On the rate of convergence of certain summability methods for Fourier integrals of L^2-functions

D. Müller and Wang Kun-yang

1. Introduction

Suppose B is a bounded convex symmetric body in $\mathbf{R}^n$ and let $|\cdot|$ be the Minkowski norm associated with B, i.e.

$$|x| = \inf\{t > 0 : t^{-1}x \in B\}, \quad x \in \mathbf{R}^n.$$

Let $m \in L^\infty(0, \infty)$. Denote by m_t the function

$$m_t(s) = m(ts) \quad (t > 0,\ s > 0).$$

We define operators T_{m_t} $(t>0)$ on $L^2(\mathbf{R}^n)$ by

$$(T_{m_t} f)\hat{\ }(\xi) = m_t(|\xi|)\hat{f}(\xi).$$

If $m \in L^\infty(0, \infty)$ and $0 < \alpha \leqq 1$, the fractional integral of order α of m is defined as in [5] (see also [6]). That is, we set

$$(1) \qquad I_\omega^\alpha(m)(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_t^\omega (s-t)^{\alpha-1} m(s)\, ds, & \text{if } 0 < t < \omega, \\ 0 & \text{if } t \geqq \omega \quad ; \end{cases}$$

and, if $0 < \alpha < 1$ and $I_\omega^{1-\alpha}(m)$ is locally absolutely continuous for every $\omega > 0$, we define the fractional derivative $m^{(\alpha)}$ by

$$(2) \qquad m^{(\alpha)}(t) = \lim_{\omega \to \infty} \left(-\frac{d}{dt} I_\omega^{1-\alpha} m(t)\right).$$

Moreover, by induction over the integer part $[\alpha]$ of α, we define for arbitrary $\alpha > 0$

$$(2') \qquad m^{(\alpha)}(t) = -\frac{d}{dt} m^{(\alpha-1)}(t),$$

provided this makes sense, i.e. that $I_\omega^{1-\delta}$, $m^{(\delta)}, \dots, m^{(\alpha-1)}$ are absolutely continuous, where $\delta=\alpha-[\alpha]$.

Notice that for m with compact support in $\mathbf{R}^+$

(3) $$(m^{(\alpha)})^\wedge(\tau) = (-i\tau)^\alpha \hat{m}(\tau),$$

where $(-i\tau)^\alpha$ is defined by the principal branch.

We will consider the *localized Riemann—Liouville spaces* $RL(2,\alpha)$ which are defined (cf. [3]) by

$$RL(2,\alpha) = \{m\in L^\infty(0,\infty)\colon \|m\|_{RL(2,\alpha)} < \infty\}, \quad \text{if} \quad \alpha > \tfrac{1}{2},$$

where

$$\|m\|_{RL(2,\alpha)} = \sup_{t>0} \|(\chi m_t)^{(\alpha)}\|_2.$$

Here $\chi\in C_0^\infty(0,\infty)$ is an arbitrary fixed non-negative and non-trivial bump function. It is known [3] that the space $RL(2,\alpha)$ does not depend on the choice of χ. For convenience we will choose χ such that

(4) $$\chi\in C_0^\infty(0,\infty),\ \chi(t)\geqq 0,\ \operatorname{supp}\chi\subset\left[\frac{1}{2},1\right], \quad \text{and} \quad \chi(t)=1 \quad \text{for} \quad \frac{5}{8}<t<\frac{7}{8}.$$

We will also consider the space of functions of *weak bounded variation* $WBV_{q,\alpha}$ in the case $q=2$ and $\alpha>0$. By definition (see [7]) $WBV_{2,\alpha}$ is the space of all $m\in L^\infty\cap C(0,\infty)$ for which $m^{(\alpha)}$ exists in the sense of (2′) and whose norm

(5) $$\|m\|_{2,\alpha} = \|m\|_\infty + \sup_{k\in\mathbf{Z}} \left\{\int_{2^{k-1}}^{2^k} |t^\alpha m^{(\alpha)}(t)|^2 \frac{dt}{t}\right\}^{1/2}$$

is finite. From [3], Theorem 2, we know that for $\alpha>1/2$

(6) $$RL(2,\alpha) = WBV_{2,\alpha},$$

with equivalent norms.

Remark. If m is supported in a compact interval $[a,b]$, $0<a<b<\infty$, then for $x<a/2$

$$m^{(\alpha)}(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_a^b (t-x)^{-\alpha-1} m(t)\,dt,$$

hence

$$|m^{(\alpha)}(x)| \leqq C_\alpha \|m\|_\infty |(b-x)^{-\alpha}-(a-x)^{-\alpha}|$$

and

$$\left(\int_{-\infty}^{a/2} |m^{(\alpha)}(x)|^2 dx\right)^{1/2} \leqq C'_\alpha \|m\|_\infty.$$

These estimates easily imply that for $0<\alpha<1$, there exist constants c, $C>0$, depending only on α and a, b, such that

(7) $$c\|m\|_{2,\alpha} \leqq \|m\|_\infty + \|m^{(\alpha)}\|_2 \leqq C\|m\|_{2,\alpha}.$$

Moreover, if $\alpha > 1/2$, by (6) we also have

$$c\|m\|_{RL(2,\alpha)} \leqq \|m^{(\alpha)}\|_2 \leqq C\|m\|_{RL(2,\alpha)}. \tag{8}$$

We denote by φ a function on $[0, \infty]$ which is non-decreasing and satisfies the following condition:

$$1 \leqq \varphi(2t) \leqq \lambda\varphi(t) \quad \text{for some} \quad \lambda \geqq 1. \tag{9}$$

We write throughout this paper

$$\mu = \log_2 \lambda, \tag{10}$$

where λ is smallest possible to satisfy (9).

It is easy to see that

$$\begin{cases} \varphi(t) \leqq \varphi(2)t^{\mu} & \text{if} \quad t \geqq 1, \\ \varphi(st) \geqq \dfrac{1}{\lambda} s^{\mu}\varphi(t) & \text{if} \quad t \geqq 0 \quad \text{and} \quad 0 \leqq s \leqq 1. \end{cases} \tag{11}$$

Corresponding to φ we define

$$\psi(t) = \varphi(1) + \left\{\int_1^{t+1} \frac{\varphi^2(s)}{s}\, ds\right\}^{1/2}.$$

Then ψ is also non-decreasing and satisfies (9), possibly with a different λ, and

$$\psi(t) \geqq \frac{1}{\lambda}\sqrt{\log 2}\, \varphi(t).$$

Given φ, we define the space L^2_φ by

$$L^2_\varphi = \{f \in L^2(\mathbf{R}^n) \colon \|f\|_{L^2_\varphi} < \infty\},$$

where

$$\|f\|_{L^2_\varphi} = \left\{\int_{\mathbf{R}^n} |\hat{f}(\xi)|^2 |\varphi(|\xi|)|^2\, d\xi\right\}^{1/2}.$$

We shall prove the following results:

Theorem 1. *Suppose* $\alpha > 1/2$, $m \in RL(2, \alpha)$. *If for some* $\beta > \mu$ *and* $\gamma > 1$

$$\|(\chi m_t)^{(\alpha)}\|_2 = O(t^{\beta}) \quad \text{as} \quad t \to 0^+, \tag{12}$$

$$\|(\chi m_t)^{(\alpha)}\|_2 = O\big((\log t)^{-\gamma}\big) \quad \text{as} \quad t \to \infty, \tag{13}$$

where χ *is a bump function as in* (4), *then*

$$\left\|\sup_{t>0} |T_{m_t} f|\, \varphi\left(\frac{1}{t}\right)\right\|_2 \leqq c\|f\|_{L^2_\varphi}.$$

Theorem 2. *Suppose* $\alpha>1/2$, $m\in RL(2,\alpha)$. *If for some* $\beta>\mu$ (12) *holds, then*

$$\left\| \sup_{0<t<1} |T_{m_t} f|\, \varphi\left(\frac{1}{t}\right)\right\|_2 \leqq c\|f\|_{L^2_\psi}.$$

Theorem 3. *Suppose* $m\in WBV_{2,\alpha}$ *for all* $0<\alpha<\frac{1}{2}$ *and* $\operatorname{supp} m\subset[\frac{1}{2},1]$. *If*

$$\|m\|^2_{2,1/2-\varepsilon} = O\left(\frac{1}{\varepsilon}\right) \quad as \quad \varepsilon\to 0^+, \tag{14}$$

which holds in particular if m is of bounded variation, then

$$\left\| \sup_{0<t<1} \frac{1}{\log\left(\frac{1}{t}+1\right)}\, \varphi\left(\frac{1}{t}\right) |T_{m_t} f| \right\|_{L^2+L^\infty} \leqq c\|f\|_{L^2_\varphi},$$

where

$$\|f\|_{L^2+L^\infty} = \inf\{\|g\|_2+\|h\|_\infty : f=g+h\}.$$

A corollary of Theorem 2 includes the result of Chen Tian-ping [4] on generalized Bochner—Riesz means of positive order.

We remark that Theorem 1 has some overlapping with the results in [6], in particular with Theorem 1 and Theorem 4 in that paper. However, in [6] Dappa and Trebels are concerned with L^p-estimates for maximal operators under no smoothness condition whatsoever on the function f (but in the more general context of quasi-radial multipliers), whereas we want to concentrate in this article on the rate of convergence of $T_{m_t} f$ as $t\to 0^+$, given f has a certain degree of smoothness, measured by some L^2_φ-norm of f. Our main result is in fact Theorem 3, which deals with the critical index of smoothness $\alpha=1/2$ for m.

As to the L^p-case, let us also mention some results due to Carbery in order to give a slightly more complete picture of what is known on the subject.

Define D^s by

$$(D^s f)^\wedge(\xi) = \|\xi\|^s \hat{f}(\xi),$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbf{R}^n$. We introduce the global Bessel potential space $L^2_\alpha = L^2_\alpha(\mathbf{R}^+)$ as in [1]: L^2_α is the completion of the C^∞ functions of compact support in $(0,\infty)$ under the norm

$$\|m\|_{L^2_\alpha} = \left\{\int_0^\infty \left| s^{\alpha+1}\left(\frac{d}{ds}\right)^\alpha \left(\frac{m(s)}{s}\right)\right|^2 \frac{ds}{s}\right\}^{1/2}.$$

Theorem (Carbery). *Let* $|\cdot|=\|\cdot\|$ *be the Euclidean norm. If* $\alpha>n\left(\frac{1}{p}-\frac{1}{2}\right)+\frac{1}{2}$

for $1<p\leqq 2$, *or* $\alpha>n\left(\frac{1}{2}-\frac{1}{p}\right)+\frac{1}{p}$ *for* $2\leqq p<\infty$, *then*

$$\left\|\sup_{t>0} t^{-s}|T_{m_t}f|\right\|_{L^p(\mathbf{R}^n)} \leqq c_\alpha \| |\cdot|^{-s} m(\cdot)\|_{L^2_\alpha} \|D^s f\|_{L^p(\mathbf{R}^n)}.$$

Furthermore, if $n=1$ *or* $n=2$, *the above estimate even holds if* $2\leqq p<\infty$ *and* $\alpha>\max\left(1/2,\ n\left(\frac{1}{2}-\frac{1}{p}\right)\right)$.

Of course, Carbery's theorem implies results on the pointwise convergence of Bochner—Riesz means of L^p functions, which will be stated later as a remark.

Throughout this paper c will denote a constant which can take different values from statement to statement.

2. Auxiliary results

Lemma 1. *Let* $0<\alpha<1$. *If* $m\in WBV_{2,\alpha}$, *then there exists a set* $E\subset(0,\infty)$ *of one-dimensional measure zero such that for any* $\beta>1$ *and every* $u\in(0,\infty)\setminus E$

$$m(u) = \frac{1}{\Gamma(\alpha)}\int_u^{\beta u}(s-u)^{\alpha-1}m^{(\alpha)}(s)\,ds+\frac{[(\beta-1)u]^\alpha}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_{\beta u}^\infty \frac{m(s)}{(s-\beta u)^\alpha(s-u)}\,ds. \tag{15}$$

Proof. We first assume that m vanishes on (a,∞) for some $0<a<\infty$. Then we have for $u\in(0,a)$

$$\int_u^a\left[\int_t^a (s-t)^{\alpha-1}m^{(\alpha)}(s)\,ds\right]dt = \int_u^a\left[\int_u^s (s-t)^{\alpha-1}\,dt\right] m^{(\alpha)}(s)\,ds$$

$$= \frac{1}{\alpha}\int_u^a (s-u)^\alpha m^{(\alpha)}(s)\,ds.$$

On the other hand, we have

$$m^{(\alpha)}(s) = -\frac{d}{ds} I_a^{1-\alpha}(m)(s), \quad s>0, \tag{16}$$

and $I_a^{1-\alpha}(m)$ is absolutely continuous on $[\varepsilon, a]$ for every $0<\varepsilon<a$. So, by plugging (16) into $\int_u^a (s-u)^\alpha m^{(\alpha)}(s)\,ds$ and integrating by parts, one obtains after some routine calculations

$$\frac{1}{\alpha}\int_u^a (s-u)^\alpha m^{(\alpha)}(s)\,ds = \Gamma(\alpha)\int_u^a m(t)\,dt.$$

By comparison with the previous formula, we see that for almost every $t>0$

$$m(t) = \frac{1}{\Gamma(\alpha)} \int_t^\infty (s-t)^{\alpha-1} m^{(\alpha)}(s)\, ds. \tag{17}$$

(Compare also [1], [7]).

Moreover, by partial integration we get from (16)

$$\int_{\beta t}^\infty (s-t)^{\alpha-1} m^{(\alpha)}(s)\, ds = [(\beta-1)t]^{\alpha-1} I_a^{1-\alpha}(m)(\beta t)$$

$$+\frac{-1}{\Gamma(1-\alpha)} \int_{\beta t}^\infty (s-t)^{\alpha-2} \int_s^\infty (u-s)^{-\alpha} m(u)\, du\, ds = \frac{[(\beta-1)t]^\alpha}{\Gamma(1-\alpha)} \int_{\beta t}^\infty \frac{m(s)\, ds}{(s-\beta t)^\alpha (s-t)}.$$

So we have proved (15) if $m(t)$ vanishes for t sufficiently large.

For general m we define m_N, $N\in\mathbf{N}$, by

$$m_N(t) = \begin{cases} m(t), & \text{if } \ 0 \leqq t \leqq N, \\ 0, & \text{if } \ t > N, \end{cases}$$

and let $m_\infty = m$. Define

$$a_N(t) = \frac{1}{\Gamma(\alpha)} \int_t^{\beta t} (s-t)^{\alpha-1} m_N^{(\alpha)}(s)\, ds,$$

$$b_N(t) = \frac{[(\beta-1)t]^\alpha}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_{\beta t}^\infty \frac{m_N(s)\, ds}{(s-\beta t)^\alpha (s-t)},$$

with $N\in\mathbf{N}$ or $N=\infty$. Then we have

$$a_\infty(t) - a_N(t) = \frac{1}{\Gamma(\alpha)} \int_t^{\beta t} (s-t)^{\alpha-1} (m-m_N)^{(\alpha)}(s)\, ds.$$

By definition

$$(m-m_N)^{(\alpha)}(s) = \lim_{\omega\to\infty} -\frac{d}{ds} \frac{1}{\Gamma(1-\alpha)} \int_s^\omega (u-s)^{-\alpha} (m-m_N)(u)\, du.$$

For any fixed $s>0$ with $N>s$ this implies

$$(m-m_N)^{(\alpha)}(s) = \frac{\alpha}{\Gamma(1-\alpha)} \int_N^\infty (u-s)^{-\alpha-1} m(u)\, du,$$

so that

$$|(m-m_N)^{(\alpha)}(s)| \leqq c\|m\|_\infty \frac{1}{(N-s)^\alpha}, \quad \text{if} \quad N > s.$$

Therefore

$$|a_\infty(t) - a_N(t)| \leqq c\|m\|_\infty \int_t^{\beta t} (s-t)^{\alpha-1} \frac{ds}{(N-s)^\alpha} \leqq c\|m\|_\infty \frac{[(\beta-1)t]^\alpha}{(N-\beta t)^\alpha}, \quad \text{if} \quad N > \beta t.$$

On the other hand, if $N>\beta t$, we also have

$$|b_\infty(t)-b_N(t)| \leqq c[(\beta-1)t]^\alpha \int_N^\infty \frac{|m(s)|}{(s-\beta t)(s-t)}\,ds \leqq c\|m\|_\infty \frac{[(\beta-1)t]^\alpha}{(N-\beta t)^\alpha},$$

since $s-t>s-\beta t$. We conclude that for every $t>0$

$$a_\infty(t)+b_\infty(t) = \lim_{N\to\infty} \big(a_N(t)+b_N(t)\big).$$

But we have proved that

$$m_N(t) = a_N(t)+b_N(t), \quad t\in(0,\infty)\setminus E_N,$$

where E_N is a set of one-dimensional measure zero. Let $E=\bigcup_{N=1}^\infty E_N$. We get

$$a_\infty(t)+b_\infty(t) = m(t), \quad \text{if} \quad t\in(0,\infty)\setminus E. \qquad \text{Q.E.D.}$$

Lemma 2. *If* $\alpha\in\left(\frac{1}{2},1\right)$ *and* $m\in RL(2,\alpha)$, *then*

$$\Big\|\sup_{1<s<2} |T_{m_s}f|\Big\|_2 \leqq c\|m\|_{RL(2,\alpha)}\|f\|_2.$$

Proof. Define operators P_s on $L^2(\mathbf{R}^n)$, with $s\in(1,3)$, by

$$(P_sf)^\wedge(\xi) = \left(\frac{d}{ds}\right)^\alpha m(s|\xi|)\hat{f}(\xi).$$

By Plancherel's theorem we have

$$\int_{\mathbf{R}^n}\int_1^3 |(P_sf)(x)|^2\,ds\,dx = \int_1^3\int_{\mathbf{R}^n} |(P_sf)^\wedge(\xi)|^2\,d\xi\,ds = \int_{\mathbf{R}^n} |\hat{f}(\xi)|^2 \int_1^3 |m_{|\xi|}^{(\alpha)}(s)|^2\,ds\,d\xi.$$

From (5) and (6) we see that for $t>0$

(18)

$$\int_1^3 |m_t^{(\alpha)}(s)|^2\,ds = \int_t^{3t} |t^\alpha m^{(\alpha)}(s)|^2 \frac{ds}{t} \leqq c\int_t^{3t} |s^\alpha m^{(\alpha)}(s)|^2 \frac{ds}{s} \leqq c\|m\|_{2,\alpha}^2 \leqq c\|m\|_{RL(2,\alpha)}^2.$$

This shows that P_s is well-defined for *a.e.* $s\in(1,3)$. Now for $s\in(1,2)$ we define two operators as follows:

$$A_sf = \frac{1}{\Gamma(\alpha)}\int_s^3 (u-s)^{\alpha-1}P_uf\,du,$$

$$B_sf = \frac{(3-s)^\alpha}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_3^\infty \frac{T_{m_u}f}{(u-3)^\alpha(u-s)}\,du.$$

Since $\alpha>1/2$ we have

$$\sup_{1<s<2} |A_sf| \leqq c\sup_{1<s<2} \Big\{\int_s^3 (u-s)^{2\alpha-2}\,du\Big\}^{1/2} \Big\{\int_1^3 |P_uf|^2\,du\Big\}^{1/2}.$$

So, using (18), we get

(19) $$\Big\|\sup_{1<s<2}|A_s f|\Big\|_2 \leqq c\|m\|_{RL(2,\alpha)}\|f\|_2.$$

On the other hand we have

(20) $$\Big\|\sup_{1<s<2}|B_s f|\Big\|_2 \leqq c\int_3^\infty \frac{\|T_{m_u}f\|_2}{(u-3)^\alpha(u-2)}\,du \leqq c\|m\|_\infty\|f\|_2.$$

For $\xi\neq 0$ and $s\in(1,2)$ one has

$$\widehat{A_s f}(\xi) = \frac{1}{\Gamma(\delta)}\int_{s|\xi|}^{3|\xi|}(t-s|\xi|)^{\alpha-1}m^{(\alpha)}(t)\,dt\,\hat f(\xi),$$

$$\widehat{B_s f}(\xi) = \frac{(3-s)^\alpha|\xi|^\alpha}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_{3|\xi|}^\infty \frac{m(t)\,dt}{(t-3|\xi|)^\delta(t-s|\xi|)}\hat f(\xi).$$

If we write $u=s|\xi|$ and $\beta=\dfrac{3}{s}>1$, then by Lemma 1, for every $u\in(0,\infty)\setminus E$, i.e. for each $|\xi|\in\dfrac{1}{s}[(0,\infty)\setminus E]$ and $s\in(1,2)$

$$(T_{m_s}f)\hat{}\,(\xi) = \widehat{A_s f}(\xi)+\widehat{B_s f}(\xi).$$

This shows that

$$T_{m_s} = A_s + B_s,$$

and so the estimate in Lemma 2 follows from (19) and (20).

Q.E.D.

Lemma 3. *Suppose* $\alpha\in(\frac{1}{2},1)$ *and* $m\in RL(2,\alpha)$. *If* $\operatorname{supp} m\subset[1,2]$, *then*

$$\left\|\sup_{t>0}|T_{m_t}f|\,\varphi\left(\frac{1}{t}\right)\right\|_2 \leqq c\|m^{(\alpha)}\|_2\|f\|_{L^2_\varphi}.$$

Proof. We have

$$\sup_{t>0}|T_{m_t}f|\,\varphi\left(\frac{1}{t}\right) \leqq c\Big\{\sum_{j=-\infty}^{\infty}\big[\sup_{1<t<2}|T_{m_{t2^j}}f|\,\varphi(2^{-j})\big]^2\Big\}^{1/2}.$$

For $j\in\mathbf{Z}$ define f_j by

(21) $$\hat f_j(\xi) = \begin{cases}\hat f(\xi), & \text{if } 2^{-j-1}<|\xi|<2^{-j+1},\\ 0\ , & \text{otherwise.}\end{cases}$$

Since $\operatorname{supp} m\subset[1,2]$ we see that

$$T_{m_{t2^j}}f = T_{m_{t2^j}}f_j \quad \text{for} \quad 1<t<2.$$

So, by Lemma 2 and (8) we get

$$\left\|\sup_{t>0}|T_{m_t}f|\varphi\left(\frac{1}{t}\right)\right\|_2^2 \leqq c\sum_{j=-\infty}^{\infty}[\|m_{2^j}\|_{RL(2,\alpha)}\|f_j\|_2\varphi(2^{-j})]^2 \leqq c\|m^{(\alpha)}\|_2^2\|f\|_{L^2_\varphi}^2.$$

Q.E.D.

3. Proof of the theorems

Since $RL(2,\beta)$ is continuously embedded in $RL(2,\alpha)$, if $\beta>\alpha$ (see [3]), we may assume without restriction in the proofs of Theorem 1 and Theorem 2 that $1/2<\alpha<1$.

Proof of Theorem 1. Choose $h\in C_0^\infty(\mathbf{R})$ such that

$$\operatorname{supp} h\subset\left[\frac{1}{2},2\right],\ \sum_{j=-\infty}^{\infty}h(2^jt)=1\quad\forall t>0.$$

Define

$$m_j(t)=m(t)h(2^jt)$$

and

$$(T_t^jf)^\wedge(\xi)=m_j(t|\xi|)\hat{f}(\xi),\quad f\in L^2(\mathbf{R}^n).$$

Then we have

$$m=\sum_{j=-\infty}^{\infty}m_j,\ T_{m_t}=\sum_{j=-\infty}^{\infty}T_t^j,$$

and

(22) $$\left\|\sup_{t>0}|T_{m_t}f|\varphi\left(\frac{1}{t}\right)\right\|_2\leqq\sum_{j=-\infty}^{\infty}\left\|\sup_{t>0}|T_t^jf|\varphi\left(\frac{1}{t}\right)\right\|_2.$$

We write

$$\tilde{m}_j(t)=m_{2^{-j}}(t)h(t)=m_j(2^{-j}t),$$

and

$$(\tilde{T}_t^jf)^\wedge(\xi)=\tilde{m}_j(t|\xi|)\hat{f}(\xi).$$

Then we see that

$$\sup_{t>0}|T_t^jf|\varphi\left(\frac{1}{t}\right)=\sup_{t>0}|\tilde{T}_t^jf|\varphi\left(\frac{2^j}{t}\right).$$

Moreover, by (9) and (10)

$$\varphi\left(\frac{2^j}{t}\right)\leqq 2^{\mu j}\varphi\left(\frac{1}{t}\right),\quad j\geqq 0.$$

So, by applying Lemma 3 to $\tilde{m}_j$ we get

$$\left\|\sup_{t>0}|T_t^jf|\varphi\left(\frac{1}{t}\right)\right\|_2\leqq c\,2^{\mu j}\|\tilde{m}_j^{(\alpha)}\|_2\|f\|_{L^2_\varphi}\quad(j\geqq 0).$$

By condition (12)

$$\|\tilde{m}_j^{(\alpha)}\|_2 \leqq c2^{-\beta j} \quad (j \geqq 0).$$

Hence

$$(23) \qquad \sum_{j=0}^{\infty} \left\| \sup_{t>0} |T_t^j f| \varphi\left(\frac{1}{t}\right) \right\|_2 \leqq c \sum_{j=0}^{\infty} 2^{-(\beta-\mu)j} \|f\|_{L_\varphi^2} \leqq c \|f\|_{L_\varphi^2}.$$

On the other hand, if $j<0$, then

$$\left\| \sup_{t>0} |T_t^j f| \varphi\left(\frac{1}{t}\right) \right\|_2 = \left\| \sup_{t>0} |\tilde{T}_t^j f| \varphi\left(\frac{2^j}{t}\right) \right\|_2$$

$$\leqq \left\| \sup_{t>0} |\tilde{T}_t^j f| \varphi\left(\frac{1}{t}\right) \right\|_2 \leqq c \|\tilde{m}_j^{(\alpha)}\|_2 \|f\|_{L_\varphi^2},$$

once again by Lemma 3. Condition (13) implies

$$\|\tilde{m}_j^{(\alpha)}\|_2 \leqq c(-j)^{-\gamma} \quad (j < 0),$$

hence

$$(24) \qquad \sum_{j=-\infty}^{-1} \left\| \sup_{t>0} |T_t^j f| \varphi\left(\frac{1}{t}\right) \right\|_2 \leqq c \sum_{j=1}^{\infty} j^{-\gamma} \|f\|_{L_\varphi^2} \leqq c \|f\|_{L_\varphi^2}.$$

The theorem now is an immediate consequence of (22), (23) and (24).

Q.E.D.

Proof of Theorem 2. By Theorem 1 we can assume supp $m \subset \left[\frac{1}{2}, \infty\right]$. We have

$$\left| \sup_{0<t<1} |T_{m_t} f| \varphi\left(\frac{1}{t}\right) \right|^2 \leqq c \sum_{k=0}^{\infty} \sup_{1<t<2} |T_{m_{t2^{-k-1}}} f|^2 \varphi^2(2^k).$$

Define f_k by

$$\hat{f}_k(\xi) = \begin{cases} \hat{f}(\xi), & \text{if } \ 2^{k-1} < |\xi| < 2^k, \\ 0 \quad , & \text{otherwise.} \end{cases}$$

Then for $t \in (1, 2)$

$$(T_{m_{t2^{-k-1}}} f)^\wedge(\xi) = m(t2^{-k-1}|\xi|) \sum_{j=k}^{\infty} \hat{f}_j(\xi),$$

i.e.

$$T_{m_{t2^{-k-1}}} f = T_{m_{t2^{-k-1}}} \left(\sum_{j=k}^{\infty} f_j\right).$$

By Lemma 2 we get

$$\left\| \sup_{1<t<2} |T_{m_{t2^{-k-1}}} f| \right\|_2^2 \leqq c \|m\|_{RL(2,\alpha)}^2 \left\| \sum_{j=k}^{\infty} f_j \right\|_2^2,$$

hence

$$\left\| \sup_{0<t<1} |T_{m_t} f| \varphi\left(\frac{1}{t}\right) \right\|_2^2 \leqq c \sum_{k=0}^{\infty} \|m\|_{RL(2,\alpha)}^2 \left\| \sum_{j=k}^{\infty} f_j \right\|_2^2 \varphi^2(2^k).$$

Since

$$\left\| \sum_{j=k}^{\infty} f_j \right\|_2^2 = \sum_{j=k}^{\infty} \|\hat{f}_j\|_2^2,$$

we obtain

$$\left\| \sup_{0<t<1} |T_{m_t} f| \varphi\left(\frac{1}{t}\right) \right\|_2^2 \leqq c \|m\|_{RL(2,\alpha)}^2 \sum_{j=0}^{\infty} \|\hat{f}_j\|_2^2 \cdot \sum_{k=0}^{j} \varphi^2(2^k).$$

Noticing that

$$\sum_{k=0}^{j} \varphi^2(2^k) \leqq c\psi^2(2^j),$$

the proof follows by another application of Plancherel's theorem.

Q.E.D.

Proof of Theorem 3. Let us first notice that if m is of bounded variation, then $\hat{m}(\tau) = O(|\tau|^{-1})$ as $|\tau| \to \infty$, which implies

$$\int_{-\infty}^{\infty} (|\hat{m}(\tau)| \, |\tau|^{1/2-\varepsilon})^2 \, dt = O\left(\frac{1}{\varepsilon}\right).$$

So, by (3), (7), m satisfies condition (14).

Now assume $m \in WBV_{2,\alpha}$ is such that (14) holds. Define operators $P_u = P_u^{k,\alpha}$ by

$$(P_u^{k,\alpha} f)^\wedge(\xi) = \left(\frac{d}{du}\right)^\alpha m\left(u \frac{|\xi|}{2^{k-2}}\right) \hat{f}(\xi), \quad u \in (1, 3).$$

By an argument similar to that in the proof of Lemma 2 we get

$$\left\| \left\{ \int_1^3 |P_u f|^2 \, du \right\}^{1/2} \right\|_2 \leqq c \|m\|_{2,\alpha} \|f\|_2. \tag{25}$$

And similarly we get for $s \in (1, 2)$

$$T_{m_t^k} f = A_t^{k,\alpha} f + B_t^{k,\alpha} f \quad \text{for every} \quad \alpha \in \left(0, \tfrac{1}{2}\right), \tag{26}$$

where

$$m_t^k(s) = m(st2^{-k+2}),$$

$$A_t^{k,\alpha} f = \frac{1}{\Gamma(\alpha)} \int_t^3 (u-t)^{\alpha-1} P_u^{k,\alpha} f \, du,$$

$$B_t^{k,\alpha} f = \frac{(3-t)^\alpha}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_3^\infty \frac{T_{m_u^k} f}{(u-3)^\alpha (u-t)} \, du.$$

Define f_j by $\hat{f}_j(\xi) = \hat{f}(\xi) \chi_{[2^{k-4}, 2^{k-2}]}(|\xi|)$ and

$$M(f)(x) = \sup_{0<t<1} \frac{1}{\log\left(\frac{1}{t} + 1\right)} \varphi\left(\frac{1}{t}\right) |T_{m_t} f(x)|.$$

We have

$$M(f)^2 \leqq c \sum_{k=3}^{\infty} \left(\frac{1}{k} \varphi(2^k) \sup_{1<t<2} |T_{m_t^k} f_k| \right)^2. \tag{27}$$

Take $\alpha=\alpha_k=\frac{1}{2}-\frac{1}{k}$, $k\geqq 3$, and split the integral of $A_t^{k,\alpha_k}f$ into the following two parts:

$$D_t^k f_k = \frac{1}{\Gamma(\alpha_k)}\int_t^{t+2^{-k(n+1)}} (u-t)^{\alpha_k-1}P_u^{k,\alpha_k}f_k\,du,$$

$$E_t^k f_k = \frac{1}{\Gamma(\alpha_k)}\int_{t+2^{-k(n+1)}}^{3^3} (u-t)^{\alpha_k-1}P_u^{k,\alpha_k}f_k\,du.$$

From (26), (27) we get

$$M(f)^2 \leqq c\sum_{k=3}^{\infty}\left(\frac{1}{k}\varphi(2^k)\right)^2 \sup_{1<t<2}\left(|B_t^{k,\alpha_k}f_k|^2+|D_t^kf_k|^2+|E_t^kf_k|^2\right),$$

hence

$$M(f) \leqq c\left\{\sum_{k=3}^{\infty}\left(\frac{1}{k}\varphi(2^k)\right)^2 \sup_{1<t<2}\left(|B_t^{k,\alpha_k}f_k|^2+|E_t^kf_k|^2\right)\right\}^{1/2} \tag{28}$$

$$+c\left\{\sum_{k=3}^{\infty}\left(\frac{1}{k}\varphi(2^k)\right)^2 \sup_{1<t<2}|D_t^kf_k|^2\right\}^{1/2}.$$

It is easy to see that

$$\left\|\sup_{1<t<2} B_t^{k,\alpha_k}f_k\right\|_2 \leqq c\|m\|_\infty\|f_k\|_2. \tag{29}$$

Next, for *a.e.* $u\in(1,3)$

$$P_u^{k,\alpha_k}f_k(x) = \int_{2^{k-4}<|\xi|<2^{k-2}}\left(\frac{d}{du}\right)^{\alpha_k} m\left(u\frac{|\xi|}{2^{k-2}}\right)\hat{f}_k(\xi)e^{ix\xi}\,d\xi.$$

So, by applying Cauchy—Schwarz' estimate and (14), we get the following uniform estimate:

$$|P_u^{k,\alpha_k}f_k(x)| \leqq c2^{k(n/2)}\left\{\int_{1/4}^3 |m^{(\alpha_k)}(s)|^2\,ds\right\}^{1/2}\|f_k\|_2 \leqq c\sqrt{k}\,2^{k(n/2)}\|f_k\|_2,$$

which implies

$$\left\|\sup_{1<t<2}|D_t^kf_k|\right\|_\infty \leqq c\sqrt{k}\,2^{-(k/2)}\|f_k\|_2. \tag{30}$$

Finally, estimating the integral in u defining $E_t^kf_k$ again by Cauchy—Schwarz, we obtain

$$|E_t^kf_k| \leqq c\sqrt{k}\left\{\int_1^3 |P_u^{k,\alpha_k}f_k|^2\,du\right\}^{1/2}\quad (1<t<2),$$

which, by (25) and (14), implies

$$\left\|\sup_{1<t<2}|E_t^kf_k|\right\|_2 \leqq ck\|f_k\|_2. \tag{31}$$

From (28)—(31) we conclude

$$\|M(f)\|_{L^2+L^\infty} \leqq c\left(\sum_{k=3}^{\infty}\left(\frac{1}{k}\varphi(2^k)\,k\|f_k\|_2\right)^2\right)^{1/2} \leqq c\|f\|_{L^2_\varphi}.$$

Q.E.D.

4. Applications

Now we can use the above estimates of maximal functions to get some results on almost everywhere convergence.

Theorem 4. *Assume m is a continuous function on $\mathbf{R}^+$, contained in $RL(2,\alpha)$ for some $\alpha>1/2$. If condition (12) holds for the multiplier $m-1$ with $\beta>\mu$, then for every $f\in L^2_\psi$*

$$T_{m_t}f(x)-f(x)=o\left(\frac{1}{\varphi\left(\frac{1}{t}\right)}\right)\quad \text{a.e.}\quad \text{as}\quad t\to 0^+.$$

Proof. Since

$$\|\chi(m_t-1)\|_\infty \leqq c\|[\chi(m_t-1)]^{(\alpha)}\|_2=O(t^\beta),$$

we have

$$m(t)-1=O(t^\beta)\quad \text{as}\quad t\to 0^+.$$

We know from (11) that

$$t^\mu\leqq c\left[\varphi\left(\frac{1}{t}\right)\right]^{-1},\quad 0<t<1.$$

Therefore

$$|m(t)-1|=O\left(\left[\varphi\left(\frac{1}{t}\right)\right]^{-1}\right)\quad \text{as}\quad t\to 0^+.$$

From this we conclude that the theorem is valid for those functions whose Fourier transforms belong to $C_0^\infty(\mathbf{R}^n)$. Since such functions are dense in L^2_ψ, the theorem will be proved provided

$$\left\|\sup_{0<t<1}|T_{(m-1)_t}f|\,\varphi\left(\frac{1}{t}\right)\right\|_2\leqq c\,\|f\|_{L^2_\psi}.$$

But this is a direct consequence of Theorem 2 applied to the multiplier $m-1$.

Q.E.D.

As a corollary of Theorem 4 we get the following result on Bochner—Riesz means of positive order, which includes the result of Chen Tian-ping [4], who only deals with the case where $|\cdot|$ is the Euclidean norm.

Corollary 1. *Let $\alpha>0$, $\ell>0$, $m(t)=(1-t^\ell)_+^\alpha$. If $f\in L^2(\mathbf{R}^n)$ satisfies the condition*

(32) $$\int_{\mathbf{R}^n}|\hat f(x)|^2|x|^{2\mu}\,dx<\infty,\quad \mu>0,$$

then for a.e. $x\in\mathbf{R}^n$

(33) $$T_{m_t}f(x)-f(x)=\begin{cases}o(t^\mu), & \text{if}\quad \ell>\mu,\\ O(t^\mu), & \text{if}\quad \ell=\mu,\end{cases}$$

as $t\to 0^+$.

Proof. First, one easily estimates

$$\|[\chi(m_t-1)]'\|_2 = O(t^\ell) \quad \text{as} \quad t \to 0^+.$$

Since $RL(2,1)\subset RL(2,\delta)$, if $1/2<\delta<1$, we get

$$\|[\chi(m_t-1)]^{(\delta)}\|_2 = O(t^\ell) \quad \text{as} \quad t \to 0^+,$$

for every $1/2<\delta<1$. Moreover, one checks easily that $m\in RL(2,\delta)$ whenever $\frac{1}{2}<\delta<\frac{1}{2}+\alpha$. Notice that the condition $\delta<\frac{1}{2}+\alpha$ is forced by the singularity of m at $t=1$.

So, if $\ell>\mu$, the required result is a direct consequence of Theorem 4 with $\varphi(t)=1+t^\mu$.

Now assume $\ell=\mu$, $\varphi(t)=1+t^\mu$. Choose a function $h\in C^\infty(0,\infty)$ such that

$$\begin{cases} h(t)=1, & \text{if} \quad 0\leqq t\leqq \frac{1}{2} \\ h(t)=0, & \text{if} \quad t>3/4. \end{cases} \tag{34}$$

If we define

$$\tilde{m}(t) = m(t)+\alpha t^\ell h(t),$$

then $\tilde{m}\in RL(2,\delta)$ for $\delta\in(\frac{1}{2},\frac{1}{2}+\alpha)$, and

$$\|[\chi(\tilde{m}_t-1)]^{(\delta)}\|_2 = O(t^{2\ell}),$$

by a similar argument as before. So, by Theorem 4, we have

$$T_{\tilde{m}_t}f(x)-f(x) = o(t^\mu) \quad \text{a.e. as} \quad t\to 0^+$$

for every $f\in L^2_\varphi$. We write $\Delta(t)=\alpha t^\ell h(t)$. Then

$$(T_{\Delta_t}f)\hat{}\,(\xi) = \alpha t^\ell h(t|\xi|)|\xi|^\ell \hat{f}(\xi).$$

For $f\in L^2_\varphi$ let $\tilde{f}$ be defined by

$$\hat{\tilde{f}}(\xi) = |\xi|^\ell \hat{f}(\xi).$$

We see that

$$t^{-\ell}T_{\Delta_t}f = \alpha T_{h_t}\tilde{f},$$

and so there only remains to prove that

$$\|\sup_{0<t<1} T_{h_t}g\|_2 \leqq c\|g\|_2, \quad g\in L^2(\mathbf{R}^n). \tag{35}$$

To this end, write $h(|\xi|)=v(\|\xi\|)+w(\xi)$, where v is smooth, $v=1$ on $[0,1/4]$ and $\operatorname{supp} v\subset[0,1/2]$.

Clearly, the maximal operator $g\mapsto\sup_{t>0}|T_{v_t(\|\cdot\|)}g|$ is dominated by the Hardy—Littlewood maximal operator, hence bounded on $L^2(\mathbf{R}^n)$. Moreover, one checks easily that w satisfies the condition (1) of the proposition in Section 3 of Carbery [2], and so also the maximal operator associated with w is bounded on $L^2(\mathbf{R}^n)$. Together this implies (35).

Q.E.D.

Corollary 2. *Let* $m(t)=\chi_{(0,1)}(t)$. *Then for every* $f\in L^2_\varphi$ *with* $\varphi(t)=1+t^\mu$ *or* $\varphi(t)=\log^{\mu+1}(e+t)(\mu>0)$ *the following estimates hold, respectively:*

$$T_{m_t}f(x)-f(x) = o\left(t^\mu \log\frac{1}{t}\right) \quad \text{a.e.} \quad \text{as} \quad t\to 0^+, \tag{36}$$

$$T_{m_t}f(x)-f(x) = o\left(\frac{1}{\log^\mu \frac{1}{t}}\right) \quad \text{a.e.} \quad \text{as} \quad t\to 0^+. \tag{37}$$

Proof. We take $h\in C^\infty(0,\infty)$ satisfying (34). Define $\tilde{m}=m-h$. Then supp $\tilde{m}\subset [\frac{1}{2}, 1]$, and $\tilde{m}$ is of bounded variation. By Theorem 3 we conclude that for $f\in L^2_\varphi$

$$T_{\tilde{m}_t}f(x) = o\left(\log\frac{1}{t}\frac{1}{\varphi\left(\frac{1}{t}\right)}\right) \quad \text{a.e.} \quad \text{as} \quad t\to 0^+. \tag{38}$$

On the other hand the multiplier h satisfies the condition of Theorem 4. So for $f\in L^2_\psi$

$$T_{h_t}f(x)-f(x) = o\left(\frac{1}{\varphi\left(\frac{1}{t}\right)}\right) \quad \text{a.e.} \quad \text{as} \quad t\to 0^+. \tag{39}$$

If $\varphi(t)=1+t^\mu$ $(\mu>0)$ then we have

$$\varphi(t) \sim \psi(t).$$

Hence, the combination of (38) and (39) yields (36).

If $\varphi(t)=\log^\mu(e+t)$, then $\psi(t)\leq c\log^{\mu+1}(e+t)$. Hence for $f\in L^2_{\log^{\mu+1}(e+t)}$ (38) and (39) imply (37).

Q.E.D.

5. Remarks

(a) Since we only consider convergence of $T_{m_t}f$ as $t\to 0^+$, it is clear that we could even replace λ in (9) by $\lambda'=\overline{\lim}_{t\to\infty}\dfrac{\varphi(2t)}{\varphi(t)}$.

(b) We do not know whether the weight function ψ in Theorem 2 could even be replaced by φ.

(c) If one choses $m(t)=(1+t^{\ell})h(t)$, h as in (34), then obviously for every $f\in L^2$ with $\operatorname{supp}\hat{f}\subset\{\xi\colon |\xi|\leqq 1/4\}$

$$T_{m_t}f-f=t^{\ell}\tilde{f}, \quad (0<t<1),$$

where $(\tilde{f})\hat{}\,(\xi)=|\xi|^{\ell}\hat{f}(\xi)$.

This example shows that the condition $\beta>\mu$ in Theorem 4 is necessary for such a theorem.

(d) By Carbery's theorem, in the case of the Euclidean norm (33) is also valid for those $f\in L^p(\mathbf{R}^n)$ for which $\|D^{\mu}f\|_{L^p(\mathbf{R}^n)}<\infty$ for the range of p's described in Carbery's theorem.

References

1. Carbery, A., Radial Fourier multipliers and associated maximal functions, in: *Recent progress in Fourier analysis,* edited by I. Peral and T.-L. Rubio de Francia, pp. 49—56, North-Holland, Amsterdam, 1985.
2. Carbery, A., An almost-orthogonality principle with applications to maximal functions associated to convex bodies, *Bull. Am. Math. Soc.* **14** (1986), 269—273.
3. Carbery, A., Gasper, G. and Trebels, W., On localized potential spaces, *J. Approximation Theory* **48** (1986), 251—261.
4. Chen Tian-ping, Generalized Bochner—Riesz means of Fourier integrals, in: *Multivariate approximation theory IV: Proceedings of the Mathematical Research Institute at Oberwolfach, Febr. 12—18, 1989,* edited by C. K. Chui, W. Schempp, K. Zeller, Birkhäuser, Basel—Boston, 1989.
5. Cossar, J., A theorem on Cesàro summability, *J. London Math. Soc.* **16** (1941), 56—68.
6. Dappa, H. and Trebels, W., On maximal functions generated by Fourier multipliers, *Ark. Mat.* **23** (1985), 241—259.
7. Gasper, G. and Trebels, W., A characterization of localized Bessel potential spaces and applications to Jacobi and Hankel multipliers, *Studia Math.* **94** (1979), 243—278.

Received February 9, 1990,
revised July 7, 1990

Detlef Müller
Fakultät für Mathematik
Universität Bielefeld
Postfach 86 40
D-4800 Bielefeld
Germany

Wang Kun-yang
Dept. of Mathematics
Beijing Normal University
Beijing
China