Convolution equations in domains of $\mathbf{C}^n$

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0. Introduction

In this paper we study convolution equations of the form $\mu * u = f$, where $\mu \in \mathscr{A}'(\mathbf{C}^n)$ is an analytic functional and $u \in \mathscr{A}(X_1)$, $f \in \mathscr{A}(X_2)$ are analytic functions in open subsets X_1 and X_2 of $\mathbf{C}^n$. The convolution $\mu * u$ of μ and u is well defined by the formula $(\mu * u)(z) = \mu(u(z - \cdot))$ and analytic in X_2, if the sets X_1 and X_2 satisfy $X_2 - \hat{K}_{\mathbf{C}^n} \subset X_1$, with K a carrier of μ and $\hat{K}_{\mathbf{C}^n}$ its polynomial hull. There are two main problems that we will be concerned with. The first one is the existence problem: to give necessary and sufficient conditions for existence of solutions $u \in \mathscr{A}(X_1)$ of $\mu * u = f$ for any $f \in \mathscr{A}(X_2)$. The second one is the approximation problem: to decide if the space E_μ, consisting of all linear combinations of exponential solutions of $\mu * u = 0$ in $\mathbf{C}^n$, is dense in the space $N(X_1, X_2)$ of all solutions $u \in \mathscr{A}(X_1)$ of $\mu * u = 0$ in X_2.

In Section 1 we introduce the concept of μ-convexity for carriers of a pair (X_1, X_2) of open subsets of $\mathbf{C}^n$. We prove that $\mu * u = f$ has a solution $u \in \mathscr{A}(X_1)$ for every $f \in \mathscr{A}(X_2)$ and that E_μ is dense in $N(X_1, X_2)$, if the pair (X_1, X_2) is μ-convex for carriers and X_2 is a Runge domain. In the rest of the paper we study the case where X_1, X_2 and K are convex sets and $X_1 = X_2 - K$. We give a sufficient condition for μ-convexity of the pair (X_1, X_2) and a necessary condition for existence of solution $u \in \mathscr{A}(X_1)$ for every $f \in \mathscr{A}(X_2)$. Both of these conditions involve growth regularities of the Fourier—Laplace transform $\hat{\mu}$ of μ in certain outward normal directions of the boundary ∂X_2 of X_2. In some important cases these conditions are equivalent, for example if ∂X_2 is a differentiable manifold or $n=1$. These are the main results of the paper and they are stated and proved in Section 4. In Section 2 we make a quite long detour from the main subject of the paper for studying growth properties of plurisubharmonic functions $p \in \mathrm{PSH}(\mathbf{C}^n)$ of finite type with respect to the order $\varrho > 0$. Our main tool for characterizing growth regularities of p is the limit set $L_\infty(p)$ of p. It is defined as the set of all $q \in \mathrm{PSH}(\mathbf{C}^n)$, such that $T_{t_j} p \to q$ in $L^1_{\mathrm{loc}}(\mathbf{C}^n)$, where $t_j \to +\infty$ and $(T_t p)(z) = t^{-\varrho} p(tz)$. We say that p is of regular growth in the direction of $z\in\mathbf{C}^n$, $z\neq 0$, if $q(z)=h_p(z)$ for all $q\in L_\infty(p)$, where h_p denotes the indicator function of p. For justifying this definition, we prove that it is equivalent to definitions given by other authors. In Section 3 we state a version of our earlier result on asymptotic approximation of functions $p\in\mathrm{PSH}(\mathbf{C}^n)$ by functions of the form $\log|\hat{\mu}|$, where $\mu\in\mathscr{A}'(\mathbf{C}^n)$. This result is of fundamental importance in our proof in Section 4. We also discuss some of our earlier results on growth regularities of Fourier—Laplace transforms of distributions.

The study of convolution equations is mainly concerned with generalizations of results from the theory of partial differential equations. In fact, every partial differential operator $P(\partial)=\sum a_\alpha\partial^\alpha$ with constant coefficients can be viewed as a convolution operator $\mu*$, where $\mu=\sum a_\alpha\partial^\alpha\delta_0$ and δ_0 is the Dirac delta distribution at the origin. Equations of the type $\mu*u=f$ with $\mu\in\mathscr{E}'(\mathbf{R}^n)$, u in some subspace of $\mathscr{D}'(X_1)$, f in some subspace of $\mathscr{D}'(X_2)$, X_1 and X_2 open in $\mathbf{R}^n$ and satisfying $X_2-\operatorname{supp}\mu\subset X_1$, were first studied by Ehrenpreis [4—5], Hörmander [6—8] and Malgrange [15—16]. The literature is now quite extensive and we refer the reader to Hörmander [10, Chapter 16].

It was Malgrange [15] who initiated the study of convolution equations in spaces of analytic functions of several variables. He proved, without any restriction on $\mu\in\mathscr{A}'(\mathbf{C}^n)$, that $\mu*u=f$ has a solution $u\in\mathscr{A}(\mathbf{C}^n)$ for every $f\in\mathscr{A}(\mathbf{C}^n)$ and that E_μ is dense in $N(\mathbf{C}^n,\mathbf{C}^n)$. Martineau [18] proved that the analogous statements hold if μ is carried by the origin and $\mathbf{C}^n$ is replaced by any open convex subset of $\mathbf{C}^n$. Moržhakov [20] was the first to study the case, where X_1, X_2 and K are convex and $X_1=X_2-K$. He proved the existence of a solution $u\in\mathscr{A}(X_1)$ of $\mu*u=f$ for any $f\in\mathscr{A}(X_2)$ and that E_μ is dense in $N(X_1,X_2)$ under the assumption that $\hat{\mu}$ is of completely regular growth (in the sense of Levin [14]), and $h_{\hat{\mu}}(i\zeta)=H_K(\zeta)$ for all $\zeta\in\mathbf{C}^n$, where H_K is the supporting function of K. Lelong and Gruman [13] and Moržhakov [21] proved that

$$\frac{1}{t}\log|\hat{\mu}(t\zeta)|\to H_K(i\bar{\zeta})\quad\text{in}\quad\mathscr{D}'(\mathbf{C}^n)\quad\text{as}\quad t\to\infty,$$

is a sufficient condition for existence of solution, and in the case where X_2 is bounded and has a differentiable boundary it is also necessary. A different type of sufficient condition has been given by Meril and Struppa [19]. The approximation problem has been studied by many authors. Krasičkov—Ternovskiĭ [12] proved, without any assumption on μ, that E_μ is dense in $N(X_1,X_2)$ if X_1 and X_2 are convex subsets of $\mathbf{C}^1$. Napalkov [22—24] has proved that E_μ is dense in $N(X_1,X_2)$ if X_1 and X_2 are convex tube domains in $\mathbf{C}^n$. We refer the reader to the recent survey by Berenstein and Struppa [3] for more information on these problems.

1. Convolution operators and μ-convexity for carriers

Let $\mu \in \mathscr{A}'(\mathbf{C}^n)$ be an analytic functional. Then there exists a compact subset ω of $\mathbf{C}^n$ and a positive constant C_ω such that

$$(1.1) \qquad |\mu(\phi)| \leqq C_\omega \sup_\omega |\phi|, \quad \phi \in \mathscr{A}(\mathbf{C}^n).$$

The analytic functional μ is said to be carried by the compact subset K of $\mathbf{C}^n$ if for every neighborhood ω of K, there exists a positive constant C_ω such that (1.1) holds. For $\phi \in \mathscr{A}(\mathbf{C}^n)$ we define the convolution $\mu * \phi$ of μ and ϕ by the formula

$$(1.2) \qquad (\mu * \phi)(z) = \mu(\phi(z-.)), \quad z \in \mathbf{C}^n.$$

By (1.1) it follows that $\mu * \phi \in \mathscr{A}(\mathbf{C}^n)$ and the operator

$$\mu *\colon \mathscr{A}(\mathbf{C}^n) \to \mathscr{A}(\mathbf{C}^n), \quad \phi \mapsto \mu * \phi$$

is continuous. The operator $\mu *$ commutes with all translations, that is for every translation operator $\tau_h\colon \mathscr{A}(\mathbf{C}^n) \to \mathscr{A}(\mathbf{C}^n)$, $h \in \mathbf{C}^n$, defined as $(\tau_h \phi)(z) = \phi(z-h)$, we have $\mu * (\tau_h \phi) = \tau_h(\mu * \phi)$. It is easily shown that for every continuous linear operator $S\colon \mathscr{A}(\mathbf{C}^n) \to \mathscr{A}(\mathbf{C}^n)$, which commutes with all translations, there exists a unique analytic functional μ such that $S\phi = \mu * \phi$ for all $\phi \in \mathscr{A}(\mathbf{C}^n)$. This property allows us to define the convolution $\mu * \nu$ of two analytic functionals μ and ν as the analytic functional corresponding to the operator

$$\mathscr{A}(\mathbf{C}^n) \to \mathscr{A}(\mathbf{C}^n), \quad \phi \mapsto \mu * (\nu * \phi).$$

We observe that $\mu * \nu$ is carried by $K + L$ if μ is carried by K and ν is carried by L. If X is an open subset of $\mathbf{C}^n$, μ is carried by a compact subset K of $\mathbf{C}^n$ with polynomial hull $\hat{K}_{\mathbf{C}^n}$ contained in X, then μ can be extended to a continuous linear functional on $\mathscr{A}(X)$, that is $\mu \in \mathscr{A}'(X)$. In fact, we can always choose a polynomially convex neighborhood of ω of $\hat{K}_{\mathbf{C}^n}$ such that $K \subset \hat{K}_{\mathbf{C}^n} \subset \omega \subset X$. Then there exists a positive constant C_ω such that (1.1) holds. If $\phi \in \mathscr{A}(X)$, then there exists a sequence ϕ_j in $\mathscr{A}(\mathbf{C}^n)$ such that $\phi_j \to \phi$ uniformly on ω. (See Hörmander [9, Theorems 2.7.4 and 2.7.7].) From (1.1) it follows that $\{\mu(\phi_j)\}$ is a Cauchy sequence. We denote its limit by $\mu(\phi)$. From (1.1) again it follows that $\mu(\phi)$ is independent of the choice of the sequence $\{\phi_j\}$ and

$$(1.3) \qquad |\mu(\phi)| \leqq C_\omega \sup_\omega |\phi|, \quad \phi \in \mathscr{A}(X).$$

Now we let X_1 and X_2 be open subsets of $\mathbf{C}^n$ and we assume that

$$(1.4) \qquad X_2 - \hat{K}_{\mathbf{C}^n} \subset X_1.$$

For every $z \in X_2$ and every $\phi \in \mathcal{A}(X_1)$, the function $w \mapsto \phi(z-w)$ is analytic in a neighborhood of $\hat{K}_{\mathbf{C}^n}$, so $(\mu * \phi)(z)$ is well defined by the formula $(\mu*\phi)(z) = \mu(\phi(z-\cdot))$. From (1.3) it follows that $\mu * \phi \in \mathcal{A}(X_2)$ and

$$\mu *\colon \mathcal{A}(X_1) \to \mathcal{A}(X_2), \quad (\mu * \phi)(z) = \mu(\phi(z-.)), \quad z \in X_2,$$

is a continuous linear operator. Its adjoint is a convolution operator between the dual spaces

$$(\mu *)^t = \check{\mu} *\colon \mathcal{A}'(X_2) \to \mathcal{A}'(X_1), \quad \nu \mapsto \check{\mu} * \nu,$$

where $\check{\mu}$ denotes the analytic functional defined by $\check{\mu}(\phi) = \mu(\check{\phi})$ and $\check{\phi}(z) = \phi(-z)$ for $\phi \in \mathcal{A}(\mathbf{C}^n)$.

Definition 1.1. Let X_1 and X_2 be open subsets of $\mathbf{C}^n$ and let μ be an analytic functional carried by the compact subset K of $\mathbf{C}^n$ satisfying $X_2 - K \subset X_1$. The pair (X_1, X_2) is said to be *μ-convex for carriers* if for every compact subset K_1 of X_1 there exists a compact subset K_2 of X_2 such that α is carried by K_2, if $\alpha \in \mathcal{A}'(\mathbf{C}^n)$ and $\check{\mu} * \alpha$ is carried by K_1.

A solution $u \in \mathcal{A}(\mathbf{C}^n)$ of the homogeneous convolution equation $\mu * u = 0$ in $\mathbf{C}^n$ is called an *exponential solution*, if it can be written of the form

$$u(z) = P(z) e^{i\langle z, \zeta\rangle}, \quad z \in \mathbf{C}^n,$$

where P is a polynomial, $\zeta \in \mathbf{C}^n$ and $\langle z, \zeta\rangle = \sum z_j \zeta_j$. We let E_μ denote the space of all linear combinations of exponential solutions of the homogeneous equation. If μ is carried by K, the sets X_1 and X_2 are open in $\mathbf{C}^n$ and satisfy (1.4), then we let $N_\mu(X_1, X_2)$ denote the space of all solutions of $u \in \mathcal{A}(X_1)$ of the homogeneous equation $\mu * u = 0$ in X_2.

Theorem 1.2. *Let X_1 and X_2 be open subsets of $\mathbf{C}^n$ and assume that X_2 is a Runge domain. Let μ be an analytic functional, carried by a compact subset K of $\mathbf{C}^n$ satisfying* (1.4), *and assume that the pair (X_1, X_2) is μ-convex for carriers. Then:*

(i) *The restrictions to X_1 of the elements in E_μ are dense in $N_\mu(X_1, X_2)$ in the topology induced by $\mathcal{A}(X_1)$.*

(ii) *The convolution equation $\mu * u = f$ has a solution $u \in \mathcal{A}(X_1)$ for all $f \in \mathcal{A}(X_2)$.*

The proof of the theorem is based on the following two lemmas, which are due to Malgrange [15]:

Lemma 1.3. *$\beta \in \mathcal{A}'(\mathbf{C}^n)$ satisfies $\beta(\phi) = 0$ for all $\phi \in E_\mu$ if and only if $\beta = \check{\mu} * \alpha$ for some $\alpha \in \mathcal{A}'(\mathbf{C}^n)$.*

Lemma 1.4. *If $\alpha_j \in \mathcal{A}'(\mathbf{C}^n)$ and $\beta_j = \check{\mu} * \alpha_j$ converges weakly to $\beta \in \mathcal{A}'(\mathbf{C}^n)$, then $\beta = \check{\mu} * \alpha$ for some $\alpha \in \mathcal{A}'(\mathbf{C}^n)$.*

Since these lemmas are well known, we do not prove them here. The proof of Theorem 1.2 (ii) is a standard duality argument. In fact, it is only a modification of the proof of Theorem 16.5.7 in Hörmander [10]. Before we give the proof, we discuss the Fourier—Laplace transformation on $\mathscr{A}'(\mathbf{C}^n)$.

Let $\mu\in\mathscr{A}'(\mathbf{C}^n)$, then the Fourier—Laplace transform $\hat{\mu}\in\mathscr{A}(\mathbf{C}^n)$ is defined as

$$\hat{\mu}(\zeta) = \mu(\exp(-i\langle .,\zeta\rangle)),\quad \zeta\in\mathbf{C}^n. \tag{1.5}$$

We observe that the euclidean inner product on $\mathbf{C}^n=\mathbf{R}^{2n}$ is given by

$$\operatorname{Re}(\langle\bar{z},\zeta\rangle) = \langle x,\xi\rangle+\langle y,\eta\rangle,\quad z = x+iy,\quad \zeta=\xi+i\eta,\quad x,y,\xi,\eta\in\mathbf{R}^n. \tag{1.6}$$

If L is a subset of $\mathbf{C}^n$, then its supporting function H_L is defined as

$$H_L(\zeta) = \sup_{z\in L}\operatorname{Re}(\langle\bar{z},\zeta\rangle),\quad \zeta\in\mathbf{C}^n. \tag{1.7}$$

The function H_L is lower semi-continuous and positively homogeneous of degree 1. If L is compact, then H_L is continuous. If μ is carried by the compact subset K of $\mathbf{C}^n$ and $\varepsilon>0$, then we take ω as an ε-neighborhood of K, and it follows from (1.3) that there exists a positive constant C_ε such that

$$|\hat{\mu}(\zeta)| \leqq C_\varepsilon\exp\left(H_K(i\bar{\zeta})+\varepsilon|\zeta|\right),\quad \zeta\in\mathbf{C}^n. \tag{1.8}$$

Conversely, the Pólya—Ehrenpreis—Martineau theorem states (see Hörmander [9, Theorem 4.5.3]), that every $f\in\mathscr{A}(\mathbf{C}^n)$ satisfying a growth estimate of the form (1.8) is the Fourier—Laplace transform $f=\hat{\mu}$ of some $\mu\in\mathscr{A}'(\mathbf{C}^n)$ carried by the convex hull of K.

Proof of Theorem 1.2. (i) Let $\beta\in\mathscr{A}'(X_1)$ and assume that $\beta(\phi)=0$ for all $\phi\in E_\mu$. By the Hahn—Banach theorem it is sufficient to prove that $\beta(\phi)=0$ for all $\varphi\in N_\mu(X_1,X_2)$. By Lemma 1.3 we have $\beta=\check{\mu}*\alpha$ for some $\alpha\in\mathscr{A}'(\mathbf{C}^n)$. Since (X_1,X_2) is μ-convex for carriers, α is carried by some compact subset of X_2. Since X_2 is a Runge domain, it follows that α can be extended to a continuous linear functional on $\mathscr{A}(X_2)$, that is $\alpha\in\mathscr{A}'(X_2)$. Hence

$$\beta(\phi) = (\check{\mu}*\alpha)(\phi) = \alpha(\mu*\phi) = 0,\quad \phi\in N_\mu(X_1,X_2).$$

(ii) In order to prove that the operator $\mu *\colon \mathscr{A}(X_1)\to\mathscr{A}(X_2)$ is surjective, it is sufficient to prove that its adjoint $\check{\mu}*\colon \mathscr{A}'(X_2)\to\mathscr{A}'(X_1)$ is injective and that its image $M=\check{\mu}*(\mathscr{A}'(X_2))$ is weakly closed in $\mathscr{A}'(X_1)$. (See Schaefer [26, Chapter IV, 6.4].) The fact that $\check{\mu}*$ is injective follows by taking Fourier—Laplace transforms. In order to prove that M is weakly closed in $\mathscr{A}'(X_1)$ it is sufficient to show that $M\cap U^0$ is weakly closed for every neighborhood U of 0 in $\mathscr{A}'(X_1)$, where U^0 denotes the polar set of U defined by $U^0=\{\beta\in\mathscr{A}'(X_1);\ |\beta(\phi)|\leqq 1 \text{ for all } \phi\in U\}$. (See Schaefer [26, Chapter IV, 7.7].) Without any restriction we may assume that

$U=\{\phi\in\mathscr{A}(X_1);\ C\sup_{K_1}|\phi|\leqq 1\}$ for some compact subset K_1 of X_1 and a positive constant C. Then U^0 is the set of all $\beta\in\mathscr{A}'(X_1)$ such that

$$|\beta(\phi)|\leqq C\sup_{K_1}|\phi|,\quad \phi\in\mathscr{A}(X_1). \tag{1.9}$$

Thus every β in U^0 is carried by K_1. Let $\beta_j=\check{\mu}*\alpha_j\in U^0\cap M$ be a sequence converging weakly to $\beta\in\mathscr{A}'(X_1)$. From (1.9) it follows that β is carried by K_1. By Lemma 1.4 there exists $\alpha\in\mathscr{A}'(\mathbf{C}^n)$ such that $\beta=\check{\mu}*\alpha$. Since (X_1, X_2) is μ-convex for carriers and X_2 is a Runge domain, it follows that $\alpha\in\mathscr{A}'(X_2)$. Hence $\beta=\check{\mu}*\alpha\in U^0\cap M$. The proof is complete.

2. Growth properties of plurisubharmonic functions

In this section we have collected definitions and basic properties of those concepts that are used for characterizing growth of plurisubharmonic and analytic functions in $\mathbf{C}^n$. Mean values of plurisubharmonic functions are very important in our proofs, so we begin by introducing a notation for them. We let X be an open subset of $\mathbf{C}^n$ and let $d(z,\partial X)$ denote the euclidean distance from $z\in X$ to the boundary ∂X of X. For $f\in L^1_{\text{loc}}(X)$ we let $(\mathscr{M}_r f)(z)$ denote the mean value of the function f over the ball with center z and radius r, where $0<r<d(z,\partial X)$,

$$(\mathscr{M}_r f)(z)=\frac{1}{\omega_{2n}r^{2n}}\int_{|w-z|\leqq r} f(w)\,d\lambda(w)=\frac{1}{\omega_{2n}}\int_{|w|\leqq 1} f(z+rw)\,d\lambda(w).$$

Here $d\lambda$ denotes the Lebesgue measure in $\mathbf{C}^n$ and ω_{2n} denotes the volume of the unit ball in $\mathbf{C}^n$. For any r, the mean value is a continuous function $L^1_{\text{loc}}(X)\times X_r\to\mathbf{R}$, $(f,z)\mapsto(\mathscr{M}_r f)(z)$, where $X_r=\{w\in X;\ d(w,\partial X)>r\}$. Moreover, the function

$$L^1_{\text{loc}}(\mathbf{C}^n)\times\mathbf{C}^n\times\mathbf{R}_+\to\mathbf{R},\quad (f,z,r)\mapsto(\mathscr{M}_r f)(z)$$

is continuous. We let $\mathrm{PSH}(X)$ denote the set of all plurisubharmonic functions not identically equal to $-\infty$ in any connected component of X. We have

$$\mathrm{PSH}(X)\subset L^1_{\text{loc}}(X)\subset\mathscr{D}'(X).$$

The topology of $L^1_{\text{loc}}(X)$ is defined by the semi-norms

$$u\mapsto\int_K |u|\,d\lambda,\quad K\subset X,\ K \text{ compact},$$

and the weak topology of $\mathscr{D}'(X)$ is defined by the semi-norms

$$u\mapsto|\langle u,\phi\rangle|,\quad \phi\in C_0^\infty(X).$$

It turns out that these two topologies are identical on $\mathrm{PSH}(X)$. Moreover, $\mathrm{PSH}(X)$

is a closed convex cone, thus a complete metrizable space. We always refer to this topology when we discuss topological properties of PSH (X).

Let $M\subset \text{PSH}(X)$ and assume that the functions in M have a uniform upper bound on any compact subset of X, that is, for every compact $K\subset X$ there exists a constant C_K such that

$$q(z) \leqq C_K, \quad q\in M, \quad z\in K. \tag{2.1}$$

In general the function $\sup_M q$ is not upper semi-continuous, thus not plurisubharmonic. On the other hand, its upper regularization $(\sup_M q)^*$, defined as the least upper semi-continuous majorant of $\sup_M q$, is plurisubharmonic. If M is bounded in PSH (X), then the functions in M have a uniform upper bound on any compact subset of X. In fact, if $K\subset X$ is compact and $0<r<d(K, \partial X)$, then

$$q(z) \leqq (\mathcal{M}_r q)(z) \leqq \frac{1}{\omega_{2n} r^{2n}} \int_{K_r} |q|\, d\lambda, \quad z\in K, \quad q\in M,$$

where K_r denotes the set of all points in X of distance $\leqq r$ from K. Since M is bounded in PSH (X), the integral is uniformly bounded for $q\in M$, and it follows that (2.1) holds for some constant C_K. Later on the case when M is compact will be important:

Proposition 2.1. *Let X be an open subset of $\mathbf{C}^n$ and let M be a compact subset of* PSH (X). *Then* $\sup_M q\in \text{PSH}(X)$.

Proof. Set $p=\sup_{q\in M} q$. We only need to prove that p is upper semi-continuous. Let $z_0\in X$, $a\in \mathbf{R}$ and assume that $p(z_0)<a$. We have to prove that there exists a neighborhood V of z_0 such that

$$p(z) < a, \quad z\in V. \tag{2.2}$$

We choose $\varepsilon>0$ such that $p(z_0)<a-\varepsilon$. If $q_0\in M$ and r_0 is chosen sufficiently small, then

$$q_0(z_0) \leqq (\mathcal{M}_{r_0} q_0)(z_0) < a-\varepsilon.$$

Since the function $L^1_{\text{loc}}(X)\times X_{r_0}\to \mathbf{R}$, $(f, z)\mapsto (\mathcal{M}_{r_0} f)(z)$ is continuous, there exists an open neighborhood U_0 of q_0 in PSH (X) and an open neighborhood V_0 of z_0 such that

$$(\mathcal{M}_{r_0} q)(z) < a-\varepsilon, \quad q\in U_0, \quad z\in V_0.$$

The mean value property implies

$$q(z) < a-\varepsilon, \quad q\in U_0, \quad z\in V_0. \tag{2.3}$$

Since q_0 is arbitrary and M is compact, there exists a finite covering $U_1, \dots, U_N$ of M and open neighborhoods $V_1, \dots, V_N$ of z_0 such that (2.3) holds for all $q\in U_j$ and $z\in V_j$. If we set $V=\cap V_j$, then (2.2) holds and the proof is complete.

Assume now that X is connected. Then the set $\mathrm{PSH}(X)$ has a certain Montel property. If $M\subset\mathrm{PSH}(X)$ and the functions in M have a uniform upper bound on any compact subset of X, then every sequence $\{q_j\}$ in M satisfies one of the following conditions:

(i) $\{q_j\}$ has a subsequence which is convergent in $\mathrm{PSH}(X)$.

(ii) $q_j\to-\infty$ uniformly in every compact subset of X.

If no sequence in M satisfies (ii), then M is relatively compact. All our unproved statements so far in this section are contained in Theorems 4.1.8 and 4.1.9 in Hörmander [10]. See also Lelong and Gruman [13, Chapter 1 and Appendix I].

Let us now turn to the growth characteristics of plurisubharmonic functions in $\mathbf{C}^n$. A function $p\in\mathrm{PSH}(\mathbf{C}^n)$ is said to be of *finite order* if there exist positive real numbers τ, σ and ϱ such that

$$p(z)\leqq\tau+\sigma|z|^{\varrho},\quad z\in\mathbf{C}^n. \tag{2.4}$$

The *order* ϱ_p of p is defined as the infimum over all ϱ, for which (2.4) holds for some τ and σ. For a given $\varrho>0$ we say that p is of *finite type with respect to the order* ϱ if (2.4) holds for some τ and σ. If p is of finite type with respect to the order ϱ_p then we define its *type* σ_p as the infimum over all σ such that (2.4) holds with some τ and $\varrho=\varrho_p$. If p is of finite type with respect to the order ϱ then we define its *indicator function* h_p *(with respect to the order* ϱ*)* as the least upper semi-continuous majorant of

$$\mathbf{C}^n\ni z\mapsto\varlimsup_{t\to+\infty}\frac{1}{t^{\varrho}}p(tz). \tag{2.5}$$

The function h_p is in $\mathrm{PSH}(\mathbf{C}^n)$ and it is positively homogeneous of degree ϱ. In order to study growth regularities of functions of finite type with respect to the order ϱ it is natural to introduce the group $\{T_t\}_{t>0}$ of mappings defined by

$$T_t\colon\ \mathrm{PSH}(\mathbf{C}^n)\to\mathrm{PSH}(\mathbf{C}^n),\quad (T_tq)(z)=\frac{1}{t^{\varrho}}q(tz),\quad z\in\mathbf{C}^n. \tag{2.6}$$

This is an example of a *continuous dynamical system* or a *flow* Φ on a metric space A, that is, a continuous mapping

$$\Phi\colon\ A\times\mathbf{R}_+\to A$$

satisfying $\Phi(\Phi(p,s),t)=\Phi(p,st)$ and $\Phi(p,1)=p$ for all $p\in A$ and $s,t\in\mathbf{R}_+=\{\tau>0\}$. We view Φ as a one parameter family of continuous mappings and write $\Phi_t p$ instead of $\Phi(p,t)$. The set $\{\Phi_t p;\ t\geqq1\}$ is called *the forward orbit of* p and the set $\{\Phi_t p;\ 0<t\leqq1\}$ is called *the backward orbit of* p. The set of all $q\in A$ that are limits of sequences of the form $\{\Phi_{t_j}p\}$, where $t_j\to+\infty$ is called the *limit set of* p *at infinity* and it is denoted by $\mathrm{L}_\infty(p)$. The set of all $q\in A$ that are limits of se-

quences of the form $\{\Phi_{t_j} p\}$, where $t_j \to 0$ is called the *limit set of p at the origin* and it is denoted by $L_0(p)$. From the continuity of Φ it follows that the sets $L_\infty(p)$ and $L_0(p)$ are Φ invariant, that is

$$\Phi_t L_\infty(p) = L_\infty(p) \quad \text{and} \quad \Phi_t L_0(p) = L_0(p), \quad t > 0.$$

The most basic properties of the one parameter family $T = \{T_t\}_{t>0}$ are:

Proposition 2.2. *Let* $p \in \mathrm{PSH}(\mathbf{C}^n)$. *Then*

(i) *p is of finite type with respect to the order ϱ if and only if the forward orbit* $\{T_t p;\ t \geqq 1\}$ *of p in* $\mathrm{PSH}(\mathbf{C}^n)$ *is relatively compact.*

(ii) *If p is of finite type with respect to the order ϱ, then $L_\infty(p)$ is T invariant, compact, connected and $q(0)=0$ for all $q \in L_\infty(p)$. We also have*

$$h_p(z) = \sup\{q(z);\ q \in L_\infty(p)\}, \quad z \in \mathbf{C}^n. \tag{2.7}$$

Proof. (i) Assume that $\{T_t p;\ t \geqq 1\}$ is relatively compact and let $\varepsilon \in (0, 1)$. If $z \in \mathbf{C}^n$ and $t = |z| \geqq 1$, then the mean value property implies

$$p(z) \leqq (\mathcal{M}_{\varepsilon t} p)(z) = t^{\varrho}(\mathcal{M}_\varepsilon T_t p)(z/|z|) \leqq \left(\frac{1}{\omega_{2n}\varepsilon^{2n}} \int_{|w| \leqq 1+\varepsilon} |T_t p|\, d\lambda\right) |z|^{\varrho}.$$

The forward orbit of p is relatively compact in $\mathrm{PSH}(\mathbf{C}^n)$, so the first factor in the right-hand side is uniformly bounded for $t \geqq 1$; we let σ be an upper bound for it and τ be an upper bound for p in the unit ball of $\mathbf{C}^n$, then p satisfies the estimate (2.4).

Conversely, if p is of finite type with respect to the order ϱ, then it follows from (2.4), that every sequence $\{T_{t_j} p\}$ with $t_j \geqq 1$ is uniformly bounded above in every compact subset of $\mathbf{C}^n$. It can not tend to $-\infty$ uniformly on every compact set, for $r \mapsto (\mathcal{M}_r p)(0)$ is an increasing function, and

$$(\mathcal{M}_\varepsilon T_t p)(0) = t^{-\varrho}(\mathcal{M}_{\varepsilon t} p)(0) \geqq t^{-\varrho}(\mathcal{M}_\varepsilon p)(0), \quad t \geqq 1. \tag{2.8}$$

Hence $\{T_{t_j} p\}$ has a subsequence converging in $\mathrm{PSH}(\mathbf{C}^n)$.

(ii) The continuity of T_t implies that the limit set $L_\infty(p)$ is T invariant. It can be written as an intersection of a decreasing sequence of compact connected sets

$$L_\infty(p) = \bigcap_{N \geqq 1} \overline{\{T_t p;\ t \geqq N\}}.$$

Hence (i) gives that it is compact and connected. From (2.4) it follows that $q(0) \leqq 0$ for all $q \in L_\infty(p)$ and the reverse inequality follows from (2.8).

For proving (2.7) we first observe that Proposition 2.1 implies that the right-hand side defines a function in $\mathrm{PSH}(\mathbf{C}^n)$. We obviously have $h_p \geqq q$ for all $q \in L_\infty(p)$, so we only need to prove that for every $z \in \mathbf{C}^n$, there exists $q \in L_\infty(p)$ such that $q(z) \geqq \overline{\lim}_{t \to +\infty} (T_t p)(z)$. We let $t_j \to +\infty$ and assume that $\lim_{j \to +\infty} T_{t_j} p(z) = \overline{\lim}_{t \to +\infty} (T_t p)(z)$. By replacing $\{t_j\}$ by a subsequence we may assume that $T_{t_j} p \to$

$q \in L_\infty(p)$. Then

$$q(z) \geqq \varlimsup_{j\to+\infty} (T_{t_j}p)(z) = \varlimsup_{t\to+\infty} (T_t p)(z)$$

and the proof is complete.

Let $f \in \mathcal{A}(\mathbf{C}^n)$ and set $p = \log|f|$. We say that f is of finite order if p is of finite order. We define the order ϱ_f, type σ_f and indicator function h_f of f as ϱ_p, σ_p and h_p, respectively. The function f is said to be of finite type with respect to the order ϱ if $p = \log|f|$ satisfies (2.4). The function f is said to be of *exponential type* if it is of finite type with respect to the order 1, that is if there exist positive constans C and σ such that

$$|f(z)| \leqq Ce^{\sigma|z|}, \quad z \in \mathbf{C}^n.$$

The problem of characterizing those subsets M of $\mathrm{PSH}(\mathbf{C}^n)$ that are limit sets of some plurisubharmonic functions was first studied by Azarin [1], Azarin and Giner [2] and Sigurdsson [27, Section 1.2]. It was completely solved by Hörmander and Sigurdsson [11]. We will not state the result here but refer the reader to [11].

If p is of finite type with respect to the order ϱ, then its forward orbit can be approximated by the forward orbit of a function of the form $\log|f|$, where f is an entire analytic function, in the following sense:

Theorem 2.4. *Let $p \in \mathrm{PSH}(\mathbf{C}^n)$ be of finite type with respect to the order ϱ. Then there exists $f \in \mathcal{A}(\mathbf{C}^n)$ such that*

$$T_t p - T_t \log|f| \to 0 \quad \text{in} \quad \mathcal{D}'(\mathbf{C}^n) \quad \text{as} \quad t \to +\infty.$$

See [27, Theorem 1.3.1]. The theorem shows that every subset M of $\mathrm{PSH}(\mathbf{C}^n)$, which is the limit set at infinity $M = L_\infty(p)$ of some $p \in \mathrm{PSH}(\mathbf{C}^n)$, is the limit set at infinity of a function of the form $p = \log|f|$ with $f \in \mathcal{A}(\mathbf{C}^n)$.

Theorem and Definition 2.5. *Let $p \in \mathrm{PSH}(\mathbf{C}^n)$ be of finite type with respect to the order ϱ and let $z \in \mathbf{C}^n$, $z \neq 0$. Then the following conditions are equivalent:*

(i) *$q(z) = h_p(z)$ for all $q \in L_\infty(p)$.*

(ii) $\varliminf_{\gamma\to 0} \varliminf_{t\to+\infty} (\mathcal{M}_\gamma T_t p)(z) = h_p(z)$.

(iii) *For every increasing sequence $\{\tau_j\}$ in $\mathbf{R}_+$ with $\tau_j \to +\infty$ and $\tau_{j+1}/\tau_j \to 1$, there exists a sequence $\{z_j\}$ in $\mathbf{C}^n$ with $z_j \to z$, such that*

$$(T_{\tau_j} p)(z_j) \to h_p(z) \quad \text{as} \quad j \to +\infty.$$

(iv) *There exists an increasing sequence $\{\tau_j\}$ in $\mathbf{R}_+$ with $\tau_j \to +\infty$ and $\tau_{j+1}/\tau_j \to 1$, and a sequence $\{z_j\}$ in $\mathbf{C}^n$ with $z_j \to z$, such that*

$$(T_{\tau_j} p)(z_j) \to h_p(z) \quad \text{as} \quad j \to +\infty.$$

(v) *If* $\varphi \in \mathrm{PSH}(\mathbf{C}^n)$ *is of finite type with respect to the order* ϱ, *then*

$$h_{p+\varphi}(z) = h_p(z) + h_\varphi(z).$$

We say that p is of regular growth in the direction of z, if these conditions hold. We say that p is of regular growth in a subset X of $\mathbf{C}^n$, *if p is of regular growth in the direction of every w in X. We say that an analytic function f of finite type with respect to the order* ϱ *is of regular growth in the direction of z, or in the set X, if* $p = \log|f|$ *is of regular growth in the direction of z, or in the set X.*

Remark. (i) Since $L_\infty(p)$ is T invariant and h_p is positively homogeneous of degree ϱ, it follows that p is of regular growth in $\mathbf{R}_+ z$ if p is of regular growth in the direction of z. This implies that

$$T_t p \to h_p \quad \text{in} \quad \mathscr{D}'(\Gamma) \quad \text{as} \quad t \to +\infty$$

if p is of regular growth in the open set X and Γ denotes the cone generated by X.

(ii) In the classical litterature on entire functions of one variable the concept of regular growth is defined differently. Translated into our notation the definition is: *A function* $p \in \mathrm{PSH}(\mathbf{C}^n)$, *of finite type with respect to the order* ϱ, *is said to be of completely regular growth in the direction of* $z \in \mathbf{C}^n$, $z \neq 0$ *if there exists a subset E of* $\mathbf{R}_+$ *such that*

$$\lim_{\substack{t \to +\infty \\ t \notin E}} T_t p(z) = h_p(z) \quad \textit{and} \quad \lim_{t \to +\infty} \frac{\lambda(E \cap (0, t))}{t} = 0.$$

Functions of completely regular growth were first studied by Levin and Pfluger, see Levin [14]. It is easy to see that if p is of completely regular growth in the direction of z, then (iv) in Theorem 2.5 holds, so p is of regular growth in the direction of z. On the other hand, S. Yu. Favorov has given an example of a function which is of regular growth in $\mathbf{C}^n$ but not of completely regular growth. See Ronkin [25] and the references given there.

(iii) The condition (ii) in Theorem 2.5 was introduced by Lelong and Gruman [13] as a definition of regular growth. The condition (iv) is the notion of *P-regular growth* introduced by Wiegerinck [28].

As an immediate consequence of Theorem 2.4 and Theorem 2.5 we get the theorem of Favorov, see Ronkin [25]:

Corollary 2.6. *Let* $f \in \mathscr{A}(\mathbf{C}^n)$ *be of finite type with respect to the order* ϱ *and let* $z \in \mathbf{C}^n$, $z \neq 0$. *Then f is of regular growth in the direction of z if and only if*

$$h_{fg}(z) = h_f(z) + h_g(z),$$

for all $g \in \mathscr{A}(\mathbf{C}^n)$ *of finite type with respect to the order* ϱ.

Proof of Theorem 2.5. (i)⇒(ii). We observe that the Fatou lemma implies

$$\varliminf_{t\to+\infty}(\mathcal{M}_\gamma T_t p)(z) \leqq \varlimsup_{t\to+\infty}(\mathcal{M}_\gamma T_t p)(z) \leqq (\mathcal{M}_\gamma h_p)(z)$$

for all $\gamma>0$, so

$$\varliminf_{\gamma\to 0}\varliminf_{t\to+\infty}(\mathcal{M}_\gamma T_t p)(z) \leqq \lim_{\gamma\to 0}(\mathcal{M}_\gamma h_p)(z) = h_p(z).$$

Assume that inequality holds and choose $\varepsilon>0$ such that

$$\varliminf_{\gamma\to 0}\varliminf_{t\to+\infty}(\mathcal{M}_\gamma T_t p)(z) < h_p(z)-\varepsilon.$$

Then there exists $\delta>0$ and $t_j\to+\infty$ such that

$$(\mathcal{M}_\delta T_{t_j} p)(z) < h_p(z)-\varepsilon, \quad j = 1, 2, 3, \dots.$$

By passing to a subsequence we may assume that $T_{t_j}p\to q\in \mathrm{L}_\infty(p)$. Then

$$q(z) \leqq (\mathcal{M}_\delta q)(z) \leqq h_p(z)-\varepsilon < h_p(z),$$

contradicting (i).

(ii)⇒(iii). For any sequence $\{z_j\}$ in $\mathbf{C}^n$ with $z_j\to z$, it follows from the mean value property and the Fatou lemma that

$$\varlimsup_{j\to+\infty}(T_{\tau_j}p)(z_j) \leqq h_p(z).$$

Hence it is sufficient to prove that there exists a sequence $z_j\to z$ such that

$$h_p(z) \leqq \varliminf_{j\to+\infty}(T_{\tau_j}p)(z_j). \tag{2.9}$$

By (ii) there exist sequences $\varepsilon_k\to+0$ and $\gamma_k\to+0$ such that

$$h_p(z)-\varepsilon_k < \varliminf_{t\to+\infty}(\mathcal{M}_{\gamma_k} T_t p)(z) \leqq \varliminf_{j\to+\infty}(\mathcal{M}_{\gamma_k} T_{\tau_j} p)(z).$$

By induction we can choose $j_1<j_2<j_3<\ldots\to+\infty$ such that

$$h_p(z)-\varepsilon_k < (\mathcal{M}_{\gamma_k} T_{\tau_j} p)(z) \quad \text{if} \quad j > j_k.$$

If $j_k\leqq j<j_{k+1}$, then there exists z_j satisfying $|z_j-z|\leqq\gamma_k$ such that

$$(\mathcal{M}_{\gamma_k} T_{\tau_j} p)(z) \leqq (T_{\tau_j}p)(z_j).$$

With this choice of z_j the inequality (2.9) holds.

(iii)⇒(iv). This is obvious.

(iv)⇒(i). Assume that $T_{t_j}p\to q\in\mathrm{L}_\infty(p)$, where $t_j\to+\infty$. By Proposition 2.2 (ii) we have $q(z)\leqq h_p(z)$, so we only need to prove $h_p(z)\leqq q(z)$. We take v_j with $\tau_{v_j-1}\leqq t_j<\tau_{v_j}$. Then $\tau_{v_j}/t_j\to 1$. For any $\gamma>0$ the function

$$\mathrm{PSH}(\mathbf{C}^n)\times\mathbf{C}^n\times\mathbf{R}_+ \to \mathbf{R}, \quad (f, w, t) \mapsto (\mathcal{M}_\gamma T_t f)(w)$$

is continuous, so

$$h_p(z) = \lim_{j\to+\infty} (T_{\tau_{v_j}} p)(z_{v_j}) \leq \lim_{j\to+\infty} \big(\mathcal{M}_\gamma T_{\tau_{v_j}/t_j}(T_{t_j}p)\big)(z_{v_j}) = (\mathcal{M}_\gamma q)(z).$$

We let $\gamma\to+0$ and get $h_p(z)\leq q(z)$.

(i)⇒(v). It is obvious that $h_{p+\varphi}(w)\leq h_p(w)+h_\varphi(w)$ for all $w\in\mathbf{C}^n$. Let $\varepsilon>0$. By Proposition 2.2 (ii), there exists $q_1\in \mathrm{L}_\infty(\varphi)$ such that $h_\varphi(z)<q_1(z)+\varepsilon$. If $T_{t_j}\varphi\to q_1$ where $t_j\to+\infty$, then we can pass to a subsequence and assume that $T_{t_j}p\to q_0\in\mathrm{L}_\infty(p)$. Then $q_0+q_1\in\mathrm{L}_\infty(p+\varphi)$ and (i) implies

$$h_p(z)+h_\varphi(z) \leq q_0(z)+q_1(z)+\varepsilon \leq h_{p+\varphi}(z)+\varepsilon.$$

Since ε is arbitrary, (v) holds.

(v)⇒(i). We assume that (i) does not hold, that is $q_0(z)<h_p(z)$ for some $q_0\in\mathrm{L}_\infty(p)$, and prove that there exists $\varphi\in\mathrm{PSH}(\mathbf{C}^n)$, of finite type with respect to the order ϱ, such that $h_{p+\varphi}(z)<h_p(z)+h_\varphi(z)$, contradicting (v). Since $\mathrm{L}_\infty(p)$ is T invariant we may assume $|z|=1$. The continuity of the mean value implies that there exist γ and δ with $0<\delta<\gamma<1$ and $\varepsilon>0$ such that

$$(T_\tau q)(w) \leq (\mathcal{M}_r T_\tau q)(w) < h_p(z)-\varepsilon|w|^\varrho, \tag{2.10}$$

for $0<r<\delta$, $|w-z|<\delta$, $1-\gamma<\tau<1+\gamma$ and q is in some neighborhood U_0 of q_0 in PSH $(\mathbf{C}^n)$. We have $T_{t_j}p\to q_0$ for some $t_j\to+\infty$. By passing to a subsequence we may assume that $T_{t_j}p\in U_0$ for all j and that the balls $B_j=\{w\in\mathbf{C}^n;\ |w-t_jz|\leq\delta t_j\}$ are disjoint.

Now we construct φ. We let $\psi\in C_0^\infty(\mathbf{C}^n)$, $0\leq\psi\leq1$, $\psi(\zeta)=1$ if $|\zeta|\leq1/2$ and $\psi(\zeta)=0$ if $|\zeta|\geq1$, and then define $\psi_j\in C_0^\infty(B_j)$ by $\psi_j(\zeta)=\psi\big((\zeta-t_jz)/t_j\delta\big)$. If ε_1 is sufficiently small, $0<\varepsilon_1<\varepsilon$, then the function φ defined by

$$\varphi(\zeta) = |\zeta|^\varrho+\varepsilon_1|\zeta|^\varrho \textstyle\sum_{j=1}^{+\infty}\psi_j(\zeta) \tag{2.11}$$

is plurisubharmonic. In fact, the smallest eigenvalue of the Levi form of the function $\zeta\mapsto|\zeta|^\varrho$ is $c|\zeta|^{\varrho-2}$ with $c=\min\{\varrho/2, \varrho^2/4\}$ and the second order partial derivatives of $\zeta\mapsto|\zeta|^\varrho\sum\psi_j(\zeta)$ are $O(|\zeta|^{\varrho-2})$. Since $\operatorname{supp}\psi_j\subset B_j$ are disjoint, $0\leq\psi_j\leq1$ and $\psi_j(t_j\zeta)=1$ if $|\zeta-z|\leq1/2$, it follows that φ is of finite type with respect to the order ϱ, $h_\varphi(\zeta)\leq(1+\varepsilon_1)|\zeta|^\varrho$ for all $\zeta\in\mathbf{C}^n$ and $h_\varphi(z)=(1+\varepsilon_1)|z|^\varrho$.

Now we take ε_2 with $0<\varepsilon_2<\varepsilon_1$. Then Hartogs' theorem implies that we can choose δ_1 and $T>0$ such that

$$(T_tp)(w) < h_p(z)+\varepsilon_2|w|^\varrho, \quad |w-z|<\delta_1, \quad t>T. \tag{2.12}$$

We choose δ_1 so small that $0<\delta_1<\delta$, $(1+\delta_1)(1-\gamma)<(1-\delta)$ and $(1-\delta_1)(1+\gamma)>(1+\delta)$. Then the inequalities

$$|w-z|<\delta_1 \text{ and } |\tau w-z|<\delta \text{ imply } (1-\gamma)<\tau<(1+\gamma). \tag{2.13}$$

In order to prove $h_{p+\varphi}(z)<h_p(z)+h_\varphi(z)$, we take $w\in\mathbf{C}^n$ with $|z-w|<\delta_1$ and $t>T$. Assume first that $|tw-t_jz|<t_j\delta$ for some j. Then (2.13) gives $1-\gamma<t/t_j<1+\gamma$, and (2.10) implies

$$(2.14)\qquad \begin{aligned}(T_tp)(w)+T_t\varphi)(w) &= (T_{t/t_j}T_{t_j}p)(w)+(T_t\varphi)(w)\\ &< h_p(z)+(1+\varepsilon_1-\varepsilon)|w|^\varrho.\end{aligned}$$

Assume now that $|tw-t_jz|\geqq t_j\delta$ for all j. Then $(T_t\varphi)(w)=|w|^\varrho$, so (2.12) gives

$$(2.15)\qquad (T_tp)(w)+(T_t\varphi)(w) < h_p(z)+(1+\varepsilon_2)|w|^\varrho.$$

By combining (2.14) and (2.15) we get

$$h_{p+\varphi}(z) \leqq \varlimsup_{w\to z}\varlimsup_{t\to\infty}(T_tp)(w)+(T_t\varphi)(w) \leqq h_p(z)+h_\varphi(z)-\min\{\varepsilon_1-\varepsilon_2, \varepsilon\}.$$

Hence (v) does not hold, and the proof is complete.

3. Growth regularities of Fourier—Laplace transforms

Let $\mu\in\mathscr{A}'(\mathbf{C}^n)$ be carried by the compact convex subset K of $\mathbf{C}^n$ and let $\hat\mu\in\mathscr{A}(\mathbf{C}^n)$ denote its Fourier—Laplace transform. Then $\hat\mu$ is of exponential type. From now on we take $\varrho=1$ in the definition (2.6) of T_t. By (1.8) we have

$$T_t(\log|\hat\mu|)(\zeta) \leqq \frac{\log C_\varepsilon}{t}+H_K(i\bar\zeta)+\varepsilon|\zeta|,\quad \zeta\in\mathbf{C}^n,$$

which implies

$$q(\zeta)\leqq H_K(i\bar\zeta),\quad q\in\mathrm{L}_\infty(\log|\hat\mu|),\quad \zeta\in\mathbf{C}^n.$$

By combining Theorem 2.4 and the Pólya—Ehrenpreis—Martineau theorem we get:

Theorem 3.1. *Let $p\in\mathrm{PSH}(\mathbf{C}^n)$ be of finite type with respect to the order 1, let K be a compact convex subset of $\mathbf{C}^n$ and assume that*

$$q(\zeta)\leqq H_K(i\bar\zeta),\quad q\in L_\infty(p),\quad \zeta\in\mathbf{C}^n.$$

Then there exists an analytic functional μ carried by K, such that

$$T_tp-T_t\log|\hat\mu|\to 0\quad in\quad \mathscr{D}'(\mathbf{C}^n)\quad as\quad t\to+\infty.$$

In the case when μ is carried by some compact subset of $\mathbf{R}^n$, there is a theorem of Martineau (see Hörmander [10, Theorem 9.1.6]), which states that there is a minimal carrier for μ in $\mathbf{R}^n$. It is called the support of μ and is denoted by $\operatorname{supp}\mu$. We let $K=\operatorname{ch}\operatorname{supp}\mu$ denote the convex hull of $\operatorname{supp}\mu$. Then $H_K(i\bar\zeta)=H_K(\operatorname{Im}\zeta)$ and it was proved in [27, Theorem 2.1.2] that

$$h_{\hat\mu}(\zeta) = H_K(\operatorname{Im}\zeta),\quad \zeta\in\mathbf{CR}^n=\{\tau\xi\in\mathbf{C}^n;\ \tau\in\mathbf{C},\ \xi\in\mathbf{R}^n\}.$$

By the Cauchy inequalities, it follows that every distribution $\mu\in\mathscr{E}'(\mathbf{R}^n)$ defines an analytic functional, $\mu\in\mathscr{A}'(\mathbf{C}^n)$. From theorems of Beurling and Vauthier, it follows that for almost every ζ in $\mathbf{C}^n$ with respect to the Lebesgue measure, we have

$$\int_L \left|T_t(\log|\hat{\mu}|)(z\eta+\zeta/t)-H_K(\operatorname{Im} z\eta)\right|d\lambda(z)\to 0 \quad \text{as} \quad t\to+\infty,$$

if L is a compact subset of $\mathbf{C}$ and $\eta\in\mathbf{R}^n$. (See Hörmander [10, Chapter 16].) This implies:

Proposition 3.2. *If* $\mu\in\mathscr{E}'(\mathbf{R}^n)$, *then* $\hat{\mu}$ *is of regular growth in* $\mathbf{CR}^n$ *and* $h_{\hat{\mu}}(\zeta)=H_K(\operatorname{Im}\zeta)=H_K(i\bar{\zeta})$ *for* $\zeta\in\mathbf{CR}^n$.

Let X be a convex subset of $\mathbf{C}^n$ and assume that $X\neq\mathbf{C}^n$. For each $z\in\partial X$ we let $N_X(z)$ denote the *outward normal cone* of ∂X at z, that is

$$\begin{aligned} N_X(z) &= \{\zeta\in\mathbf{C}^n;\ \operatorname{Re}(\langle\bar{w},\zeta\rangle)\leqq\operatorname{Re}(\langle\bar{z},\zeta\rangle) \quad \text{for all } w\in X\} \\ &= \{\zeta\in\mathbf{C}^n;\ H_X(\zeta)=\operatorname{Re}(\langle\bar{z},\zeta\rangle)\}. \end{aligned}$$

The set $N_X(z)$ is a closed convex cone in $\mathbf{C}^n$. The *tangent cone* $T_X(z)$ of ∂X at z is defined as the inverse dual cone of $N_X(z)$,

$$T_X(z)=\{w\in\mathbf{C}^n;\ \operatorname{Re}(\langle\bar{w},\zeta\rangle)\leqq 0 \quad \text{for all } \zeta\in N_X(z)\}.$$

It is a closed convex cone in $\mathbf{C}^n$ and $\bar{X}\subset\{z\}+T_X(z)$ for all $z\in\partial X$.

If $X\subset\mathbf{R}^n\subset\mathbf{C}^n$, then $N_X(x)=N_X^{\mathbf{R}}(x)+i\mathbf{R}^n$, for all $x\in\partial X$, where

$$N_X^{\mathbf{R}}(x)=\{\xi\in\mathbf{R}^n;\ \langle y,\xi\rangle\leqq\langle x,\xi\rangle \quad \text{for all } y\in X\}.$$

We let $V_X^{\mathbf{R}}(x)$ denote the subspace of $\mathbf{R}^n$ generated by $N_X^{\mathbf{R}}(x)$. From Proposition 3.2 and the maximum principle we now get:

Proposition 3.3. *Let* $\mu\in\mathscr{E}'(\mathbf{R}^n)$ *and let* $K=\operatorname{ch}\operatorname{supp}\mu$. *Then* $\hat{\mu}$ *is of regular growth in* $V_K^{\mathbf{R}}(x)+iN_K^{\mathbf{R}}(x)$ *and* $h_{\hat{\mu}}(\zeta)=H_K(\operatorname{Im}\zeta)=H_K(i\bar{\zeta})$ *for* $\zeta\in V_K^{\mathbf{R}}(x)+iN_K^{\mathbf{R}}(x)$ *and* $x\in\partial K$, *the boundary of* K *in* $\mathbf{R}^n$.

We state an interesting special case:

Proposition 3.4. *Let* K *be a compact convex polyhedron in* $\mathbf{R}^n$ *and let* μ *be a distribution with* $\operatorname{ch}\operatorname{supp}u=K$. *Then* $\hat{\mu}$ *is of regular growth in* $\mathbf{C}^n$ *and* $h_{\hat{\mu}}(\zeta)=H_K(\operatorname{Im}\zeta)=H_K(i\bar{\zeta})$ *for all* $\zeta\in\mathbf{C}^n$.

For a discussion of the last three propositions see [27, Section 3.1].

4. Convolution equations in convex domains

In this section we give a relation between the growth regularities of the Fourier—Laplace transform of μ and the existence of solutions of $\mu * u = f$ in convex subsets of $\mathbf{C}^n$.

Theorem 4.1. *Let μ be an analytic functional carried by the compact convex subset K of $\mathbf{C}^n$. Let X_1 and X_2 be open convex subsets of $\mathbf{C}^n$ and assume that $X_1 = X_2 - K$. Consider the following conditions:*

(i) *For every $z \in \partial X_2$ and every $\zeta \in N_{X_2}(z)$, such that ζ lies in the relative boundary of $N_{X_2}(z) \cap \ell_\zeta$ in $\ell_\zeta = \{\tau\zeta \in \mathbf{C}^n; \tau \in \mathbf{C}\}$ the complex line through ζ and 0, the Fourier—Laplace transform $\hat{\mu}$ of μ is of regular growth in the direction of $-i\bar{\zeta}$ and $h_{\hat{\mu}}(-i\bar{\zeta}) = H_K(-\zeta)$.*

(ii) *The pair (X_1, X_2) is μ-convex for carriers.*

(iii) *The convolution equation $\mu * u = f$ has a solution $\mu \in \mathscr{A}(X_1)$ for every $f \in \mathscr{A}(X_2)$.*

(iv) *For every ζ in the closure of $\cup N_{X_2}(z)$, where the union is taken over all differentiable boundary points z of X_2, the Fourier—Laplace transform $\hat{\mu}$ of μ is of regular growth in the direction of $-i\bar{\zeta}$ and $h_{\hat{\mu}}(-i\bar{\zeta}) = H_K(-\zeta)$.*

We have (i)⇒(ii)⇒(iii)⇒(iv).

It is clear that (i) and (iv) are the same condition if X_2 has a differentiable boundary. If $n=1$, then every ζ in the boundary of $N_{X_2}(z)$ in $\mathbf{C}$, is the limit of a sequence of outward normals at differentiable boundary points, so (i) and (iv) are equivalent. We have seen in Section 3, that Fourier—Laplace transforms of distributions are of regular growth in certain directions. As an immediate consequence of Proposition 3.3 we get:

Theorem 4.2. *Let $\mu \in \mathscr{E}'(\mathbf{R}^n)$ and set $K = \operatorname{ch}\operatorname{supp}\mu$. Let X_1 and X_2 be open convex subsets of $\mathbf{C}^n$ satisfying $X_1 = X_2 - K$. If $\zeta \in \mathbf{C}^n$ and ζ lies in the relative boundary of $N_{X_2}(z) \cap \ell_\zeta$ in ℓ_ζ for some $z \in \partial X_2$, implies $-i\bar{\zeta} \in V_K^{\mathbf{R}}(x) + iN_K^{\mathbf{R}}(x)$ for some $x \in \partial K$, then the conditions in Theorem 4.1 hold.*

Remark. The statement of the conditions (i) and (iv) is simply

$$q(-i\bar{\zeta}) = H_K(-\zeta), \quad q \in \mathrm{L}_\infty(\log|\hat{\mu}|).$$

For the proof of Theorem 4.1 we need:

Lemma 4.3. *Let μ be an analytic functional carried by the compact convex subset K of $\mathbf{C}^n$, let X_1 and X_2 be open convex subsets of $\mathbf{C}^n$ and assume that $X_1 = X_2 - K$. Let K_1 be a compact convex subset of X_1 and let M denote the closure in $\operatorname{PSH}(\mathbf{C}^n)$ of the union of all $L_\infty(\log|\hat{\alpha}|)$, where $\alpha \in \mathscr{A}'(\mathbf{C}^n)$ and $\check{\mu} * \alpha$ is carried by K_1. Then M is* *compact in* PSH $(\mathbf{C}^n)$, $h=\sup_{q\in M} q\in \mathrm{PSH}(\mathbf{C}^n)$ *and* h *is positively homogeneous of degree* 1. *Moreover, there exists a compact subset* L *of* X_2, *such that for every* $q\in M$, *there exists* $p\in L_\infty(\log|\hat{\mu}|)$ *such that*

$$q(i\zeta) < H_L(\zeta)+H_K(-\zeta)-p(-i\zeta), \quad \zeta\in\mathbf{C}^n\setminus\{0\}. \tag{4.1}$$

Proof. We set $N=\{q\in\mathrm{PSH}(\mathbf{C}^n);\ q(0)=0,\ q(i\zeta)\leqq H_{K_1}(\zeta),\ \zeta\in\mathbf{C}^n\}$. Then N is a compact subset of $\mathrm{PSH}(\mathbf{C}^n)$ and $\mathrm{L}_\infty(\log|\hat{\beta}|)\subset N$, if $\beta\in\mathscr{A}'(\mathbf{C}^n)$ is carried by K_1. Assume that $\beta=\check{\mu}*\alpha$ for some $\alpha\in\mathscr{A}'(\mathbf{C}^n)$, then $\log|\hat{\alpha}|=\log|\hat{\beta}|-\log|\hat{\check{\mu}}|$. Since the forward orbits of the functions $\log|\hat{\alpha}|$, $\log|\hat{\beta}|$ and $\log|\hat{\check{\mu}}|$ are relatively compact, it follows that every $q\in\mathrm{L}_\infty(\log|\hat{\alpha}|)$ can be written of the form

$$q(\zeta) = q_1(\zeta)-p(-\zeta) \quad \zeta\in\mathbf{C}^n, \tag{4.2}$$

where $q_1\in N$ and $p\in\mathrm{L}_\infty(\log|\hat{\mu}|)$. Since N and $\mathrm{L}_\infty(\log|\hat{\mu}|)$ are compact it follows that M is compact and that (4.2) holds for all $q\in M$. Proposition 2.1 gives that the function h is plurisubharmonic in $\mathbf{C}^n$, and the T invariance of $\mathrm{L}_\infty(\log|\hat{\alpha}|)$ implies that h is positively homogeneous of degree 1. Since $X_1=X_2-K$ there exists a compact convex subset L of X_2 such that $K_1\subset L-K$. The supporting function of $L-K$ is $\zeta\mapsto H_L(\zeta)+H_K(-\zeta)$, so (4.1) follows from (4.2). The proof is complete.

Proof of Theorem 4.1. (i)⇒(ii). Since X_1 is convex it is sufficient to show that the condition in Definition 1.1 holds for every compact convex subset K_1 of X_1. Let the function $h\in\mathrm{PSH}(\mathbf{C}^n)$ be chosen as in Lemma 4.3. It is sufficient to show that for every $w\in\mathbf{C}^n\setminus\{0\}$, there exists a compact convex subset L_w of X_2 and an open conic neighborhood Γ_w of w in $\mathbf{C}^n\setminus\{0\}$ such that

$$h(i\bar{\zeta}) \leqq H_{L_w}(\zeta), \quad \zeta\in\Gamma_w. \tag{4.3}$$

In fact, we then take a finite covering $\Gamma_{w_1},\dots,\Gamma_{w_k}$ of $\mathbf{C}^n\setminus\{0\}$ and set $K_2=\mathrm{ch}(L_{w_1}\cup\dots\cup L_{w_k})$. If $\alpha\in\mathscr{A}'(\mathbf{C}^n)$ and $\check{\mu}*\alpha$ is carried by K_1, then $h_2(i\zeta)\leqq h(i\zeta)\leqq H_{K_2}(\zeta)$, $\zeta\in\mathbf{C}^n$, which implies that α is carried by K_2.

Assume first that $H_{X_2}(w)=+\infty$. We have $H_{X_2}=\sup H_L$, where the supremum is taken over all compact convex subsets L of X_2, so there exists a compact subset L_w of X_2 such that $h(i\bar{w})<H_{L_w}(w)$. Since h is upper semi-continuous, H_{L_w} is continuous and both h and H_{L_w} are homogeneous of degree 1, there exists an open conic neighborhood Γ_w of w such that (4.3) holds.

Assume now that $H_{X_2}(w)<+\infty$. Then $w\in N_{X_2}(z)$ for some $z\in\partial X_2$. We let $\partial(N_{X_2}(z)\cap\ell_w)$ denote the relative boundary of $N_{X_2}(z)\cap\ell_w$ in ℓ_w and let the compact set L be chosen as in Lemma 4.3. From (i) (see the remark above) and (4.1) we get

$$h(i\bar{\zeta}) \leqq H_L(\zeta), \quad \zeta\in\partial(N_{X_2}(z)\cap l_w). \tag{4.4}$$

Since L is a compact subset of X_2 and $\overline{X_2}\subset\{z\}+T_{X_2}(z)$, we can choose an open convex cone G and a point z' in $X_2\setminus L$ such that $\bar{G}\setminus\{0\}$ is contained in the interior of $T_{X_2}(z)$ and $L\subset\{z'\}+G$. We now set $L_w=\mathrm{ch}\,(\{z'\}\cup L)$. Then

$$T_{L_w}(z')\setminus\{0\}\subset G\subset\bar{G}\setminus\{0\}\subset \operatorname{int} T_{X_2}(z),$$

which implies $N_{X_2}(z)\setminus\{0\}$ is contained in the the interior of $N_{L_w}(z')$ and from (4.4) it follows that

$$h(i\bar{\zeta})\leqq H_{L_w}(\zeta),\qquad \zeta\in\partial(N_{X_2}(z)\cap l_w). \tag{4.5}$$

We have $H_{L_w}(\zeta)=\mathrm{Re}\,(\langle\bar{z'},\zeta\rangle)$ for $\zeta\in N_{L_w}(z')$, so H_{L_w} is pluriharmonic in a neighborhood of $N_{X_2}(z)\setminus\{0\}$. Furthermore, both h and H_{L_w} are positively homogeneous of degree 1. Hence the Phragmén—Lindelöf principle implies that (4.3) holds for ζ in $N_{X_2}(z)\cap\ell_w$. We replace L_w by a larger compact convex subset of X_2 such that strict inequality holds in (4.3) for $\zeta\in(N_{X_2}(z)\cap\ell_w)\setminus\{0\}$. Then the upper semicontinuity and the homogeneity give that (4.3) even holds for some conic neighborhood Γ_w of w.

(ii)⇒(iii). This was already proved as Theorem 1.2 (ii).

(iii)⇒(iv). It is sufficient to prove the statement for $\zeta_0\in N_{X_2}(z_0)$, where it is assumed that z_0 is a differentiable boundary point of X_2, $|\zeta_0|=1$ and that there exists a closed ball $B(w_0, r_0)$ contained in $\bar{X}_2$ such that $B(w_0, r_0)\cap\partial X_2=\{z_0\}$. In fact, every outward normal at a differentiable boundary point is the limit of a sequence of outward normals at boundary points with this property and the set $\{\zeta\in\mathbf{C}^n;\ q(-i\bar{\zeta})=H_K(-\zeta)$ for all $q\in\mathrm{L}_\infty(\log|\hat{\mu}|)\}$ is a closed cone. We have $z_0=w_0+r_0\zeta_0$ and by replacing $B(w_0, r_0)$ by a smaller ball we may assume that there exists $\gamma_0>0$ such that $B(w_0-\gamma\zeta_0, r_0)\subset X_2$ if $0<\gamma<\gamma_0$.

We assume that $q_0(-i\bar{\zeta}_0)<H_K(-\zeta_0)$ for some $q_0\in\mathrm{L}_\infty(\log|\hat{\mu}|)$ and prove that it implies that there exists a sequence $\alpha_j\in\mathscr{A}'(X_2)$ such that $\beta_j=\check{\mu}*\alpha_j\in\mathscr{A}'(X_1)$ converges weakly to $\beta\in\mathscr{A}'(X_1)$, $\beta=\check{\mu}*\alpha$ for some $\alpha\in\mathscr{A}'(\mathbf{C}^n)$ but α is not carried by any compact subset of X_2. Then the image $\check{\mu}*(\mathscr{A}'(X_2))$ is not weakly closed in $\mathscr{A}'(X_1)$, and $\mu *\colon \mathscr{A}(X_1)\to\mathscr{A}(X_2)$ is not surjective, contradicting (iii). (See the proof of Theorem 1.2.)

There exists a sequence $t_j\to+\infty$ such that $T_{t_j}(\log|\hat{\mu}|)\to q_0$. Since $q_0(-i\bar{\zeta}_0)<H_K(-\zeta_0)$ and H_K is continuous, there exist $\delta_0>0$ and $\varepsilon_0>0$ such that

$$T_{t_j}(\log|\hat{\mu}|)(-i\bar{\zeta})<H_K(-\zeta)-\varepsilon_0|\zeta|,\quad |\zeta-\zeta_0|<\delta_0, \tag{4.6}$$

holds for all sufficiently large j. By replacing t_j by a subsequence we may assume that the balls $B_j=B(t_j\zeta_0, t_j\delta_0)$ are disjoint and that (4.6) holds for all j. If we combine (1.8) with (4.6), then it follows that for every $\varepsilon>0$ there exists c_ε such that

$$\log|\hat{\mu}(-i\bar{\zeta})|\leqq c_\varepsilon+\varepsilon|\zeta|+H_K(-\zeta)-\varepsilon_0|\zeta|\textstyle\sum_{j=1}^{\infty}\chi_j(\zeta),\quad \zeta\in\mathbf{C}^n, \tag{4.7}$$

where χ_j denotes the characteristic function of the ball B_j. Let $\psi_j \in C_0^\infty(B_j)$ be chosen as in the proof of Theorem 2.5, with ζ_0 in the role of z. We define the function φ by

(4.8) $$\varphi(\zeta) = \mathrm{Re}\,(-i\langle w_0, \zeta\rangle)+r_1|\zeta|+\varepsilon_1|\zeta| \textstyle\sum_{j=1}^{\infty} \psi_j(i\bar{\zeta}), \quad \zeta \in \mathbf{C}^n,$$

where the positive constants ε_1 and r_1 satisfy $r_1+\varepsilon_1=r_0$ and $0<\varepsilon_1<\varepsilon_0$. If ε_1 is chosen sufficiently small then it follows as in the proof of Theorem 2.5, that $\varphi \in \mathrm{PSH}\,(\mathbf{C}^n)$ and φ is of finite type with respect to the order 1. By Theorem 3.1, there exists $\alpha \in \mathscr{A}'(\mathbf{C}^n)$ such that

(4.9) $$T_t\varphi - T_t \log|\hat{\alpha}| \to 0 \quad \text{in} \quad \mathscr{D}'(\mathbf{C}^n) \quad \text{as} \quad t \to +\infty.$$

The functional α is carried by the ball $B(w_0, r_0)$, for

$$h_{\hat{\alpha}}(i\bar{\zeta}) = h_\varphi(i\bar{\zeta}) \leqq \mathrm{Re}\,(\langle \bar{w}_0, \zeta\rangle)+r_0|\zeta| = H_{B(w_0,r_0)}(\zeta), \quad \zeta \in \mathbf{C}^n.$$

Furthermore, α is not carried by any compact subset of X_2, for $|\zeta_0|=1$, $z_0=w_0+r_0\zeta_0 \in \partial X_2$ and

$$h_{\hat{\alpha}}(i\bar{\zeta}_0) = \mathrm{Re}\,(\langle \bar{w}_0, \zeta_0\rangle)+r_0 = \mathrm{Re}\,(\langle \bar{z}_0, \zeta_0\rangle).$$

On the other hand, $\beta=\check{\mu}*\alpha$ is carried by the compact convex subset $K_1=B(w_0, r_1)-K$ of X_1. In fact, we have $\log|\hat{\beta}|=\log|\hat{\mu}|+\log|\hat{\alpha}|$ and (4.9), so for every $q \in \mathrm{L}_\infty(\log|\hat{\beta}|)$ there exists $\tau_j \to +\infty$ such that $T_{\tau_j}(\log|\hat{\mu}|+\varphi) \to q$ as $j \to +\infty$. We add (4.7) and (4.8) and use the fact that $\varepsilon_1<\varepsilon_0$ and $\psi_j \leqq \chi_j$. Then

$$q(i\bar{\zeta}) = \overline{\lim_{w\to\zeta}}\ \overline{\lim_{j\to+\infty}}\,(T_{\tau_j}(\log|\hat{\mu}|+\varphi)(w)$$
$$\leqq \mathrm{Re}\,(\langle \bar{w}_0, \zeta\rangle)+r_1|\zeta|+H_K(-\zeta) = H_{K_1}(\zeta), \quad \zeta \in \mathbf{C}^n.$$

Now we set $z_j=-\gamma_j\zeta_0$, where $0<\gamma_j<\gamma_0$, $\gamma_j \to 0$. Then the sequence $\alpha_j=\delta_{z_j}*\alpha$ has the desired properties. The proof is complete.

Note added in proof. In a recent paper in Izvestija Akad. Nauk SSSR **54:3** (1990), A. S. Krivošeev gave a necessary and sufficient condition for the existence of solution of the inhomogeneous convolution equation. His results are thus stronger than our Theorem 4.1.

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