Section 2. The Calderón method of interpolation is a particular case of this more general method.

We call the reader's attention to the fact that some of the methods used here are valid only for Banach spaces, so that we only treat the interpolation of the spaces in the case $1\leqq q\leqq\infty$. The case $0<q<1$ requires other techniques which will be the object of a forthcoming paper.

2. A reiteration theorem

A relation between the real interpolation theory with a function parameter and the complex method of interpolation for families of Banach spaces is proved in this section (see Theorem 2.6). For the reader's convenience we start by reviewing the methods of interpolation used below.

The complex method of interpolation

We describe the complex method of interpolation for families of Banach spaces as given in [4]. Let $\mathbf{T}=\{z\in\mathbf{C}:\ |z|=1\}$; to simplify notation we shall write $\theta\in\mathbf{T}$ instead of $e^{i\theta}\in\mathbf{T}$. Let $\{B(\theta)\}_{\theta\in\mathbf{T}}$ be a family of Banach spaces. We say that this family is an *interpolation family of Banach spaces* (or *interpolation family*, for short) if

i) each $B(\theta)$ is continuously embedded in a Banach space $(U, \|\ \|_U)$,
ii) the function $\theta\to\|b\|_{B(\theta)}$ is measurable for each $b\in\bigcap_{\theta\in\mathbf{T}}B(\theta)$,
and if

$$\mathscr{B}=\Big\{b\in\bigcap_{\theta\in\mathbf{T}}B(\theta)\colon \int_0^{2\pi}\log^+\|b\|_{B(\theta)}\,d\theta<\infty\Big\}$$

we have
iii) $\|b\|_U\leqq k(\theta)\|b\|_{B(\theta)}$ for all $b\in\mathscr{B}$, with $\log^+ k(\theta)\in L^1(\mathbf{T})$.

The space $\mathscr{B}$ is called the *log-intersection space* of the given family and U is called the *containing space*.

We let $N^+(\mathscr{B})$ be the space of all $\mathscr{B}$-valued analytic functions of the form $g(z)=\sum_{j=1}^m\chi_j b_j$ for which $\|g\|_\infty=\sup_\theta\|g(\theta)\|_{B(\theta)}<\infty$, where $\chi_j\in N^+$ and $b_j\in\mathscr{B}$, $j=1,2,\dots,m$. (N^+ denotes the positive Nevalinna class for $D=\{z\in\mathbf{C}:\ |z|\leqq1\}$.) The completion of the space $N^+(\mathscr{B})$ with respect to $\|\ \|_\infty$ is denoted by $\mathscr{F}(\mathscr{B})$. For $z\in D$, the space $[B(\theta)]_z$ will consist of all elements of the form $f(z)$ for $f\in\mathscr{F}(\mathscr{B})$. A Banach space norm is defined on $[B(\theta)]_z$ by $\|v\|_z=\inf\{\|f\|_\infty\colon f\in\mathscr{F}(\mathscr{B}), f(z)=v\}$,

$v \in [B(\theta)]_z$, $z \in D$. It can be proved that $([B(\theta)]_z, \| \|_z)$ is a Banach space and $\mathscr{B}$ is dense in each $[B(\theta)]_z$. Other properties of these spaces, such as the interpolation property and reiteration can be found in [4] and [5]. The only one we shall need in this paper is the following subharmonicity property, which is contained in proposition (2.4) of [4]:

Proposition 2.1. *For each* $f \in \mathscr{F}(\mathscr{B})$ *and each* $z \in D$,

$$\|f(z)\|_z \leqq \exp \int_T [\log \|f(\theta)\|_{B(\theta)}] P_z(\theta)\, d\theta$$

where $P_z(\theta) = \mathrm{Re}\left(\frac{1}{2\pi} \frac{1+ze^{-i\theta}}{1-ze^{-i\theta}}\right)$ *is the Poisson Kernel of* D *for evaluation at* z.

Another type of subharmonicity property which will be used in the sequel is the "fundamental inequality" of [9] (Proposition 3.1) which we state below.

Suppose that the function $p\colon \bar{D} \to [1, \infty]$ is such that $1/p(z)$ is harmonic on D. A measurable function $F\colon T \times M \to R$ is called *p-admissible* if

$$\int_T \|F(\theta, \cdot)\|_{L^{p(\theta)}} P_z(\theta)\, d\theta < \infty$$

for some $z \in D$ (and hence for all z).

Proposition 2.2. *For a p-admissible function* F *we have*

$$\log \|u_F(z, \cdot)\|_{L^{p(z)}} \leqq \int_T \left(\log \|F(\theta, \cdot)\|_{L^{p(\theta)}}\right) P_z(\theta)\, d\theta$$

where $u_F(z, x) = \exp\left[\int_T \left(\log |F(\theta, x)|\right) H_z(\theta)\, d\theta\right]$, $z \in D$, *and* $H_z(\theta)$ *is the analytic function whose real part is* $P_z(\theta)$ *and* $H_z(0) = P_z(0)$.

The real method of interpolation with a function parameter

The class B_ψ consists of all continuously differentiable functions $\varphi\colon \mathbf{R}^+ \to \mathbf{R}^+$ such that $0 < \alpha_\varphi \equiv \inf_{t>0} \frac{t\varphi'(t)}{\varphi(t)} \leqq \sup_{t>0} \frac{t\varphi'(t)}{\varphi(t)} \equiv \beta_\varphi < 1$. We may assume, without loss of generality, that $\varphi(1) = 1$. We notice that if $0 < \theta < 1$, $t^\theta \in B_\psi$ and $\alpha_\varphi = \beta_\varphi = \theta$.

Given a pair of compatible Banach spaces (A_0, A_1), $\varphi \in B_\psi$ and $q \in [1, \infty]$, we define the space $(A_0, A_1)_{\varphi, q} \equiv \bar{A}_{\varphi, q}$ as the set of all $a \in A_0 + A_1$ such that

$$\|a\|_{\varphi, q} = \left(\int_0^\infty [\varphi(t)^{-1} K(t, a)]^q \frac{dt}{t}\right)^{1/q} < \infty$$

where $K(a, t)$ is the K-functional used in the classical real method of interpolation (see [1]). Notice that when $\varphi(t) = t^\theta$, $0 < \theta < 1$, $(A_0, A_1)_{\varphi, q} = (A_0, A_1)_{\theta, q}$. Several properties of these intermediate spaces can be found in [6], [11], [12] and [13].

In what follows we shall need several properties of the class B_ψ. It turns out that B_ψ is contained in a class of functions B_K introduced by T. F. Kalugina in [11] (see [6]). The class B_K consists of all continuous and non-decreasing functions $\varphi\colon \mathbf{R}^+ \to \mathbf{R}^+$ such that $\bar\varphi(s) = \sup_{t>0} \frac{\varphi(st)}{\varphi(t)} < \infty$ for every s, and

$$\int_0^\infty \min\left(1, \frac{1}{t}\right) \bar\varphi(t) \frac{dt}{t} < \infty.$$

Properties of the functions in $B_K \supset B_\psi$ are recorded in the following proposition:

Proposition 2.3. ([6].) *If* $\varphi \in B_K$ *we have*

(1) $\varphi(t) \leqq \bar\varphi(t) < \infty$;

(2) $\bar\varphi(st) \leqq \bar\varphi(s)\bar\varphi(t), \quad s, t \in \mathbf{R}^+$ *($\bar\varphi$ is submultiplicative)*;

(3) $\int_0^\infty \left[\min\left(1, \frac{1}{t}\right)\bar\varphi(t)\right]^p \frac{dt}{t} < \infty$ *for all* $p > 0$;

(4) $\underline{\varphi}(s) = 1/\bar\varphi(1/s)$, *where* $\underline{\varphi}(s) = \inf_{t>0} \frac{\varphi(st)}{\varphi(t)}$;

(5) $\frac{\varphi(s)}{s}\left(\int_0^s \left[\frac{t}{\varphi(t)}\right]^p \frac{dt}{t}\right)^{1/p} \leqq \left(\int_1^\infty \left[\frac{\bar\varphi(t)}{t}\right]^p \frac{dt}{t}\right)^{1/p}, \quad p > 0$;

(6) $\varphi(s)\left(\int_s^\infty \left[\frac{1}{\varphi(t)}\right]^p \frac{dt}{t}\right)^{1/p} \leqq \left(\int_0^1 [\bar\varphi(t)]^p \frac{dt}{t}\right)^{1/p}, \quad p > 0$;

(7) *if* $\varphi \in B_K$ *then there is a function* $g \in B_\psi$ *such that* φ *and* g *are "equivalent" in the sense that there are two positive constants* c_1 *and* c_2 *such that* $c_1 g(t) \leqq \varphi(t) \leqq c_2 g(t)$, $t > 0$.

We shall now give some properties of the spaces $\bar A_{\varphi,q}$ which we shall use below and which cannot be found in the literature.

Proposition 2.4. *Let* $\varphi \in B_\psi$, $a \in A_0 + A_1$ *and* $1 \leqq q \leqq \infty$. *Then*

$$\|a\|_{A_0+A_1} \equiv K(1, a) \leqq \left(\int_0^\infty [\bar\varphi(s)^{-1} \min(1, s)]^q \frac{ds}{s}\right)^{-1/q} \|a\|_{\varphi,q}.$$

Proof. Since $\min\left(1, \frac{s}{t}\right) K(t, a) \leqq K(s, a)$, we have

$$\|a\|_{\varphi,q} \geqq K(t, a)\left(\int_0^\infty \left[\varphi(s)^{-1} \min\left(1, \frac{s}{t}\right)\right]^q \frac{ds}{s}\right)^{1/q}.$$

A change of variables and the definition of $\underline{\varphi}$ imply

$$\|a\|_{\varphi,q} \geqq K(t,a)\left(\int_0^\infty \left[\varphi\left(\frac{t}{u}\right)^{-1} \min\left(1, \frac{1}{u}\right)\right]^q \frac{du}{u}\right)^{1/q}$$
$$\geqq \frac{K(t,a)}{\varphi(t)}\left(\int_0^\infty \left[\underline{\varphi}(u) \min\left(1, \frac{1}{u}\right)\right]^q \frac{du}{u}\right)^{1/q}.$$

Property (4) given in Proposition 2.3 and a change of variables imply

$$\|a\|_{\varphi,q} \geqq \frac{K(t,a)}{\varphi(t)}\left(\int_0^\infty \left[\bar{\varphi}\left(\frac{1}{u}\right)^{-1} \min\left(1, \frac{1}{u}\right)\right]^q \frac{du}{u}\right)^{1/q}$$
$$= \frac{K(t,a)}{\varphi(t)}\left(\int_0^\infty [\bar{\varphi}(u)^{-1} \min(1,u)]^q \frac{du}{u}\right)^{1/q},$$

from where the result follows by taking $t=1$. ∎

Proposition 2.5. *Let* $\varphi \in B_\psi$ *and* $1 \leqq q \leqq \infty$. *Then for each* $a \in \bar{A}_{\varphi,q}$ *and for all* $\varepsilon > 0$, *there exists a sequence* $\{u_n\} \subset A_0 \cap A_1$ *such that* $a = \sum_{n=-\infty}^{\infty} u_n$, *with convergence in* $A_0 + A_1$ *and*

$$J(2^n, u_n) \leqq 3(1+\varepsilon) K(2^n, a) \quad n \in \mathbf{Z}$$

where $J(2^n, u_n) = \max\{\|u_n\|_{A_0}, 2^n \|u_n\|_{A_1}\}$.

Proof. Since $\varphi(t) \leqq \max(t^{\alpha_\varphi}, t^{\beta_\varphi})$ ([6]) and $K(t,a) \leqq C\varphi(t)\|a\|_{\varphi,q}$, we obtain $\min\left(1, \frac{1}{t}\right) K(t,a) \to 0$ as $t \to 0$ or ∞. We can then apply the *fundamental lemma of interpolation theory* (Chapter 3 of [1]) to obtain the result. ∎

Proposition 2.6. *Let* $\varphi \in B_\psi$ *and* $1 \leqq q \leqq \infty$. *Then for* $a \in \bar{A}_{\varphi,q}$ *and any decomposition of* a *of the form* $a = \sum_n u_n$, $u_n \in A_0 \cap A_1$, *with convergence in* $A_0 + A_1$, *we have*

$$(1) \qquad \|a\|_{\bar{A}_{\varphi,q}} \leqq C\left(\int_0^\infty \bar{\varphi}(s) \min\left(1, \frac{1}{s}\right) \frac{ds}{s}\right) \left[\sum_{n=-\infty}^{\infty} (\varphi(2^n)^{-1} J(2^n, u_n))^q\right]^{1/q}.$$

Let $\varphi \in B_\psi$ *and* $1 \leqq q \leqq \infty$. *Then for* $a \in \bar{A}_{\varphi,q}$ *we have*

$$(2) \qquad \|a\|_{\bar{A}_{\varphi,q}} \geqq [\log(2)/\bar{\varphi}(2)] \left[\sum_{n=-\infty}^{\infty} (\varphi(2^n)^{-1} K(2^n, a))^q\right]^{1/q}.$$

The proof is the same as in the case of $\varphi(t) = t^\theta$, $0 < \theta < 1$, so that details are left to the reader. Observe that (1) and (2) of Proposition 2.6 can be used to obtain discrete characterizations of $\bar{A}_{\varphi,q}$.

Main result and consequences

In this section we shall deal with functions $F\colon \mathbf{T}\times\mathbf{R}^+\to\mathbf{R}^+$ such that $F(\theta, t)\equiv F_\theta(t)$ is measurable on θ for every $t>0$ and $F_\theta\in B_\psi$ for every $\theta\in\mathbf{T}$. Given $q\colon T\to[1,\infty)$ measurable, the basic assumption on F is the following condition S:

$$\int_0^{2\pi}\log^+\left(\int_0^\infty\left[\bar{F}_\theta(t)\min\left(1,\frac{1}{t}\right)\right]^{q(\theta)}\frac{dt}{t}\right)^{1/q(\theta)}d\theta<\infty \tag{2.1}$$

$$\int_0^{2\pi}\log\left(\int_0^\infty\left[\bar{F}_\theta(t)\min\left(1,\frac{1}{t}\right)\right]\frac{dt}{t}\right)d\theta<\infty \tag{2.2}$$

$$\int_0^{2\pi}\log^+\left(\int_0^\infty[(\bar{F}_\theta(t))^{-1}\min(1,t)]^{q(\theta)}\frac{dt}{t}\right)^{-1/q(\theta)}d\theta<\infty. \tag{2.3}$$

Theorem 2.7. *Let* (A_0, A_1) *be a pair of compatible Banach spaces and F and q as above satisfying condition S. Set* $A(\theta)=(A_0, A_1)_{F_\theta, q(\theta)}$. *Then* $\{A(\theta)\}_{\theta\in\mathbf{T}}$ *is an interpolation family of Banach spaces and* $[A(\theta)]_z=(A_0, A_1)_{F_z, q(z)}$, *with equivalent norms, where* $F_z(t)=\exp\left\{\int_{\mathbf{T}}[\log F(\theta,t)]P_z(\theta)d\theta\right\}$ *and* $(1/q(z))=\int_T(1/q(\theta))P_z(\theta)d\theta$.

Note. As in Proposition 2.1, $P_z(\theta)$ denotes the Poisson kernel of D for evaluation at z. Observe that if $F(\theta, t)=t^{\alpha(\theta)}$, where α is a measurable function from T to $(0,1)$, $F_z(t)=t^{\alpha(z)}$ with $\alpha(z)$ the harmonic extension of $\alpha(\theta)$.

Proof. We begin by showing that $F_z\in B_\psi$ for all $z\in D$ so that it makes sense to write $(A_0, A_1)_{F_z, q(z)}$. For $F_\theta\in B_\psi$ we set $G_\theta(t)=(tF_\theta'(t))/F_\theta(t)$ so that a simple exercise in ordinary differential equations shows

$$F_\theta(t)=\exp\left(\int_1^t(G_\theta(s)/s)\,ds\right). \tag{2.4}$$

Thus

$$F_z(t)=\exp\left\{\int_{\mathbf{T}}\left(\int_1^t(G_\theta(s)/s)\,ds\right)P_z(\theta)\,d\theta\right\}=\exp\left\{\int_1^t(G_z(s)/s)\,ds\right\}$$

where $G_z(s)\equiv G(z,s)$ is the Poisson integral of $G(.,s)$. Hence $G_z(t)=(tF_z'(t)/F_z(t))$. The maximum principle and the continuity of $F(\theta, .)$ show that $\sup_{t>0}(G_z(t))<\sup_{t>0}\sup_{\theta\in\mathbf{T}}G_\theta(t)<1$ and similarly $\inf_{t>0}G_z(t)>0$. This shows that $F_z\in B_\psi$, as we wanted.

Our next step is to prove that $\{A(\theta)\}_{\theta\in\mathbf{T}}$ *is an interpolation family*. To see this, observe that $A(\theta)\hookrightarrow A_0+A_1$ and by Proposition 2.4. $\|a\|_{A_0+A_1}\leq k(\theta)\|a\|_{A(\theta)}$, with

$$k(\theta)=\left(\int_0^\infty[(\bar{F}_\theta(s))^{-1}\min(1,s)]^{q(\theta)}\frac{ds}{s}\right)^{-1/q(\theta)},$$

so that by (2.3), $\log^+k(\theta)\in L^1(T)$, which is all we needed to show.

We now come to the main part of the proof. We first prove the inclusion

$$[A(\theta)]_z \subset (A_0, A_1)_{F_z, q(z)} \tag{2.5}$$

and the corresponding norm inequality.

Let $a \in [A(\theta)]_z$ and take $\varepsilon > 0$. We can find $f \in \mathscr{F}(A)$ with $f(z) = a$ such that

$$\|f\|_\infty \leqq \|a\|_z (1+\varepsilon). \tag{2.6}$$

By the subharmonicity of $\log K(t, f(z))$ (see Lemma 4.1 of [10]) and the definition of $F_z(t)$, we obtain:

$$\|a\|_{F_z, q(z)} = \left(\int_0^\infty [F_z(t)^{-1} K(t, f(z))]^{q(z)} \frac{dt}{t}\right)^{1/q(z)}$$

$$\leqq \left(\int_0^\infty \left[F_z(t)^{-1} \exp\left(\int_{\mathbf{T}} [\log K(t, f(\theta))] P_z(\theta)\, d\theta\right)\right]^{q(z)} \frac{dt}{t}\right)^{1/q(z)}$$

$$= \left(\int_0^\infty \left[\exp\left(\int_T \log [F_\theta(t)^{-1} K(t, f(\theta))] P_z(\theta)\, d\theta\right)\right]^{q(z)} \frac{dt}{t}\right)^{1/q(z)}.$$

Using Proposition 2.2 and the inequality (2.6), we deduce

$$\|a\|_{F_z, q(z)} \leqq \exp\left(\int_T \log\left[\left(\int_0^\infty [F_\theta(t)^{-1} K(t, f(\theta))]^{q(\theta)} \frac{dt}{t}\right)^{1/q(\theta)}\right] P_z(\theta)\, d\theta\right)$$

$$= \exp\left(\int_T [\log \|f(\theta)\|_{F_\theta, q(\theta)}] P_z(\theta)\, d\theta\right) \leqq \|f\|_\infty \leqq \|a\|_z (1+\varepsilon).$$

The inclusion (2.5) now follows with norm less than or equal to 1 upon letting $\varepsilon \to 0$.

In order to prove the inclusion

$$[A(\theta)]_z \supset (A_0, A_1)_{F_z, q(z)} \tag{2.7}$$

and the corresponding norm inequality, we need the following lemma:

Lemma 2.8. *Under condition S, $A_0 \cap A_1 \subset \mathscr{A}$, where $\mathscr{A}$ denotes the log-intersection space of the family $\{A(\theta)\}_{\theta \in \mathbf{T}}$.*

Proof. Since $K(t, a) \leqq \min(1, t) \|a\|_{A_0 \cap A_1}$, we deduce

$$\|a\|_{F_\theta, q(\theta)}^{q(\theta)} \leqq \|a\|_{A_0 \cap A_1}^{q(\theta)} \left(\int_0^1 [F_\theta(t)^{-1} t]^{q(\theta)} \frac{dt}{t} + \int_1^\infty [F_\theta(t)^{-1}]^{q(\theta)} \frac{dt}{t}\right).$$

Using (5) and (6) of Proposition 2.3 with $s=1$ we obtain

$$\|a\|^{q(\theta)}_{F_\theta, q(\theta)} \leqq \|a\|^{q(\theta)}_{A_0\cap A_1}\left(\int_1^\infty \left[\frac{\bar{F}_\theta(t)}{t}\right]^{q(\theta)}\frac{dt}{t}+\int_0^1 [\bar{F}_\theta(t)]^{q(\theta)}\frac{dt}{t}\right)$$

$$= \|a\|^{q(\theta)}_{A_0\cap A_1}\left(\int_0^\infty \left[\bar{F}_\theta(t)\min\left(1, \frac{1}{t}\right)\right]^{q(\theta)}\frac{dt}{t}\right).$$

The desired result now follows from condition (2.1). ∎

To prove (2.7) let $a\in(A_0, A_1)_{F_z, q(z)}$ and $\varepsilon>0$. By Proposition 2.5 there is a representation of a of the form $a=\sum_n u_n$ (convergence in A_0+A_1) with $u_n\in A_0\cap A_1$ such that

(2.8) $$J(2^n, u_n) \leqq 3(1+\varepsilon)K(2^n, a), \quad n\in\mathbf{N}.$$

Fix $t>0$ and let $\tilde{G}(\xi, t)$, $\xi\in D$, be the harmonic conjugate of $G(., t)$ normalized by $\tilde{G}(z, t)=0$. Similarly let $(1/q(\xi))^{\sim}$ be the harmonic conjugate of $1/q(\xi)$ such that $(1/q(z))^{\sim}=0$. Set $W(\xi, t)=G(\xi, t)+i\tilde{G}(\xi, t)$, $\xi\in D$, and $(1/s(\xi))=(1/q(\xi))+i(1/q(\xi))^{\sim}$, $\xi\in D$. Let $H(\xi, t)$ be so that $W(\xi, t)=\dfrac{tH'(\xi,t)}{H(\xi,t)}$, that is, $H(\xi, t)=\exp\left(\int_1^t \frac{W(\xi,s)}{s}\,ds\right)$. Define

$$A_n(\xi) = \frac{H(\xi, 2^n)}{F_z(2^n)}\left[\frac{J(2^n, u_n)}{F_z(2^n)}\right]^{-1+(q(z)/s(\xi))}, \quad n\in\mathbf{Z}.$$

We are going to show that A_n is bounded for every n. In fact,

$$|H(\xi, 2^n)| = \exp\left(\int_1^{2^n}\frac{G(\xi, s)}{s}\,ds\right) \leqq 2^n,$$

since $G(\xi, s)\leqq 1$, and

$$\left|\left[\frac{J(2^n, u_n)}{F_z(2^n)}\right]^{-1+(q(z)/s(\xi))}\right| = \left[\frac{J(2^n, u_n)}{F_z(2^n)}\right]^{-1+(q(z)/q(\xi))}$$

$$\leqq \frac{F_z(2^n)}{J(2^n, u_n)}\max\left(1, \left[\frac{J(2^n, u_n)}{F_z(2^n)}\right]^{q(z)}\right).$$

These two estimates give the desired result.

Set now $g_N(\xi)=\sum_{n=-N}^{N} f_n(\xi)$, $\xi\in D$, where $f_n(\xi)=u_n\cdot A_n(\xi)\in A_0\cap A_1$ so that by Lemma 2.8 and the boundedness of A_n we have $g_N\in N^+(A)$ for all positive integers N. Write $C(\theta)=C\int_0^\infty \bar{F}_\theta(s)\min\left(1, \frac{1}{s}\right)\frac{ds}{s}$, with C as in the first part of

Proposition 2.6. Using this proposition and the definition of A_n, we obtain

$$\|g_N(\theta)\|_{F_\theta,q(\theta)} \leqq C(\theta)\left(\sum_{n=-N}^{N}\left[F_\theta(2^n)^{-1}J(2^n,f_n(\theta))\right]^{q(\theta)}\right)^{1/q(\theta)}$$

$$= C(\theta)\left(\sum_{n=-N}^{N}\left[F_\theta(2^n)^{-1}\left|\frac{H(\theta,2^n)}{F_z(2^n)}\left[\frac{J(2^n,u_n)}{F_z(2^n)}\right]^{-1+(q(z)/s(\theta))}\right|J(2^n,u_n)\right]^{q(\theta)}\right)^{1/q(\theta)}$$

$$= C(\theta)\left(\sum_{n=-N}^{N}\left[F_z(2^n)^{-1}J(2^n,u_n)\right]^{q(z)}\right)^{1/q(\theta)}.$$

Using (2.8) we obtain

$$\|g_N(\theta)\|_{F_\theta,q(\theta)} \leqq 3(1+\varepsilon)C(\theta)\left(\sum_{n=-N}^{N}\left[F_z(2^n)^{-1}K(2^n,a)\right]^{q(z)}\right)^{1/q(\theta)}.$$

Proposition 2.1 now implies

$$\|g_N(z)\|_z \leqq 3(1+\varepsilon)\exp\left(\int_T [\log C(\theta)]P_z(\theta)\,d\theta\right)\left(\sum_{n=-N}^{N}\left[F_z(2^n)^{-1}K(2^n,a)\right]^{q(z)}\right)^{1/q(z)},$$

where $C(z)=\exp\left(\int_T [\log C(\theta)]P_z(\theta)d\theta\right)$ is finite due to (2.2) of condition S, the only property we had not used so far.

Notice that $\lim_{N\to\infty} g_N(z)$ coincides formally with $\sum_{n=-\infty}^{\infty} u_n=a$ (convergence in A_0+A_1) so that a density argument will give $\|a\|_z\leqq k(1+\varepsilon)C(z)\|a\|_{F_z,q(z)}$, after using the second part of Proposition 2.6. The details of this density argument are similar to the ones given on page 89 of [10] and, therefore, omitted. The inclusion (2.7) follows upon letting $\varepsilon\to 0$, and hence Theorem 2.7 is proved. ∎

Corollary 2.9. *Let $q\colon T\to[1,\infty)$ and $w\colon \mathbf{T}\times\mathbf{R}^+\to\mathbf{R}^+$ be measurable functions on $\mathbf{T}$ such that $\varphi_\theta(t)=t^{(1-(1/q(\theta)))}/w_\theta(t)$ belongs to B_ψ for every $\theta\in\mathbf{T}$ and satisfy condition S given before Theorem 2.7. Then $\{\Lambda^{q(\theta)}(w(\theta,.))\}_{\theta\in\mathbf{T}}$ is an interpolation family of Banach spaces and $[\Lambda^{q(\theta)}(w(\theta,.))]_z=\Lambda^{q(z)}(w(z,.))$, with equivalent norms, where*

$$\frac{1}{q(z)}=\int_{\mathbf{T}}\frac{1}{q(\theta)}P_z(\theta)\,d\theta \quad \text{and} \quad w(z,s)=\exp\left\{\int_{\mathbf{T}}[\log w(\theta,s)]P_z(\theta)\,d\theta\right\}.$$

Proof. Use (1.2), that is, $(L^1,L^\infty)_{\varphi_\theta,q(\theta)}=\Lambda^{q(\theta)}(w(\theta,.))$, and Theorem 2.7. ∎

We now show that (1.3) can be deduced from Theorem 2.7. To prove this we need to observe that for any pair B_0 and B_1 of Banach spaces, if

$$B(\xi)=\begin{cases}B_0 & \text{if } 0\leqq\xi<(1-\theta)2\pi\\ B_1 & \text{if } (1-\theta)2\pi\leqq\xi<2\pi\end{cases}$$

then

$$[B_0,B_1]_\theta=[B(\xi)]_0 \tag{2.9}$$

(see [5]). Take

$$q(\xi)=\begin{cases}q_0 & \text{if } 0\leqq\xi<(1-\theta)2\pi\\ q_1 & \text{if } (1-\theta)2\pi\leqq\xi<2\pi,\end{cases}\qquad F_\xi(s)=\begin{cases}\varphi_0(s) & \text{if } 0\leqq\xi<(1-\theta)2\pi\\ \varphi_1(s) & \text{if } (1-\theta)2\pi\leqq\xi<2\pi\end{cases}$$

and observe that $F_0(s)=[\varphi_0(s)]^{1-\theta}[\varphi_1(s)]^{\theta}=\varphi(s)$ and $\frac{1}{q(\theta)}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}=\frac{1}{q}$. Hence, using (2.9) and theorem (2.7) we obtain

$$[(A_0, A_1)_{\varphi_0,q_0}, (A_0, A_1)_{\varphi_1,q_1}]_\theta = [(A_0, A_1)_{F_\xi,q(\xi)}]_0 = (A_0, A_1)_{F_0,q(0)} = (A_0, A_1)_{\varphi,q}.$$

A similar argument shows that (1.4) can be deduced from Corollary 2.9. In fact, with $q(\xi)$ as above and

$$w(\xi, s) = \begin{cases} w_0(s) & \text{if } \ 0 \leqq \xi < (1-\theta)2\pi \\ w_1(s) & \text{if } \ (1-\theta)2\pi \leqq \xi < 2\pi \end{cases}$$

we have $\frac{1}{q(0)}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}=\frac{1}{q}$ and $w(0, s)=w_0(s)^{1-\theta}w_1(s)^{\theta}=w(s)$. Hence, using (2.9) and Corollary 2.9 we obtain:

$$[\Lambda^{q_0}(w_0), \Lambda^{q_1}(w_1)]_\theta = [\Lambda^{q(\xi)}(w(\xi, \cdot))]_0 = \Lambda^{q(0)}(w(0, \cdot)) = \Lambda^q(w).$$

3. Weighted inequalities for the Hardy operator and interpolation

In this section we shall identify the intermediate spaces of the family $\{\Lambda^{q(\theta)}(w(\theta, .))\}_{\theta\in\mathbf{T}}$ using weighted inequalities for the Hardy operator and its dual. We first summarize a method of interpolation for families of Banach lattices which was introduced in [9] and prove some results needed in the sequel.

The Calderón product for families of Banach lattices

A subclass X of a class of measurable functions on a σ-finite measure space (M, dx) is called a *Banach lattice* if there exists a norm $\|\ \|_X$ on X such that $(X, \|\ \|_X)$ is a Banach space and if $f\in X$ and g is a measurable function such that $|g(x)|\leqq|f(x)|$ almost everywhere on M, then $g\in X$ and $\|g\|_X\leqq\|f\|_X$.

Let $\{X(\theta)\}_{\theta\in\mathbf{T}}$ be a family of Banach lattices on (M, dx). For $z\in D$ we define $[X(\theta)]^z$ to be the class of measurable functions f on M for which there exists $\lambda>0$ and a measurable function $F\colon T\times M\to\mathbf{R}$ with $\|F(\theta, .)\|_{X(\theta)}\leqq 1$ almost everywhere such that

$$|f(x)| \leqq \lambda \exp\left\{\int_{\mathbf{T}} [\log|F(\theta, x)|] P_z(\theta)\, d\theta\right\}.$$

We let $\|f\|^z\equiv\|f\|_{[X(\theta)]^z}$ be the infimum of the values of λ for which such an inequality holds. Several properties of the spaces $[X(\theta)]^z$ can be found in [9].

The Banach lattices we are interested in here are the following: for $q\in[1,\infty]$ and $w\geqq 0$ on $(0,\infty)$, $X_q(w)$ will denote the class of functions f on $(0,\infty)$ such that

(3.1) $$\|f\|_{X_q(w)}=\Big\{\int_0^\infty [w(s)|f(s)|]^q\,ds\Big\}^{1/q}<\infty.$$

Proposition 3.1. *Let* $q\colon \mathbf{T}\to[1,\infty)$ *and* $w\colon \mathbf{T}\times\mathbf{R}^+\to\mathbf{R}^+$ *be measurable functions. If* $w(z,s)=\exp\{\int_{\mathbf{T}}[\log w(\theta,s)]P_z(\theta)d\theta\}$ *we have* $[X_{q(\theta)}(w(\theta,.))]^z=X_{q(z)}(w(z,.))$, *with equal norms, where* $(1/q(z))=\int_{\mathbf{T}}(1/q(\theta))P_z(\theta)d\theta$.

Proof. $\subset$. Given $f\in[X_{q(\theta)}(w(\theta,.))]^z$ an $\varepsilon>0$, the definition of $[\;]^z$ allows us to choose $F(\theta,.)\in X_{q(\theta)}(w(\theta,.))$ with $\|F(\theta,.)\|_{X_{q(\theta)}(w(\theta,.))}\leqq 1$ and

$$|f(s)|\leqq(1+\varepsilon)\|f\|^z\exp\Big\{\int_T[\log|F(\theta,s)|]P_z(\theta)\,d\theta\Big\}.$$

We use this inequality together with the fundamental inequality of [9] (Proposition 3.1) to obtain

$$\|f\|_{X_{q(z)}(w(z,\cdot))}\leqq(1+\varepsilon)\|f\|^z\Big[\int_0^\infty w(z,s)^{q(z)}\exp\Big\{q(z)\int_T[\log|F(\theta,s)|]P_z(\theta)\,d\theta\Big\}\,ds\Big]^{1/q(z)}$$

$$\leqq(1+\varepsilon)\|f\|^z\exp\Big\{\int_T[\log\|F(\theta,\cdot)\|_{X_{q(\theta)}(w(\theta,\cdot))}]P_z(\theta)\,d\theta\Big\}\leqq(1+\varepsilon)\|f\|^z.$$

The desired inclusion follows upon letting ε approach 0.

$\supset$. Take $f\in X_{q(z)}(w(z,.))$ and write

$$|f(s)|=\|f\|_{X_{q(z)}(w(z,\cdot))}\{[(|f(s)|)/w(z,s)][(w(z,s))/\|f\|_{X_{q(z)}(w(z,\cdot))}]\}.$$

The term inside the brackets coincides with

$$\exp\Big\{\int_T\Big[\log\Big(\frac{w(z,s)w(\theta,s)^{-1}|f(s)|}{\|f\|_{X_{q(z)}(w(z,\cdot))}}\Big)\Big]P_z(\theta)\,d\theta\Big\}$$

and the norm, in the space $X_{q(\theta)}(w(\theta,.))$, of the term inside the parentheses in this last expression is 1. By definition of $[\;]^z$ we have $f\in[X_{q(\theta)}(w(\theta,.))]^z$ and the inclusion norm is less than or equal to 1. This finishes the proof of Proposition 3.1. ∎

Let (M,dx) be a measure space and define $f^{**}(t)=(1/t)\sup\int_E|f(x)|\,dx$, where $0<t<\infty$, $f\in L_{\mathrm{loc}}(M)$ and the supremum is taken over all measurable sets E in M such that $|E|\leqq t$. If X is a Banach lattice on $(0,\infty)$ we denote by X^* the class of measurable functions f on M such that $f^{**}\in X$ and write $\|f\|_{X^*}=\|f^{**}\|_X$. We shall prove that under some conditions on M and on w, $\Lambda^q(w)$ is a particular case of an X^* space.

Proposition 3.2. *Let* (M,dx) *be a non-atomic measure space,* $1\leqq q\leqq\infty$ *and* $w\in W_q(S)$. *Then* $\Lambda^q(w)=(X_q(w))^*$, *with equivalent norms.*

Proof. The proof is based on the equality

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds = S(f^*)(t) \tag{3.2}$$

which holds for non-atomic measure spaces and can be found in [15]. Let $f\in(X_q(w))^*$. Since f^* is non-increasing, (3.2) implies $f^{**}(t)\geqq f^*(t)$ and hence

$$\|f\|_{\Lambda^q(w)} \leqq \Big(\int_0^\infty [w(s)f^{**}(s)]^q\,ds\Big)^{1/q} = \|f\|_{(X_q(w))^*}$$

so that we have proved one inclusion. To prove the other one take $f\in\Lambda^q(w)$ and use (3.2) to write

$$\|f\|_{(X_q(w))^*} = \Big(\int_0^\infty [w(s)S(f^*)(s)]^q\,ds\Big)^{1/q}.$$

Since $w\in W_q(S)$ we deduce

$$\|f\|_{(X_q(w))^*} \leqq C_q(w)\Big(\int_0^\infty [w(s)f^*(s)]^q\,ds\Big)^{1/q} = C_q(w)\|f\|_{\Lambda^q(w)}$$

so that Proposition 3.2 is proved. ∎

Main result and consequences

Theorem 3.3. *Let* $q\colon \mathbf{T}\to[1,\infty)$ *and* $w\colon \mathbf{T}\times\mathbf{R}^+\to\mathbf{R}^+$ *be measurable functions. If*

$$\int_T \Big[\log^+\Big(\int_0^1 [w(\theta,s)]^{-q'(\theta)}\,ds\Big)^{1/q'(\theta)}\Big]\,d\theta<\infty, \tag{*}$$

the family $\{\Lambda^{q(\theta)}(w(\theta,.))\}_{\theta\in\mathbf{T}}$ *is an interpolation family of Banach spaces. Moreover if* $w(\theta,.)\in W_{q(\theta)}(S)\cap W_{q(\theta)}(S^*)$ *and*

$$(**)\quad \int_{\mathbf{T}}[\log C(S;\,q(\theta),w(\theta,\cdot))]\,d\theta<\infty,\quad \int_{\mathbf{T}}[\log C(S^*;\,q(\theta),w(\theta,\cdot))]\,d\theta<\infty$$

we have $[\Lambda^{q(\theta)}(w(\theta,.))]_z=\Lambda^{q(z)}(w(z,.))$, *with equivalent norms, where*

$$\frac{1}{q(z)}=\int_{\mathbf{T}}\frac{1}{q(\theta)}P_z(\theta)\,d\theta \quad \text{and} \quad w(z,s)=\exp\Big\{\int_{\mathbf{T}}[\log(w(\theta,s))]P_z(\theta)\,d\theta\Big\}.$$

Proof. We start by proving that $\{\Lambda^{q(\theta)}(w(\theta,.))\}_{\theta\in\mathbf{T}}$ is an interpolation family of Banach spaces. We take L^1+L^∞ as the containing space and for generic q and w we shall prove $\Lambda^q(w)\subset L^1+L^\infty$, finding an upper bound for the inclusion norm. For $f\in\Lambda^q(w)$, let

$$g(x)=\begin{cases} f(x) & \text{if } |f(x)|>f^*(1)\\ 0 & \text{otherwise}\end{cases}$$

and $h(x)=f(x)-g(x)$. Then, (see [16]),

$$g^*(s) \leqq \begin{Bmatrix} f^*(s) & \text{if} & 0<s<1 \\ 0 & \text{if} & s>1 \end{Bmatrix}; \quad h^*(s) \leqq \begin{Bmatrix} f^*(1) & \text{if} & s \leqq 1 \\ f^*(s) & \text{if} & s>1 \end{Bmatrix}.$$

Thus $g \in L^1$ since

$$\int_0^\infty |g(s)|\, ds \leqq \int_0^1 f^*(s)\, ds \leqq \Big(\int_0^1 [w(s) f^*(s)]^q\, ds\Big)^{1/q} \Big(\int_0^1 [w(s)]^{-q'}\, ds\Big)^{1/q'}$$

$$\leqq \|f\|_{A^q(w)} \Big(\int_0^1 [w(s)]^{-q'}\, ds\Big)^{1/q'}.$$

Also $h \in L^\infty$ since

$$|h(s)| \leqq f^*(1) \leqq \int_0^1 f^*(s)\, ds \leqq \|f\|_{A^q(w)} \Big(\int_0^1 [w(s)]^{-q'}\, ds\Big)^{1/q'}.$$

It follows from here that (*) is the right condition for $\{A^{q(\theta)}(w(\theta, .))\}_{\theta \in T}$ to be an interpolation family of Banach spaces.

We now prove the equality of the spaces. By Proposition 3.2 we have $A^{q(\theta)}(w(\theta, .)) = (X_{q(\theta)}(w(\theta, .)))^*$ and by theorem (9.2) of [9] we obtain

$$[A^{q(\theta)}(w(\theta, .))]^z = ([X_{q(\theta)}(w(\theta, .))]^z)^*$$

(observe that (**) is the hypothesis we need to apply this theorem). It follows from here and Proposition 3.1 that $[A^{q(\theta)}(w(\theta, .))]^z = (X_{q(z)}(w(z, .)))^*$. Using Proposition 2.2 we deduce $w(z, .) \in W_{q(z)}(S)$ and hence Proposition 3.2 shows

$$[A^{q(\theta)}(w(\theta, \cdot))]^z = A^{q(z)}(w(z, \cdot)). \tag{3.3}$$

Since $q(\theta) < \infty$, we also have $q(z) < \infty$ and, by (3.3), $[A^{q(\theta)}(w(\theta, .))]^z$ has the dominated convergence property needed to apply Theorem 6.1 of [9]. Upon applying this theorem and using (3.3) we obtain $[A^{q(\theta)}(w(\theta, .))]_z = A^{q(z)}(w(z, .))$, which is the desired result. ∎

Notice that (1.11) follows from Theorem 3.3 by an argument similar to the one given at the end of Section 2.

Corollary 3.4. (Proposition 9.4 of [9].) *Let $p\colon T \to [1, \infty)$ and $q\colon T \to [1, \infty)$ be two measurable functions on T such that*

$$\int_T [\log p(\theta)]\, d\theta < \infty \quad \textit{and} \quad \int_T \Big[\log \frac{1}{p(\theta)-1}\Big]\, d\theta < \infty. \tag{3.4}$$

Then $\{L^{p(\theta), q(\theta)}\}$ is an interpolation family of Banach spaces and $[L^{p(\theta), q(\theta)}]_z = L^{p(z), q(z)}$, with equivalent norms, where

$$\frac{1}{q(z)} = \int_T \frac{1}{q(\theta)} P_z(\theta)\, d\theta \quad \textit{and} \quad \frac{1}{p(z)} = \int_T \frac{1}{p(\theta)} P_z(\theta)\, d\theta.$$

Proof. We apply Theorem 3.3 with $w(\theta,s)=s^{(1/p(\theta))-(1/q(\theta))}$. Since

$$\int_0^1[w(\theta,s)]^{-q'(\theta)}\,ds = p'(\theta)/q'(\theta),$$

(*) follows from (3.4). We now find the constants $C(S;q(\theta),w(\theta,.))$ and $C(S^*;q(\theta),w(\theta,.))$ for this particular choice of w. We start by finding the value of K in (1.8):

$$\sup_{r>0}\Big(\int_r^\infty s^{-q(\theta)}s^{(q(\theta)/p(\theta))-1}\,ds\Big)^{1/q(\theta)}\Big(\int_0^r s^{-(q'(\theta)/p(\theta))+(q'(\theta)/q(\theta))}\,ds\Big)^{1/q'(\theta)}$$

$$= p'(\theta)[1/q(\theta)]^{1/q(\theta)}[1/q'(\theta)]^{1/q'(\theta)}.$$

There is an inequality between C of (1.7) and K of (1.8), namely $K\leqq C\leqq K(q)^{1/q}(q')^{1/q'}$ (see [14], Theorem 1). Using this, we obtain $C(S;q(\theta),w(\theta,.))\leqq p'(\theta)=p(\theta)/(p(\theta)-1)$ and hence the first part of (**) follows from (3.4). Similarly,

$$\sup_{r>0}\Big(\int_0^r s^{(q(\theta)/p(\theta))-1}\,ds\Big)^{1/q(\theta)}\Big(\int_r^\infty s^{-q'(\theta)}s^{-(q'(\theta)/p(\theta))+(q'(\theta)/q(\theta))}\,ds\Big)^{1/q'(\theta)}$$

$$= p(\theta)[1/q(\theta)]^{1/q(\theta)}[1/q'(\theta)]^{1/q'(\theta)}$$

and hence $C(S^*;q(\theta),w(\theta,.))\leqq p(\theta)$. The second part of (**) now follows from (3.4) and therefore Theorem 3.3 can be applied to prove the corollary. ∎

4. Final result

We shall prove that Theorem 3.3 is more general than Corollary 2.9. This is contained in the following result:

Theorem 4.1. *If* $1\leqq q<\infty$ *and* $\varphi\in B_\psi$ *then* $w(s)=s^{(1-(1/q))}/\varphi(s)\in W_q(S)\cap W_q(S^*)$.

Proof. We shall prove that if $\varphi\in B_K$ then $w\in W_q(S)\cap W_q(S^*)$, and since $B_\psi\subset B_K$ Theorem 4.1 will follow (this is not a great improvement since B_ψ and B_K are equivalent classes of functions in the sense of (7), Proposition 2.3). We have

$$I = \sup_{r>0}\Big(\int_r^\infty s^{-q}[w(s)]^q\,ds\Big)^{1/q}\Big(\int_0^r[w(s)]^{-q'}\,ds\Big)^{1/q'}$$

$$= \sup_{r>0}\Big(\int_r^\infty\Big[\frac{1}{\varphi(s)}\Big]^q\frac{ds}{s}\Big)^{1/q}\Big(\int_0^r[\varphi(s)]^{q'}\frac{ds}{s}\Big)^{1/q'}$$

$$= \sup_{r>0}\Big(\int_r^\infty\Big[\frac{1}{\varphi(s)}\Big]^q\frac{ds}{s}\Big)^{1/q}\Big(\int_0^1\Big[\frac{\varphi(rs)}{\varphi(r)}\Big]^{q'}\frac{ds}{s}\Big)^{1/q'}\varphi(r).$$

From (6) of Proposition 2.3 we obtain

$$I \leqq \left(\int_0^1 [\bar{\varphi}(s)]^q \frac{ds}{s}\right)^{1/q} \left(\int_0^1 [\bar{\varphi}(s)]^{q'} \frac{ds}{s}\right)^{1/q'},$$

which is finite by (3) of Proposition 2.3. Similarly, but using (5) instead of (6) of Proposition 2.3 we obtain

$$\sup_{r>0} \left(\int_0^r [w(s)]^q \, ds\right)^{1/q} \left(\int_r^\infty s^{-q'} [w(s)]^{-q'} \, ds\right)^{1/q'}$$

$$= \sup_{r>0} \left(\int_0^r \left[\frac{s}{\varphi(s)}\right]^q \frac{ds}{s}\right)^{1/q} \left(\int_r^\infty \left[\frac{\varphi(s)}{s}\right]^{q'} \frac{ds}{s}\right)^{1/q'}$$

$$= \sup_{r>0} \left(\int_0^r \left[\frac{s}{\varphi(s)}\right]^q \frac{ds}{s}\right)^{1/q} \left(\int_1^\infty \left[\frac{\varphi(rs)}{s\varphi(r)}\right]^{q'} \frac{ds}{s}\right)^{1/q'} \frac{\varphi(r)}{r}$$

$$\leqq \left(\int_1^\infty \left[\frac{\bar{\varphi}(s)}{s}\right]^q \frac{ds}{s}\right)^{1/q} \left(\int_1^\infty \left[\frac{\bar{\varphi}(s)}{s}\right]^{q'} \frac{ds}{s}\right)^{1/q'},$$

which is finite by (3) of Proposition 2.3. ∎

The converse of Theorem 4.1 is not true in the sense that there exists $w \in W_q(S) \cap W_q(S^*)$, $1 < q < \infty$, such that for all $\varphi \in B_\psi$, w is *not* equivalent to $t^{1/q'}/\varphi(t)$ (see (7), Proposition 2.3). This function w is defined by

$$w(t) = \begin{cases} t^\alpha & \text{if } 0 < t \leqq 1/2 \\ (1-t)^\alpha & \text{if } 1/2 < t \leqq 1 \\ (t-1)^\alpha & \text{if } 1 < t \leqq 2 \\ 1 & \text{if } t > 2 \end{cases}$$

where $0 < \alpha < 1/q'$. Observe that $w(1) = 0$ while $\varphi(1) = 1$. The conditions on α is what is required to show that $w \in W_q(S) \cap W_q(S^*)$, $1 < q < \infty$.

In view of the above result we conjecture that a theory of real interpolation with a function parameter $w \in W_q(S) \cap W_q(S^*)$ for all $q \in [1, \infty)$ could be developed. This theory would be more general than the existing ones.

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