Duality of space curves and their tangent surfaces in characteristic $p>0$

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0. Introduction

Let X be a nondegenerate complete irreducible curve in projective N-space $\mathbf{P}^N$ over an algebraically closed field k of characteristic p. Let $\pi\colon \tilde{X}\to X$ be the normalization of X and $\mathfrak{G}$ the linear system on $\tilde{X}$ corresponding to the subspace $V_{\mathfrak{G}}=\text{Image}\,[H^0(\mathbf{P}^N, \mathcal{O}(1))\to H^0(\tilde{X}, \pi^*\mathcal{O}_X(1))]$. Let $\tilde{P}$ be a point on $\tilde{X}$. Since X is nondegenerate, there are $N+1$ integers $\mu_0(\tilde{P})<\ldots<\mu_N(\tilde{P})$ such that there are $D_0, \ldots, D_N\in\mathfrak{G}$ with $v_{\tilde{P}}(D_i)=\mu_i(\tilde{P})$ $(i=0, \ldots, N)$, where $v_{\tilde{P}}(D_i)$ is the multiplicity of D_i at $\tilde{P}$. When $p=0$, the sequence $\mu_0(\tilde{P}), \ldots, \mu_N(\tilde{P})$ coincides with $0, 1, \ldots, N$ except for finitely many points. On the contrary, this is not always valid in positive characteristic. However, F. K. Schmidt [12] (when $\mathfrak{G}$ is the canonical linear system) and other authors [8], [9], [10], [13] (for any linear systems) showed that there are $N+1$ integers $b_0<\ldots<b_N$ such that $\mu_0(\tilde{P}), \ldots, \mu_N(\tilde{P})$ coincides with $b_0, \ldots, b_N$ except for finitely many points.

From now on, we denote by $B(\mathfrak{G})$ the set of integers $\{b_0, \ldots, b_N\}$. Since we take an interest in the invariant $B(\mathfrak{G})$, we always assume that $p>0$.

What geometric phenomena does the invariant $B(\mathfrak{G})$ reflect? Roughly speaking, this invariant reflects the duality of osculating developables of X. Let Y be a closed subvariety of $\mathbf{P}^N$. We define the conormal variety $C(Y)$ of Y by the Zariski closure of

$$\{(y, H^*)\in Y\times\check{\mathbf{P}}^N \mid y \text{ is smooth},\ \ T_y(Y)\subset H\},$$

where $\check{\mathbf{P}}^N$ is the dual N-space of $\mathbf{P}^N$ and $T_y(Y)$ is the (embedded) tangent space at y to Y. The image of the second projection $C(Y)\to\check{\mathbf{P}}^N$ is denoted Y^*, which is called the dual variety of Y. The original variety Y is said to be reflexive if $C(Y)\to Y^*$ is generically smooth (The Monge—Segre—Wallace criterion; see [6, page 169]). In the previous paper [5], we proved the following theorem.

Theorem 0.0 [5; Theorem 3.3]. *Let ν be an integer with $0 \leqq \nu \leqq N-2$. Assume that $b_{\nu+1} \not\equiv 0 \mod p$. Then the ν-th osculating developable of X is reflexive if and only if $b_{\nu+2} \not\equiv 0 \mod p$.*

In the present paper, we prove a more precise theorem on this line for space curves, i.e., nondegenerate curves in $\mathbf{P}^3$.

For a space curve X, one has the following five possibilities:

(RR) $p>3$ and $B(\mathfrak{G})=\{0, 1, 2, 3\}$;

(RN) $p>2$ and $B(\mathfrak{G})=\{0, 1, 2, q\}$;

($\mathrm{NR_I}$) $B(\mathfrak{G})=\{0, 1, q, q+1\}$;

($\mathrm{NR_{II}}$) $p>2$ and $B(\mathfrak{G})=\{0, 1, q, 2q\}$;

(NN) $B(\mathfrak{G})=\{0, 1, q, q'q\}$,

where q and q' are powers of p (see proposition 1.2 below). Moreover, any case can be shown to occur (see example 2.6 below). Our theorem is as follows.

Theorem 0.1. *Let X be a nondegenerate space curve and* Tan X *be the tangent surface of X.*

(i) *$B(\mathfrak{G})$ is of type* (RR) $\Leftrightarrow X$ *and* Tan X *are reflexive.*

(ii) *$B(\mathfrak{G})$ is of type* (RN) $\Leftrightarrow X$ *is reflexive and* Tan X *is nonreflexive.*

(iii) *$B(\mathfrak{G})$ is of type* ($\mathrm{NR_I}$) $\Leftrightarrow X$ *is nonreflexive and* Tan X *is ordinary.*

(iv) *$B(\mathfrak{G})$ is of type* ($\mathrm{NR_{II}}$) $\Leftrightarrow X$ *is nonreflexive and* Tan X *is semiordinary (of reflexive type).*

(v) *$B(\mathfrak{G})$ is of type* (NN) $\Leftrightarrow X$ *and* Tan X *are nonreflexive.*

The main tool of our proof of the theorem is the Hessian criterion of reflexivity obtained by Hefez—Kleiman [2].

1. Type of $B(\mathfrak{G})$

We will use some knowledge of the theory of Weierstrass points in positive characteristic. Surveys of this theory can be found in [3; §§1—2] and/or [13; §1].

This section is a sort of elementary number theory. Let p be a prime number. Then a nonnegative integer u can be written uniquely as $u=\sum_{i\geqq 0} u_i p^i$, where u_i are integers with $0 \leqq u_i < p$. We denote by $u \underset{p}{>} v$ (or $v \underset{p}{<} u$) if $u>v$ and $u_i \geqq v_i$ for all $i \geqq 0$.

Lemma 1.0. *Let u, v be nonnegative integers with $u \underset{p}{>} v$. If $u \in B(\mathfrak{G})$, then $v \in B(\mathfrak{G})$.*

Proof. See [11; Satz 6] or [13; Cor. 1.9].

Corollary 1.1 (cf. [1; Prop. 2]). *Let* $B(\mathfrak{G})=\{b_0<b_1<b_2<b_3\}$ *and* $i_0=\text{Max}\,\{i|b_i=i\}$. *Then*

(0) $i_0\geqq 1$, *i.e.,* $b_0=0$ *and* $b_1=1$.

Moreover, we assume that $i_0<3$. *Then we have that*

(i) $b_{i_0+1}\equiv 0 \mod p$,
(ii) *if* $i_0<p$, *then* b_{i_0+1} *is a power of* p.

Proof. (0) The condition $b_0=0$ is valid for any linear system. Since the morphism corresponding to $\mathfrak{G}$ coincides with $\pi\colon \tilde{X}\to X$ which is birational (hence separable), we have $b_1=1$.

(i) Write $b_{i_0+1}=ap+r$ with $0\leqq r<p$. If $r>0$, then $b_{i_0+1}-1 \underset{p}{<} b_{i_0+1}$. This implies $b_{i_0+1}-1\in B(\mathfrak{G})$ by (1.0). Hence we have $b_{i_0+1}-1=b_{i_0}=i_0$, which contradicts to the choice of i_0.

(ii) From the above, we may write as $b_{i_0+1}=up^m$ with $m>0$ and $(u,p)=1$. If $u>1$, then $(u-1)p^m \underset{p}{<} b_{i_0}$. Hence $(u-1)p^m\in B(\mathfrak{G})$ by (1.0). Hence we have $(u-1)p^m\leqq b_{i_0}=i_0<p$, which is a contradiction. □

The next proposition is the main purpose of this section.

Proposition 1.2. *The invariant* $B(\mathfrak{G})$ *of a space curve over a field of characteristic* $p>0$ *must be one of the following 5-types:*

(RR) $p>3$ *and* $B(\mathfrak{G})=\{0,1,2,3\}$;

(RN) $p>2$ *and* $B(G)=\{0,1,2,q\}$;

(NR$_\text{I}$) $B(\mathfrak{G})=\{0,1,q,q+1\}$;

(NR$_\text{II}$) $p>2$ *and* $B(\mathfrak{G})=\{0,1,q,2q\}$;

(NN) $B(\mathfrak{G})=\{0,1,q,q'q\}$,

where q *and* q' *are powers of* p.

Proof. We know that $b_0=0$ and $b_1=1$. First we assume that $p>2$ and $b_2=2$. If $B(\mathfrak{G})$ is not of type (RR), then b_3 is a power of p by (1.1). This case is of type (RN).

Next we assume that $b_2>2$ or $p=2$. In this case, b_2 is a power of p, say q, by (1.1). Write $b_3=aq+r$ with $0\leqq r<q$. Since $b_3>b_2=q$, we have $a\geqq 1$. Since $r\underset{p}{<}aq+r$, we have $r\in B(\mathfrak{G})$. Hence $r=0$ or 1. If $r=1$, then $aq\underset{p}{<}aq+1$. Hence

$aq \in B(\mathfrak{G})$ and hence $b_2 = aq$. So we have $a=1$. This case is of type (NR_1). Next we consider the case $r=0$. Write $a=up^m$ with $(u,p)=1$. If $u=1$, then this case is of type (NN). Suppose that $u>1$. Write $u=u'p+u''$ with $0 \leqq u'' < p$. Since $(u,p)=1$, we have $u''>0$. Hence we have

$$b_3 = aq = up^m q = u'p^{m+1}q + u''p^m q \underset{p}{>} u'p^{m+1}q + (u''-1)p^m q.$$

Hence $u'p^{m+1}q+(u''-1)p^m q \in B(\mathfrak{G})$. Since $u'p^{m+1}q+(u''-1)p^m q \equiv 0 \mod q$, this must coincide with b_2. Hence we have

$$u'p^{m+1}q+(u''-1)p^m q = q,$$

and hence we have $u'=0$, $u''=2$, i.e., $b_3=2q$. This completes the proof.

Remark 1.3. In the next section, we will show that for each type of $B(\mathfrak{G})$ described in (1.2), there is a nondegenerate space curve whose $B(\mathfrak{G})$ has the assigned type.

2. Some properties of $B(\mathfrak{G})$

Let $\operatorname{Reg} X$ be the open set of smooth points of X. We will identify $\operatorname{Reg} X$ with $\pi^{-1}(\operatorname{Reg} X)$. Let $P \in \operatorname{Reg} X$ be a general point. Choose a plane section G_0 of X such that $P \notin \operatorname{Supp} G_0$. Let $\tilde{G}_0 \in \mathfrak{G}$ corresponding to G_0 via the isomorphism

$$(1) \qquad H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(1)) \simeq V_{\mathfrak{G}} \subset H^0(\tilde{X}, \pi^* \mathcal{O}_X(1)).$$

Then we have the commutative diagram:

$$(2) \qquad \begin{array}{ccc} L(\tilde{G}_0) \overset{\text{def}}{=} \{f \in k(\tilde{X})^\times \mid \operatorname{div} f + \tilde{G}_0 > 0\} \cup \{0\} & \simeq & H^0(\tilde{X}, \pi^*\mathcal{O}_X(1)) \\ \cup\!\uparrow & & \cup\!\uparrow \\ L(\mathfrak{G};\, \tilde{G}_0) \overset{\text{def}}{=} \{f \in k(\tilde{X})^\times \mid \operatorname{div} f + \tilde{G}_0 \in \mathfrak{G}\} \cup \{0\} & \simeq & V_{\mathfrak{G}}. \end{array}$$

2.0. A characterization of $B(\mathfrak{G})$. Let $t \in \mathcal{O}_{X,P}$ be a local parameter at P. Identifying the field of fractions of $\hat{\mathcal{O}}_{X,P}$ with $k((t))$ and viewing $k(X) \subset k((t))$ via this identification, we can define iterative derivations $\{D_t^{(\nu)} \mid \nu = 0, 1, 2, \dots\}$ on $k(X)$ such that $D_t^{(\nu)}(t^m) = \binom{m}{\nu} t^{m-\nu}$ (see, [5; appendix]). Let $f_0, \dots, f_3$ be a basis of $L(\mathfrak{G};\, \tilde{G}_0)$. Then the sequence $\{b_0 < b_1 < b_2 < b_3\}$ coincides with the minimal element of

$$\{\mu_0 < \mu_1 < \mu_2 < \mu_3 \mid \det(D_t^{(\mu_i)} f_j)_{(i,j)} \neq 0\}$$

by lexicographic order (see [3; § 1] or [13; page 5]).

Remark 2.1. Let us consider the vector space $\oplus^4 k(X)$ over $k(X)$ and denote by V_m the subspace generated by

$$\{(D_t^{(\nu)}f_0, D_t^{(\nu)}f_1, D_t^{(\nu)}f_2, D_t^{(\nu)}f_3) \mid 0 \leqq \nu \leqq m\}.$$

Then we have

$$V_0 \subsetneqq V_1 = \dots = V_{b_2-1} \subsetneqq V_{b_2} = \dots = V_{b_3-1} \subsetneqq V_{b_3} = \overset{4}{\oplus} k(X) \tag{3}$$

by the preceding characterization of $B(\mathfrak{G})$.

2.2. Standard coordinates on $\mathbf{P}^3$ *with respect to* P. It is obvious that we can choose a basis x_0, x_1, x_2, x_3 of $L(\mathfrak{G}; \tilde{G}_0)$ such that

$$0 = v_P(x_0) < v_P(x_1) < v_P(x_2) < v_P(x_3),$$

where v_P is the valuation of $\mathcal{O}_{X,P}$. Note that since P is a general point, this sequence is nothing but $\{b_0<b_1<b_2<b_3\}$. The sections $X_0, \dots, X_3$ of $H^0(\mathbf{P}^3, \mathcal{O}(1))$ corresponding to $x_0, \dots, x_3$ via isomorphisms (1) and (2) are called standard coordinates on $\mathbf{P}^3$ with respect to P.

Remark 2.3. With the above notations,

(a) the plane section $X_0=0$ on X is G_0;
(b) the rational function on X obtained by X_i/X_0 is x_i.

Remark 2.4. Let $X_0, \dots, X_3$ be standard coordinates on $\mathbf{P}^3$ with respect to a general point P of X and x_i be the restriction of X_i/X_0 to X $(i=0, \dots, 3)$. Since $1=b_1=v_P(x_1)$, we may consider x_1 itself as a local parameter at P. Moreover replacing, if necessary, X_2 and X_3 by c_2X_2 and c_3X_3 for suitable $c_2, c_3 \in k^{\times}$, $x_0, \dots, x_3$ can be expanded by $t=x_1$ as:

$$\begin{cases} x_0 = 1 \\ x_1 = t \\ x_2 = \quad t^{b_2} + (\text{higher order terms}) \\ x_3 = \qquad t^{b_3} + (\text{higher order terms}). \end{cases} \tag{4}$$

Lemma 2.5. *Under the above notations, suppose that* $b_2>2$. *Then we have*

$$D_t^{(\nu)} x_2 = 0 \quad \textit{and} \quad D_t^{(\nu)} x_3 = 0 \quad \textit{for} \quad 2 \leqq \forall \nu < b_2.$$

Proof. From (3) in remark 2.1, we have that the rank of

$$\begin{pmatrix} D_t^{(0)}x_0 & D_t^{(0)}x_1 & D_t^{(0)}x_2 & D_t^{(0)}x_3 \\ D_t^{(1)}x_0 & D_t^{(1)}x_1 & D_t^{(1)}x_2 & D_t^{(1)}x_3 \\ D_t^{(\nu)}x_0 & D_t^{(\nu)}x_1 & D_t^{(\nu)}x_2 & D_t^{(\nu)}x_3 \end{pmatrix}$$

is 2, if $2 \leqq \nu < b_2$. Since

$$(D_t^{(i)} x_j)_{\substack{i=0,1,\nu \\ j=0,1,2,3}} = \begin{pmatrix} 1 & x_1 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & D_t^{(\nu)} x_2 & D_t^{(\nu)} x_3 \end{pmatrix}$$

by (4), we have $D_t^{(\nu)} x_2 = D_t^{(\nu)} x_3 = 0$ if $2 \leqq \nu < b_2$. □

Example 2.6. Let us consider the morphism

$$\pi \colon \mathbf{P}^1 \to \mathbf{P}^3 \quad \text{with homogeneous coordinates} \quad Y_0, \dots, Y_3$$

$$t \mapsto (1 : t : t^u : t^v)$$

with

$$2 \leqq u < v,$$

and let us denote by X the image of π. Then $\pi\colon \mathbf{P}^1 \to X$ is the normalization of X. Let $\mathfrak{G}$ be the linear system on $\mathbf{P}^1$ corresponding to the plane sections on X. Let $c \in \mathbf{P}^1$ be a general point. Let $s = t - c$. Then the coordinate functions $y_i (= (Y_i/Y_0)|_X)$ can be expressed by using s as follows;

$$\text{(5)} \qquad \begin{cases} y_0 = 1 \\ y_1 = c + s \\ y_2 = (c+s)^u \\ y_3 = (c+s)^v. \end{cases}$$

Let q be a power of p. If $u = 2$ and $v = q$, then (5) is rewritten as

$$y_0 = 1$$
$$y_1 - c y_0 = s$$
$$y_2 - c^2 y_0 - 2c(y_1 - c y_0) = s^2$$
$$y_3 - c^q y_0 = s^q.$$

Let $\tilde{G}_0 = q \cdot \infty$ on $\mathbf{P}^1$. Then $\tilde{G}_0 \in \mathfrak{G}$ and

$$L(\mathfrak{G}; \tilde{G}_0) = \langle y_0, y_1 - c y_0, y_2 - c^2 y_0 - 2c(y_1 - c y_0), y_3 - c^q y_0 \rangle.$$

Hence we have $B(\mathfrak{G}) = \{0, 1, 2, q\}$.

By arguments similar to that of the above case, we can show that

if $u = 2$, $v = 3$, then $B(\mathfrak{G}) = \{0, 1, 2, 3\}$;

if $u = q$, $v = q+1$, then $B(\mathfrak{G}) = \{0, 1, q, q+1\}$;

if $u = q$, $v = 2q$, then $B(\mathfrak{G}) = \{0, 1, q, 2q\}$;

if $u = q$, $v = q'q$, then $B(\mathfrak{G}) = \{0, 1, q, q'q\}$,

where q and q' are powers of p.

Note that $B(\mathfrak{G})$ is not always $\{0, 1, u, v\}$, which is the "gap sequence" at the origin (properly speaking, $\{1, 2, u+1, v+1\}$ is the gap sequence at the origin). The origin may be a $\mathfrak{G}$-Weierstrass point. For example, if $u=2$, $v=q+1$, then $B(\mathfrak{G})=\{0, 1, 2, q\}$.

3. Tangential properties of a space curve

First we review the Hessian criterion of reflexivity of projective varieties (for details, see [2; 3.2] or [6; page 176]).

Let Y be a closed subvariety of dimension n, $C(Y)$ the conormal variety of Y and Y^* the dual variety of Y.

Let $P\in \operatorname{Reg} Y$ and $t_1, \ldots, t_n$ a system of local parameters of $\mathcal{O}_{Y,P}$. Let H be a hyperplane with $T_P(Y)\subset H$ and $h\in\mathcal{O}_{Y,P}$ a local equation of H at P.

The Hessian rank at $(P, H^*)\in C(Y)^0$ is defined as the rank of the matrix $\left(\frac{\partial^2 h}{\partial t_i \partial t_j}(P)\right)_{(i,j)}$, where $C(Y)^0=C(Y)\cap(\operatorname{Reg} Y)\times Y^*$. Since the Hessian rank is lower semicontinuous on $C(Y)^0$, we may define the Hessian rank h_Y of Y by the Hessian rank at a general point $(P, H^*)\in C(Y)^0$.

The duality codefect c_Y of Y is defined by

$$c_Y = \dim Y+\dim Y^*-(N-1).$$

Note that the inequality $h_Y\leqq c_Y$ holds.

Hessian criterion (Hefez—Kleiman). *Y is reflexive if and only if $h_Y=c_Y$.*

When Y is a hypersurface, the matrix $\left(\frac{\partial^2 h}{\partial t_i \partial t_j}(P)\right)$ is an $(N-1)\times(N-1)$ matrix. Hence $h_Y\leqq N-1$. If $h_Y=N-1$, Y is said to be ordinary. In this case, Y is reflexive and Y^* is a hypersurface, since $c_Y=\dim Y^*\leqq N-1$. If $h_Y=N-2$, Y is said to be semiordinary. If Y is semiordinary, then

(i) Y is reflexive$\Leftrightarrow\dim Y^*=N-2$

(ii) Y is nonreflexive$\Leftrightarrow\dim Y^*=N-1$.

When the first case occurs, Y is said to be semiordinary of reflexive type. When the second case occurs, Y is said to be semiordinary of nonreflexive type.

Now let us return to that problem of space curves. Our result can be summarized in the following table.

Table 3.0. Let X be a nondegenerate space curve, $\mathfrak{G}$ the linear system on $\tilde{X}$ corresponding to the plane sections of X and $\operatorname{Tan} X$ the tangent surface of X.

Type of $B(\mathfrak{G})$	Reflexivity of X	$h_{\operatorname{Tan} X}$	$\dim(\operatorname{Tan} X)^*$	Reflexivity of $\operatorname{Tan} X$
(RR)	Reflexive	1	1	Reflexive
(RN)	Reflexive	0	1	Nonreflexive
($\mathrm{NR_I}$)	Nonreflexive	2	2	Reflexive
($\mathrm{NR_{II}}$)	Nonreflexive	1	1	Reflexive
(NN)	Nonreflexive	0	1	Nonreflexive

3.1. Notes to accompany table 3.0.

(0) Table 3.0 with Proposition 1.2 implies our main theorem (Theorem 0.1).

(i) Since $\operatorname{Tan} X$ is a surface in $\mathbf{P}^3$, the duality codefect of $\operatorname{Tan} X$ coincides with $\dim(\operatorname{Tan} X)^*$. Therefore the last column in the table follows from the preceding two columns.

(ii) The first two rows and the column of reflexivity of X result from the previous paper (see theorem 0.0 and [5; (3.1)]). Therefore, to complete the table, it suffices to show the following theorem.

Theorem 3.2. *Notations are same as in* (3.0).

(i) *If* $B(\mathfrak{G})$ *of* X *is of type* ($\mathrm{NR_I}$), *then* $h_{\operatorname{Tan} X}=2$ *and* $\dim(\operatorname{Tan} X)^*=2$.

(ii) *If* $B(\mathfrak{G})$ *of* X *is of type* ($\mathrm{NR_{II}}$), *then* $h_{\operatorname{Tan} X}=1$ *and* $\dim(\operatorname{Tan} X)^*=1$.

(iii) *If* $B(\mathfrak{G})$ *of* X *is of type* (NN), *then* $h_{\operatorname{Tan} X}=0$ *and* $\dim(\operatorname{Tan} X)^*=1$.

This theorem will be proved in the next section.

4. Proof of theorem 3.2.

In this section, we give a proof of theorem 3.2. Let X be a space curve whose $B(\mathfrak{G})$ is of type ($\mathrm{NR_I}$) or ($\mathrm{NR_{II}}$) or (NN).

Choose a general point Q of $\operatorname{Tan} X$. We may assume that there is a point $P\in\operatorname{Reg}^0(X)$ with $Q\in T_P(X)$, where $\operatorname{Reg}^0(X)=\{P\in\operatorname{Reg} X\,|\,\mu_i(P)=b_i\ (0\leqq\forall i\leqq 3)\}$.

Let G_0 be a hyperplane section of X such that $\operatorname{Supp} G_0\not\ni P, Q$ and let $\tilde{G}_0$ be the divisor on $\tilde{X}$ corresponding to G_0 (cf. § 2).

Choose $x_0, x_1, x_2, x_3\in L(\mathfrak{G};\tilde{G}_0)$ such that $x_0=1$, $x_1=t$ is a local parameter at P and

$$(6)\qquad \begin{cases} x_2 = t^q+\sum_{i>q}\alpha_i t^i \\ x_3 = t^{b_3}+\sum_{i>b_3}\beta_i t^i \end{cases}$$

in $\hat{\mathcal{O}}_{X,P}=k[[t]]$ (cf. 1.3 and 1.5). The system of coordinates of $\mathbf{P}^3$ corresponding to $x_0, \dots, x_3$ via a natural isomorphism $L(\mathfrak{G}; \tilde{G}_0)\cong H^0(\mathbf{P}^3, \mathcal{O}(1))$ (cf. 2.2) will be denoted $X_0, \dots, X_3$.

Lemma 4.0. *In the expression* (6), *if* $\alpha_i\neq 0$ *or* $\beta_i\neq 0$, *then* $i\equiv 0$ *or* $1 \mod q$.

Proof. Let i be a positive integer with $i\not\equiv 0, 1 \mod q$. Hence we may write as $i=aq+r$ with $2\leqq r<q$. Letting $D_t^{(r)}$ operate on x_2, we have

$$D_t^{(r)} x_2 = \dots + \binom{aq+r}{r} \alpha_i t^{aq} + \dots$$

in $\hat{\mathcal{O}}_{X,P}=k[[t]]$. Since $D_t^{(r)} x_2=0$ (by 2.5) and $\binom{aq+r}{r}\equiv 1 \mod p$, we have $\alpha_i=0$. Similarly, we have $\beta_i=0$ if $i\not\equiv 0, 1 \mod q$. □

Choose an open subset V of X such that

(a) $P\in V\subset \mathrm{Reg}^0(X)$,

(b) $t_{|V}\colon V\to t(V)(\subset \mathbf{P}^1)$ is an étale covering,

(c) x_1, x_2, x_3 are regular on V.

Then the morphism

$$\psi = \psi_P\colon V\times\mathbf{A}^1 \to \mathrm{Tan}\, X\cap\{X_0\neq 0\} \subset \mathbf{A}^3_{(X_1/X_0, X_2/X_0, X_3/X_0)} \subset \mathbf{P}^3 \tag{7}$$

$$(\eta, c) \mapsto \vec{x}(\eta)+y(c)D_t^{(1)}\vec{x}(\eta)$$

is well-defined and generically surjective, where $\vec{x}=(x_1, x_2, x_3)$ and y is a coordinate function of $\mathbf{A}^1$ (cf. [5; § 2]).

Since $X_0(Q)\neq 0$ and $Q\in T_P(X)$, there is a point $c\in\mathbf{A}^1$ such that $Q=\psi(P, c)=(c, 0, 0)$.

Put $s=y-c\in\mathcal{O}_{\mathbf{A}^1,c}$. Then we have

$$\hat{\mathcal{O}}_{V\times\mathbf{A}^1,(P,c)} = k[[s, t]].$$

Let us consider the functions on Tan X

$$z_1 = \left(\frac{X_1}{X_0}-c\right)\Big|\, \mathrm{Tan}\, X$$

$$z_2 = \frac{X_2}{X_0}\,\Big|\, \mathrm{Tan}\, X$$

$$z_3 = \frac{X_3}{X_0}\,\Big|\, \mathrm{Tan}\, X.$$

Since the maximal ideal of $\mathcal{O}_{\mathbf{P}^3,Q}$ is generated by $\frac{X_1}{X_0}-c$, $\frac{X_2}{X_0}$, $\frac{X_3}{X_0}$, that of $\mathcal{O}_{\mathrm{Tan}\,X,Q}$ is generated by z_1, z_2, z_3.

Lemma 4.1. *Let*

$$\psi^*\colon \hat{\mathcal{O}}_{\mathrm{Tan}\,X,Q} \to \hat{\mathcal{O}}_{V\times\mathbf{A}^1,(P,c)} = k[[s,t]]$$

be the homomorphism induced by ψ_P. *Then*

$$\psi^* z_1 = t+s$$

$$\psi^* z_2 = t^q+\textstyle\sum_{k\geqq 2}\alpha_{kq}t^{kq}+\sum_{k\geqq 1}\alpha_{kq+1}(t+s+c)t^{kq}$$

$$\psi^* z_3 = \begin{cases} (t+s+c)\,t^q+\sum_{k\geqq 2}\beta_{kq}t^{kq}+\sum_{k\geqq 2}\beta_{kq+1}(t+s+c)\,t^{kq} \\ \qquad\qquad \textit{if } B(\mathfrak{G}) \textit{ is of type } (\mathrm{NR_I}) \\ t^{2q}+\sum_{k\geqq 3}\beta_{kq}t^{kq}+\sum_{k\geqq 2}\beta_{kq+1}(t+s+c)\,t^{kq} \\ \qquad\qquad \textit{if } B(\mathfrak{G}) \textit{ is of type } (\mathrm{NR_{II}}) \\ t^{q'q}+\sum_{k>q'}\beta_{kq}t^{kq}+\sum_{k\geqq q'}\beta_{kq+1}(t+s+c)\,t^{kq} \\ \qquad\qquad \textit{if } B(\mathfrak{G}) \textit{ is of type } (\mathrm{NN}). \end{cases}$$

Proof. By definitions of z_1, z_2, z_3 and ψ, we have

$$\psi^* z_1 = x_1+yD_t^{(1)}x_1-c$$

$$\psi^* z_2 = x_2+yD_t^{(1)}x_2$$

$$\psi^* z_3 = x_3+yD_t^{(1)}x_3.$$

Using the expression (6) and lemma 4.0, we get the expression of $\psi^* z_1$, $\psi^* z_2$ and $\psi^* z_3$ as above. ∎

Put

$$\text{(8)}\qquad \begin{cases} u = t+s \\ v = t^q. \end{cases}$$

Then the expressions of $\psi^* z_1$, $\psi^* z_2$, $\psi^* z_3$ may be rewritten as:

$$\text{(9)}\quad \begin{cases} \psi^* z_1 = u \\ \psi^* z_2 = (1+\alpha_{q+1}c)v+(\alpha_{2q}+\alpha_{2q+1}c)v^2+\alpha_{q+1}uv+(\text{higher order terms on } u \;\&\; v) \\ \psi^* z_3 = \begin{cases} cv+(\beta_{2q}+\beta_{2q+1})v^2+uv+(\text{higher order terms on } u \;\&\; v), \\ \qquad\qquad \text{if } B(\mathfrak{G}) \text{ is of type } (\mathrm{NR_I}) \\ (1+\beta_{2q+1}c)v^2+(\text{higher order terms on } u \;\&\; v), \\ \qquad\qquad \text{if } B(\mathfrak{G}) \text{ is of type } (\mathrm{NR_{II}}) \\ v^{q'}+(\text{higher order terms on } u \;\&\; v), \\ \qquad\qquad \text{if } B(\mathfrak{G}) \text{ is of type } (\mathrm{NN}). \end{cases} \end{cases}$$

Lemma 4.2. *If Q was chosen as sufficiently general, then $\hat{\mathcal{O}}_{\mathrm{Tan}\,X,Q} \simeq k[[u,v]]$ via ψ^*.*

Proof. There is the tower of rings;

$$k[[s,t]] \supset k[[u,v]] \supset \psi^* \hat{\mathcal{O}}_{\mathrm{Tan}\,X,Q} \supset k[[\psi^* z_1, \psi^* z_2]].$$

Since Q is sufficiently general, we may assume that $1+\alpha_{q+1}c \neq 0$. Hence, in the expression (9), the linear terms of $\psi^* z_1$ and $\psi^* z_2$ are linearly independent over k. Hence we have $k[[u,v]] = k[[\psi^* z_1, \psi^* z_2]]$ (see [14; VII cor. 2 to lemma 2]). Hence ψ^*: $\hat{\mathcal{O}}_{\mathrm{Tan}\,X,Q} \to k[[u,v]]$ is surjective. Since both sides are formal power series rings over k of two variables, this is an isomorphism. ∎

From now on, we assume that Q was chosen as sufficiently general and identify $\hat{\mathcal{O}}_{\mathrm{Tan}\,X,Q}$ with $k[[u,v]]$ via ψ^*.

The following lemma is elementary.

Lemma 4.3. *Let Y be a surface in $\mathbf{A}^3_{(Z_1,Z_2,Z_3)}$ with a smooth point at the origin O and $\{u,v\}$ a system of local parameters of $\hat{\mathcal{O}}_{Y,O}$. Let z_1, z_2, z_3 be images in $\hat{\mathcal{O}}_{Y,O}$ of coordinate functions Z_1, Z_2, Z_3, respectively and let*

$$z_1 = p_1(u,v)$$

$$z_2 = p_2(u,v)$$

$$z_3 = p_3(u,v)$$

in $k[[u,v]] = \hat{\mathcal{O}}_{Y,O}$.

If $h(Z_1,Z_2,Z_3)=0$ is an equation of the tangent plane to Y at O, then we have

$$h(z_1,z_2,z_3) = \gamma \begin{vmatrix} z_1 & z_2 & z_3 \\ \dfrac{\partial p_1}{\partial u}(0) & \dfrac{\partial p_2}{\partial u}(0) & \dfrac{\partial p_3}{\partial u}(0) \\ \dfrac{\partial p_1}{\partial v}(0) & \dfrac{\partial p_2}{\partial v}(0) & \dfrac{\partial p_3}{\partial v}(0) \end{vmatrix}$$

in $\hat{\mathcal{O}}_{Y,O} = k[[u,v]]$, where $\gamma \in k^\times$.

Let us return to our proof.

Let $h(Z_1,Z_2,Z_3)=0$ be an equation of the tangent plane to $\mathrm{Tan}\,X$ at Q.

Using (9), we have

$$\frac{\partial z_1}{\partial u}(0) = 1, \quad \frac{\partial z_1}{\partial v}(0) = 0,$$

$$\frac{\partial z_2}{\partial u}(0) = 0, \quad \frac{\partial z_2}{\partial v}(0) = 1+\alpha_{q+1}c,$$

$$\frac{\partial z_3}{\partial u}(0) = 0, \quad \frac{\partial z_3}{\partial v}(0) = \begin{cases} c & \text{if } B(\mathfrak{G}) \text{ is of type } (\mathrm{NR_I}) \\ 0 & \text{otherwise.} \end{cases}$$

Hence, by lemma 4.3, we may assume that

$$h(z_1, z_2, z_3) = \begin{cases} -cz_2+(1+\alpha_{q+1}c)z_3 & \text{if } B(\mathfrak{G}) \text{ is of type } (\mathrm{NR}_I) \\ z_3 & \text{otherwise.} \end{cases}$$

Now, we compute the Hessian rank of Tan X.

Put

$$H = \begin{pmatrix} \dfrac{\partial^2 h}{\partial u^2}(0) & \dfrac{\partial^2 h}{\partial u\,\partial v}(0) \\ \dfrac{\partial^2 h}{\partial u\,\partial v}(0) & \dfrac{\partial^2 h}{\partial v^2}(0) \end{pmatrix}.$$

Case 1. $B(\mathfrak{G})$ is of type ($\mathrm{NR_I}$).

Since

$$h(z) = \{(1+\alpha_{q+1}c)(\beta_{2q}+\beta_{2q+1})-c(\alpha_{2q}+\alpha_{2q+1}c)\}v^2 + uv + (\text{higher order terms on } u \ \& \ v),$$

we have $H=\begin{pmatrix} 0 & 1 \\ 1 & * \end{pmatrix}$. Therefore $h_{\mathrm{Tan}\,X}=2$.

Case 2. $B(\mathfrak{G})$ is of type ($\mathrm{NR_{II}}$).

In this case,

$$h(z) = (1+\beta_{2q}c)v^2 + (\text{higher order terms on } u \ \& \ v).$$

Hence we have $H=\begin{pmatrix} 0 & 0 \\ 0 & 2(1+\beta_{2q}c) \end{pmatrix}$. Since Q is general and $p>2$, we have rank $H=1$, i.e., $h_{\mathrm{Tan}\,X}=1$.

Case 3. $B(\mathfrak{G})$ is of type (NN).

Since

$$h(z) = v^{q'} + (\text{higher order terms on } u \ \& \ v)$$

and q' is a power of p, we have rank $H=0$.

When the first case occur, Tan X is ordinary. Hence $\dim(\mathrm{Tan}\,X)^*=2$.

When one of the remaining two cases occur, the tangent planes to Tan X along the line $T_P(X)$ are constant, equal to the plane $X_3=0$. Therefore, a general fibre of q: Tan $X \to (\text{Tan } X)^*$ has positive dimension. This means $\dim(\text{Tan } X)^* \leqq 1$. Since Tan X is not a plane (because X is nondegenerate), $\dim(\text{Tan } X)^* \geqq 1$. Hence we have $\dim(\text{Tan } X)^*=1$. This completes the proof.

5. Miscellaneous remarks

The first remark is concerned with example 2.6.

Remark 5.1. Example 2.6 can be generalized as follows. Let

$$A = \{a_0 < a_1 < \ldots < a_N\}$$

be a set of nonnegative integers with $a_0=0$ and $a_1=1$. Let X be the image of

$$\mathbf{P}^1 \to \mathbf{P}^N$$

$$t \mapsto (t^{a_0} : t^{a_1} : \ldots : t^{a_N}).$$

Then the invariant $B(\mathfrak{G})$ of X coincides with A if and only if A has the following property: Let u, v be nonnegative integers with $u \underset{p}{\geq} v$. If $u \in A$, then $v \in A$.

This can be proved by using a characterization of $B(\mathfrak{G})$ similar to (2.0).

Remark 5.2. We give here examples of smooth curves in $\mathbf{P}^3$ whose invariants have the assigned type.

(a) The invariant $B(\mathfrak{G})$ of a smooth curve X in $\mathbf{P}^3$ with $\deg X<p$ is of type (RR).

(b) (Schmidt [12]) Let $p=5$ and X be the smooth model of the plane curve $y^5+y-x^3=0$, which is nonhyperelliptic of genus 4. Hence X can be embedded in $\mathbf{P}^3$ by means of the canonical linear system $\mathfrak{K}$. Then $B(\mathfrak{K})=\{0, 1, 2, 5\}$, which is of type (RN).

(c) Invariants $B(\mathfrak{G})$ of the curves described in [4] are of type (NR_I).

(d) (Komiya [7]) Let $p=2$ and X be the complete intersection of $Y_1Y_2-Y_0Y_3=0$ and $\lambda Y_0^3+Y_1^3+Y_2^3+Y_3^3=0$ in $\mathbf{P}^3$, where $\lambda \neq 0, 1$. This curve is smooth of genus 4 and the linear system $\mathfrak{G}$ of line sections is canonical. Then we have $B(\mathfrak{G})=\{0, 1, 2, 4\}$, which is of type (NN).

Recently, Hajime Kaji (private communication, April, 1989) gave an example of a smooth curve of type (NR_II).

Example 5.3. (Kaji) Assume that $p>3$ and $q=p^e$ $(e>0)$. Let g: $\mathbf{P}^1 \to \mathbf{P}^1 \times \mathbf{P}^1$ be the graph morphism of the Frobenius morphism of degree q. Consider the morphism

$$\varphi\colon \mathbf{P}^1\times\mathbf{P}^1 \to \mathbf{P}^3$$
$$(s:t)\times(u:v) \mapsto s(3u^2:2uv:v^2:0)+t(0:u^2:2uv:3v^2).$$

Let X be the image of $\varphi\circ g$. Then X is smooth of type ($\mathrm{NR_{II}}$).

Proof. Since $p_2\circ g\colon \mathbf{P}^1\to\mathbf{P}^1$ is purely inseparable, the Zariski tangent space $\mathcal{T}_{g(s:t)}(g(\mathbf{P}^1))$ coincides with $\mathcal{T}_{g(s:t)}(\mathbf{P}^1\times(s^q:t^q))$ in $\mathcal{T}_{g(s:t)}(\mathbf{P}^1\times\mathbf{P}^1)$. Since the morphism $\varphi_{|\mathbf{P}^1\times(u:v)}\colon \mathbf{P}^1\times(u:v)\to\mathbf{P}^3$ is an embedding, $d(\varphi\circ g)_{(s:t)}\neq 0$.

On the other hand, if $(u_1:v_1)\neq(u_2:v_2)$, then $\varphi(\mathbf{P}^1\times(u_1:v_1))\cap\varphi(\mathbf{P}^1\times(u_2:v_2))=\emptyset$. To prove this, we consider the twisted cubic

$$\psi\colon \mathbf{P}^1\ni(u:v)\mapsto(u^3:u^2v:uv^2:v^3)\in\mathbf{P}^3.$$

Then we have $T_{\psi(u:v)}(C)=\varphi(\mathbf{P}^1\times(u:v))$, where $C=\psi(\mathbf{P}^1)$. If $T_{\psi(u_1:v_1)}(C)\cap T_{\psi(u_2:v_2)}(C)\neq\emptyset$, then there is a plane $H\subset\mathbf{P}^3$ with $H\supset T_{\psi(u_i:v_i)}(C)$ $(i=1,2)$. Then we have $(H.C)\geqq 4$, which is a contradiction.

In particular, the morphism $\varphi\circ g$ is injective. Hence $\varphi\circ g$ is an embedding.

From the arguments of the previous paragraphs, we have $T_{\varphi\circ g(s:t)}(X)=\varphi(\mathbf{P}^1\times(s^q:t^q))$. Hence we have $\operatorname{Tan} X=\varphi(\mathbf{P}^1\times\mathbf{P}^1)$. Since $\varphi(\mathbf{P}^1\times\mathbf{P}^1)=\operatorname{Tan} C$ and C is of type (RR), $\varphi(\mathbf{P}^1\times\mathbf{P}^1)$ is reflexive and its dual is of dimension 1 (cf. [5; (4.2)]). Therefore $\operatorname{Tan} X$ is semiordinary.

Since for a general point $(s:t)\in\mathbf{P}^1$

$$i(X\cdot T_{\varphi\circ g(s:t)}(X);\ \varphi\circ g(s:t)) = i(g(\mathbf{P}^1)\cdot\mathbf{P}^1\times(s^q:t^q);\ g(s:t)) = \deg p_2\circ g = q,$$

X is nonreflexive. This completes the proof.

Concerning the example in (5.2.c), the referee posed the following problem.

Problem 5.4. Is the tangent surface of a (smooth) rational curve of type ($\mathrm{NR_I}$) always a quadric surface?

Remark 5.5. It is easy to show that if $p>2$ and if X is a nonreflexive smooth curve on a smooth quadric surface, then X is one of the curves discribed in (5.2.c).

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