A Blaschke-type product and random zero sets for Bergman spaces

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0. Introduction

Let $L^p(\mathbf{D})$ $(p \geqq 1)$ be the Banach space of all measurable functions f on the open unit disk $\mathbf{D}=\{z \in \mathbf{C}\colon |z|<1\}$ such that

(0.1) $$\|f\|_p = \left\{\int_{\mathbf{D}} |f(z)|^p \, dA(z)\right\}^{1/p} < \infty,$$

where dA is the normalized Lebesgue area measure. Let A^p be the subspace of $L^p(\mathbf{D})$ consisting of analytic functions. The A^p are usually called the Bergman spaces.

Definition. We say that a subset Λ of the disk $\mathbf{D}$ is a zero-set for the space A^p if there exists a nonzero function $f \in A^p$ such that $f|_\Lambda = 0$.

The purpose of this paper twofold. First we introduce a Blaschke type product whose factors have an extremal property in A^2 similar to the extremal property enjoyed by the classical Blaschke factors for the Hardy spaces H^p. It will be shown that our Blaschke type products converge for all A^p-zero sets, and they are contractive divisors of zeros for A^2. In the second part of the paper we apply these Blaschke type products (or rather their modification) to obtain a result concerning probabilistic characterization of A^p-zero sets. The probabilistic approach to the study of A^p-zero sets was apparently initiated by Emile LeBlanc [B] who obtained the following result:

Theorem ([B]). *Let* $\{r_n\}_{n=1}^\infty$ *be a sequence in* $(0, 1)$ *that satisfies the condition:*

(0.2) $$\limsup_{\varepsilon \to 0} \frac{\sum (1-r_n)^{1+\varepsilon}}{\log 1/\varepsilon} < \frac{1}{2p}.$$

Then for almost all independent choices of $\{\theta_n\}_{n=1}^\infty$ *the set* $\{r_n e^{i\theta_n}\}_{n=1}^\infty$ *is an* A^p-*zero set.*

The proof of this theorem was based on a Blaschke type product introduced by C. Horowitz in [Hor]. It will be shown that our Blaschke type product is more effective and allows us to sharpen the result of E. LeBlanc to its natural limits:

Theorem 1. *Let* $1\leqq p\leqq 2$ *and* $\{r_n\}_{n=1}^{\infty}$ *be a sequence in* $(0, 1)$ *satisfying the condition*

$$\limsup_{\varepsilon\to 0}\frac{\sum_{r_n<1-\varepsilon}(1-r_n)}{\log 1/\varepsilon}<\frac{1}{p}. \tag{0.3}$$

Then for almost all independent choices of $\{\theta_n\}_{n=1}^{\infty}$ *the set* $\{r_n e^{i\theta_n}\}_{n=1}^{\infty}$ *is an* A^p*-zero set. The constant* $1/p$ *is sharp.*

In order to see how this result is related to the condition (0.2) we prove the following

Proposition 0.4. *Let* $\{d_k\}$ *be a sequence of positive numbers, and let*

$$\gamma_1=\limsup_{\varepsilon\to 0}\frac{\sum d_k^{1+\varepsilon}}{\log 1/\varepsilon}, \tag{0.5}$$

$$\gamma_2=\limsup_{\varepsilon\to 0}\frac{\sum_{d_k>\varepsilon} d_k}{\log\log 1/\varepsilon}. \tag{0.6}$$

Then $\gamma_1\leqq\gamma_2\leqq e\gamma_1$.

Proof. Consider the function $m(x)=\sum_{d_k>x} d_k$. It follows from (0.6) that for every $\gamma>\gamma_2$

$$m(x)<\gamma\log\log 1/x$$

for all sufficiently small x. Hence we have

$$\begin{aligned}\sum d_k^{1+\varepsilon}&=-\int_0^1 x^{\varepsilon}\,dm(x)=\varepsilon\int_0^1 x^{\varepsilon-1}m(x)\,dx\\&\leqq\varepsilon\gamma\int_0^1 x^{\varepsilon-1}\log\log 1/x\,dx+c=\varepsilon\gamma\int_0^{\infty}e^{-t\varepsilon}\log t\,dt+c\\&=\gamma\int_0^{\infty}e^{-s}\log(s/\varepsilon)\,dt+c\leqq\gamma\log 1/\varepsilon+c,\end{aligned}$$

where c is some constant which depends only on m and γ. We conclude that $\gamma_1\leqq\gamma$. Since γ can be chosen arbitrarily close to γ_2, we have $\gamma_1\leqq\gamma_2$.

Now let $\gamma>\gamma_1$ and $\varepsilon>0$ be sufficiently small. It follows from (0.5) that

$$\varepsilon\int_0^1 x^{\varepsilon-1}m(x)\,dx\leqq\gamma\log 1/\varepsilon+c$$

for some $c>0$. The function $m(x)$ is decreasing, hence for every $0<a<1$ we get

$$\varepsilon\int_0^1 x^{\varepsilon-1}m(x)\,dx>m(a)a^{\varepsilon}$$

and $m(a) \leqq \gamma a^{-\varepsilon} \log 1/\varepsilon + c$. We can now choose $\varepsilon = (\log 1/a)^{-1}$ and this gives the inequality $\gamma_2 \leqq e\gamma$. ▮

The proposition above shows that E. LeBlanc's condition (0.2) is essentially the double-logarithmic growth of the function

$$\varphi(r) = \sum_{r_n < r} (1 - r_n) \tag{0.7}$$

and the gap between logarithmic and double-logarithmic conditions cannot be overcome with the help of a Horowitz type product.

Theorem 1 (in a slightly stronger form) will be proved in Section 2. In Section 3 a similar result is established for the case $p > 2$; however, in this case we are unable to obtain a sharp constant. Section 3 also contains some open problems and other discussions.

Acknowledgement. The author is grateful to professor B. Korenblum for stimulating discussions and for his attention to this work.

1. A Blaschke-type product

Some notations and definitions. Symbols **C** and **D** will stand for the complex plane and the open unit disk in **C**. $\mathbf{T} = \partial\mathbf{D} = \{z \in \mathbf{C}: |z| = 1\}$ is the unit circle. Let $dA(z)$ be the area measure on **D** normalized so that the area of **D** is 1:

$$dA(z) = \frac{1}{\pi} dx\, dy = \frac{1}{\pi} r\, dr\, d\theta.$$

For $1 \leqq p \leqq \infty$, H^p will denote the classical Hardy space and $A^p(\mathbf{D})$ (or simply A^p) the Bergman space of functions analytic in **D** with the norm

$$\|f\|_p = \left\{\int_{\mathbf{D}} |f(z)|^p \, dA(z)\right\}^{1/p} < \infty.$$

For basic facts about spaces H^p and A^p see [Dur], [Zhu].

$A^2(\mathbf{D})$ is a Hilbert space with the scalar product

$$\langle f, g \rangle = \int_{\mathbf{D}} f(z)\overline{g(z)}\, dA(z).$$

$K_\lambda(z) = (1 - \bar{\lambda} z)^{-2}$ is the reproducing kernel of A^2 so that for every $\lambda \in \mathbf{D}$ and $f \in A^2$

$$\langle f, K_\lambda \rangle = f(\lambda).$$

It is easy to see that $\|K_\lambda\|_2 = (1 - |\lambda|^2)^{-1}$.

Let $d_\lambda = 1 - |\lambda|^2$ and define the function $s_\lambda(z) = d_\lambda^2 K_\lambda(z)$.

For a fixed $\lambda \in \mathbf{D}$ we consider the following problem:

$$\sup\{\Re f(0) : f \in A^2, \|f\|_2 = 1, f(\lambda) = 0\}. \tag{1.1}$$

The solution to an analogous problem for the spaces H^p (for every p, $1\leqq p\leqq\infty$) is the Blaschke factor

$$B_\lambda(z)=\frac{|\lambda|}{\lambda}\frac{\lambda-z}{1-\bar{\lambda}z}.$$

For the space A^2 the extremal function exists and is given in the following

Proposition 1.2. *Let f_λ be the solution to the extremal problem* (1.1). *Then*

$$f_\lambda(0)f_\lambda(z)=1-d_\lambda^2K_\lambda(z).$$

Proof. Let $g\in A^2$ and $g(\lambda)=0$. The function

$$\varphi(t)=\Re\langle f_\lambda+tg,1\rangle\|f_\lambda+tg\|_2^{-1}$$

must have its maximum at $t=0$. This implies that $\Re\langle g,1\rangle=f_\lambda(0)\Re\langle g,f_\lambda\rangle$. Since g can be multiplied by any constant we conclude that the function $1-f_\lambda(0)f_\lambda$ is orthogonal to the space $A_\lambda^2=\{h\in A^2\colon h(\lambda)=0\}$ and hence is equal to cK_λ with some constant $c\in\mathbf{C}$. This constant can be determined from the condition $f_\lambda(\lambda)=0$. ∎

The function f_λ can be written in any of the following forms:

$$f_\lambda(z)=\frac{1-d_\lambda^2K_\lambda(z)}{\sqrt{1-d_\lambda^2}}=\frac{\bar{\lambda}^2(\lambda-z)\left(\frac{2-|\lambda|^2}{\bar{\lambda}}-z\right)}{|\lambda|\sqrt{2-|\lambda|^2}\,(1-\bar{\lambda}z)^2}=\frac{1}{\sqrt{2-|\lambda|^2}}B_\lambda(z)(2-|\lambda|B_\lambda(z)). \tag{1.3}$$

At $\lambda=0$ it is natural to define f_λ by continuity:

$$f_0(z)=\sqrt{2}\,z.$$

We proceed now to the contractive properties of f_λ.

Definition. Let $A_\lambda^2=\{f\in A^2\colon f(\lambda)=0\}$. A function $f\in A_\lambda^2$ is said to be a contractive (or λ-contractive) divisor for the space A^2 if $\|f\|_2\leqq 1$ and for every $g\in A_\lambda^2$, we have g/f in A^2 and

$$\|g/f\|_2\leqq\|g\|_2.$$

Proposition 1.4. *For every $\lambda\in\mathbf{D}$ the function f_λ defined by* (1.3) *is a contractive divisor for the space A^2. Moreover, for every $g\in A^2$*

$$\|f_\lambda g\|_2^2=\|g\|_2^2+\frac{|\lambda|^2d_\lambda^2}{1-d_\lambda^2}\int_{\mathbf{D}}(1-|z|^2)^2|K_\lambda(z)|\,|g'(z)|^2\,dA(z). \tag{1.5}$$

Remarks. 1. The fact that f_λ is a contractive divisor follows immediately from (1.5) and the simple observation that $\|f_\lambda\|_2=1$.

2. Equality (1.5) is an analog of formula (1) [Car] for functions with the finite Dirichlet integral. It is a special case of the formula obtained by H. Hedenmalm (cf. Corollary 4.2 [Hed]).

Proof. We begin with the following:

$$(1-d_\lambda^2)\|f_\lambda g\|_2^2 = \langle (1-d_\lambda^2 K_\lambda)g, (1-d_\lambda^2 K_\lambda)g\rangle$$
$$= (1-d_\lambda^2)\|g\|_2^2 + d_\lambda^2 \int_{\mathbf{D}} \{d_\lambda^2 |K_\lambda(z)|^2 - 2\Re K_\lambda(z)+1\}|g(z)|^2\, dA(z).$$

The expression in braces can be represented as a linear combination of moduli of analytic functions:

$$d_\lambda^2|K_\lambda|^2 - 2\Re K_\lambda + 1 = |K_\lambda - 1|^2 - 2|\lambda|^2|K_\lambda|^2 + |\lambda|^4|K_\lambda|^2$$

and it is easy to check that the last expression is equal to

$$\tfrac{1}{4}|\lambda|^2 \Delta_z\{(1-|z|^2)^2|K_\lambda(z)|\}.$$

Now we can apply Green's formula to obtain (1.5). ▮

Given any function $f\in A^2$ with the zero set $\Lambda_f=\{\lambda\in\mathbf{D}: f(\lambda)=0\}$ we can divide this function successively by the factors $\{f_\lambda\}_{\lambda\in\Lambda_f}$ without increasing the norm of f. Moreover, since for every A^2-zero set Λ

$$\textstyle\sum_{\lambda\in\Lambda}(1-|\lambda|^2)^2<\infty,$$

(this follows from (2.3) below), the product $\prod_{\lambda\in\Lambda} f_\lambda$ converges. We have thus proved the following result:

Theorem 1.6. *Every non-zero function $f\in A^2$ admits a factorization*

$$f = BF \tag{1.7}$$

where $F\in A^2$, F has no zeros in $\mathbf{D}$, $\|F\|_2\leqq\|f\|_2$, and $B=\prod f_\lambda$ is a Blaschke-type product whose zeros coincide with those of f. ▮

This factorization (as well as that of C. Horowitz [Hor] and B. Korenblum [Kor2]) is not quite satisfactory because we can hardly control the A^2-norm of the Blaschke-type product. Nevertheless this product admits good probabilistic estimates, and under some assumptions it turns out to be almost surely in A^2.

On the other hand the factorization discovered recently by Håkan Hedenmalm [Hed] features both factors B and F belonging to A^2. It is interesting to note that Hedenmalm's Blaschke-type factor is the solution (for an arbitrary set Λ) to the extremal problem $\sup\{\Re f(0)\colon \|f\|_2\leqq 1, f|_\Lambda=0\}$, which corresponds to the situation for H^2 and the classical Blaschke product.

2. Random zero sets

For the definition of a random set we will use the probability space $\Omega=\prod_{n=1}^{\infty}\Omega_n$, where Ω_n is the interval $[0, 2\pi)$ for each n. A_n is the σ-field of Lebesgue measurable sets and P_n is the (normalized) Lebesgue measure. An element of Ω is denoted by $\omega=(\theta_1, \theta_2, \dots)$ where $0\leqq\theta_n<2\pi$ for all n. $\{\theta_1, \theta_2, \dots\}$ is a sequence of random independent variables defined on Ω.

For every countable set $\Lambda=\{\lambda_n\}_{n=1}^{\infty}\subset\mathbf{D}$ define a random set Λ_ω as a map $\Omega\to 2^{\mathbf{D}}$, where for every $\omega\in\Omega$ the set Λ_ω is obtained by a rotation of each point $\lambda_n\in\Lambda$ through the angle θ_n:

$$\Lambda_\omega = \{\lambda_n e^{i\theta_n}\}_{n=1}^{\infty}. \tag{2.1}$$

We denote by Λ_r the intersection of the set Λ with the disk $\mathbf{D}_r=\{z\colon |z|\leqq r\}$:

$$\Lambda_r = \{\lambda\in\Lambda\colon |\lambda| \leqq r\}$$

and define the following functions:

$$\varphi(r) = \textstyle\sum_{\lambda\in\Lambda_r}(1-|\lambda|),$$

$$\varphi_1(r) = \textstyle\sum_{\lambda\in\Lambda_r}\log\dfrac{r}{|\lambda|},$$

$$\varphi_2(r) = \textstyle\sum_{\lambda\in\Lambda_r}(1-|\lambda|^2),$$

$$n(r) = \operatorname{card}\Lambda_r. \tag{2.2}$$

It is well-known that any A^p-zero set Λ satisfies the condition

$$\textstyle\sum_{\lambda\in\Lambda} d_\lambda \log^{-1-\varepsilon} 1/d_\lambda < \infty \tag{2.3}$$

for all $\varepsilon>0$ (see [Hor]). From now on we restrict our considerations to the sets Λ which satisfy (2.3). We need the following technical result.

Lemma 2.4. *Let* $\Lambda\subset\mathbf{D}$ *be a discrete set satisfying condition* (2.3). *Then*

$$2\varphi(r)-\varphi_2(r) = O(1) \quad \textit{as} \quad r\to 1, \tag{2.5}$$

$$\varphi_1(r)+(1-r)n(r)-\varphi(r) = O(1) \quad \textit{as} \quad r\to 1, \tag{2.6}$$

$$n(r) = \int_0^r \frac{d\varphi(t)}{1-t}, \quad r\in(0, 1) \quad \textit{and} \quad n(r)\leqq\frac{\varphi(r)}{1-r}. \tag{2.7}$$

Proof. (2.5) and (2.7) are direct consequences of the definition (2.2). The expression in (2.6) is equal to

$$\textstyle\sum_{\Lambda_r}(\log(1/|\lambda|)-1+|\lambda|)+n(r)(\log r-1+r).$$

The first term is bounded by the finite sum $\sum(1-|\lambda|)^2$. The second term is

$O((1-r)^2n(r))$ and hence $O((1-r)\varphi(r))$. The function $\varphi(r)$ admits the following estimate

$$\varphi(r) \leqq \log^{1+\varepsilon}\frac{1}{1-r}\sum_{\Lambda_r}(1-|\lambda|)\log^{-1-\varepsilon}\left(\frac{1}{1-|\lambda|}\right).$$

We conclude that $(1-r)\varphi(r)=o(1)$ as $r\to 1$, and (2.6) is thus proved. ∎

We proceed now to the construction of the Blaschke-type product. For every $\lambda\in\mathbf{D}$ and $s\geqq 1$ define

(2.8) $$b_\lambda^{(s)}(z) = 1-\frac{(1-|\lambda|^2)^s}{(1-\bar{\lambda}z)^s}.$$

When $s=1$, this is equal to

$$b_\lambda^{(1)}(z) = B_\lambda(0)B_\lambda(z),$$

where B_λ is the classical Blaschke factor. For $s=2$ the function $b_\lambda^{(s)}$ coinscides with $f_\lambda(0)f_\lambda(z)$, where f_λ is the extremal function described in Section 1.

For every set $\Lambda\subset\mathbf{D}$ and every function $s=s(\lambda)$ we can define an infinite product

(2.9) $$b_\Lambda^{(s)} = \prod_{\lambda\in\Lambda} b_\lambda^{(s(\lambda))}.$$

Suppose that the functions s satisfies

(2.10) $$\sum_{\lambda\in\Lambda} d_\lambda^{s(\lambda)} < \infty.$$

Then the product (2.9) represents a function holomorphic in $\mathbf{D}$ whose zeros are precisely on Λ.

These Blaschke type products $B_{\Lambda_\omega}^{(s)}$ are instrumental in proving the following result which is somewhat more general then Theorem 1:

Theorem 2.11. *Let* $1\leqq p\leqq 2$ *and* $\Lambda=\{\lambda_n\}_{n=1}^\infty$ *be a discrete subset of the unit disk* $\mathbf{D}$ *that satisfies the condition*

(2.12) $$\int_0^1 e^{p\varphi(r)}\log^\sigma\frac{1}{1-r}\,dr < \infty$$

for some $\sigma>1$. *Then for almost all independent choices of* $\{\theta_n\}_{n=1}^\infty$ *the set* $\Lambda_\omega=\{\lambda_n e^{i\theta_n}\}$ *is an* A^p*-zero set.*

Corollary 2.13 (see Theorem 1). *If* $1\leqq p\leqq 2$ *and*

$$\limsup_{\varepsilon\to 0}\frac{\sum_{r_k<1-\varepsilon}(1-r_k)}{\log 1/\varepsilon} < \frac{1}{p},$$

then for almost all $\omega\in\Omega$ *the set* $\{r_n e^{i\theta_n}\}_{n=1}^\infty$ *is an* A^p*-zero set.*

Proof of Theorem 2.11. Consider the Banach space $L^p(\Omega, A^p)$ of all A^p-valued measurable functions on Ω with the norm

$$\|f\|_{\Omega,p} = \left(\int_\Omega \|f(\omega)\|_{A^p}^p \, d\omega\right)^{1/p}. \tag{2.14}$$

Let Λ be a subset of $\mathbf{D}$ that satisfies (2.12), and Λ_ω be the random set defined by (2.1). Our aim is to construct a sequence $s=\{s_n\}$ so that the product

$$B_{\Lambda_\omega}^{(s)}(z) = \Pi_{n\geqq 1}\left(1-\left(\frac{d_{\lambda_n}}{1-\bar{\lambda}_n\zeta_n z}\right)^{s_n}\right) \tag{2.15}$$

(where $\zeta_n=e^{i\theta_n}$) converges to a holomorphic function in $\mathbf{D}$, which belongs to the space $L^p(\Omega, A^p)$. When this is done, the conclusion of the theorem will follow because the finiteness of the norm (2.14) for the product (2.15) implies that for almost all $\omega\in\Omega$ the function $B_{\Lambda_\omega}^{(s)}$ belongs to A^p. Hence for these ω's the set Λ_ω is an A^p-zero set.

Define a function g on $\mathbf{D}$ by

$$g(z) = \int_\Omega |B_{\Lambda_\omega}^{(s)}(z)|^p \, d\omega. \tag{2.16}$$

We can apply Fubini's theorem to obtain

$$\|B_{\Lambda_\omega}^{(s)}\|_{\Omega,p}^p = \int_{\mathbf{D}} g(z)\, dA(z). \tag{2.17}$$

Our goal is to established that $g\in L^1(\mathbf{D})$.

Lemma 2.18. *Let* $0<p\leqq 2$, $\lambda\in\mathbf{D}$, *and* $s\geqq 1$. *Then*

$$\frac{1}{2\pi}\int_0^{2\pi} |b_\lambda^{(s)}(re^{i\theta})|^p \, d\theta \leqq \left(1+\frac{\Gamma(2s-1)}{\Gamma^2(s)}\frac{d_\lambda^{2s}}{(1-|\lambda|^2r^2)^{2s-1}}\right)^{p/2}. \tag{2.19}$$

Proof. The function $b_\lambda^{(s)}$ defined by (2.8) has the following Taylor expansion:

$$b_\lambda^{(s)}(z) = 1-d_\lambda^s \sum_{n=0}^\infty \frac{\Gamma(n+s)}{n!\,\Gamma(s)}(\bar{\lambda}z)^n. \tag{2.20}$$

Using this expansion we can easily compute the integral (2.19) when $p=2$:

$$\frac{1}{2\pi}\int |b_\lambda^{(s)}(re^{i\theta})|^2 d\theta = (1-d_\lambda^s)^2+\sum_{n=1}^\infty d_\lambda^{2s}\left(\frac{\Gamma(n+s)}{n!\,\Gamma(s)}\right)^2 |\lambda r|^{2n}.$$

Since $\Gamma^2(n+s)\leqq\Gamma(n+1)\Gamma(n+2s-1)$, we have

$$\|b_\lambda^{(s)}(rz)\|_{H^2}^2 \leqq 1+\frac{\Gamma(2s-1)}{(\Gamma(s))^2}\frac{d_\lambda^{2s}}{(1-|\lambda|^2r^2)^{2s-1}},$$

which completes the proof of (2.19) in the case $p=2$. To obtain (2.19) for $p\in(0,2)$ one has to use the inequality $\|f\|_{H^p}\leqq\|f\|_{H^2}$. ∎

Proposition 2.22. *The function* $h(s)=\Gamma(2s-1)/\Gamma^2(s)$ *has the following properties:* (a) $h(1)=1$; (b) $h'(1)=0$; (c) $h''(1)=\frac{\pi^2}{3}$; (d) $h(s)\leqq 1+2(s-1)^2$ *for* $1\leqq s\leqq 2$.

Proof. This follows from the basic properties of the Γ-function. ∎

Proposition 2.23. *Let* $r\in(0,1)$, $\varepsilon>0$, *and* $r_1<1$ *satisfy*

$$1-r_1=(1-r)\log^{-1-\varepsilon}\frac{1}{1-r}. \tag{2.24}$$

Then for every $|\lambda|\geqq r_1$ *and* $s\geqq 1$

$$d_\lambda^{2s}(1-|\lambda|^2r^2)^{1-2s}\leqq c\,d_\lambda\log^{-1-\varepsilon}(1/d_\lambda) \tag{2.25}$$

with some constant c independent of λ *and r.*

Proof. Condition (2.24) and $|\lambda|\geqq r_1$ imply that

$$d_\lambda\leqq 2(1-r_1)=2(1-r)\log^{-1-\varepsilon}1/(1-r).$$

The last inequality is equivalent to

$$d_\lambda(1-r)^{-1}\leqq c_1\log^{-1-\varepsilon}(1/d_\lambda) \tag{2.26}$$

with some $c_1>0$.

We can also deduce from (2.24) that for $|\lambda|>r_1$

$$(1-|\lambda|^2r^2)^{-1}\leqq c_2(1-r)^{-1}. \tag{2.27}$$

Combining (2.26) and (2.27) we obtain

$$d_\lambda(1-|\lambda|^2r^2)^{-1}\leqq c\log^{-1-\varepsilon}1/d_\lambda.$$

Now (2.25) follows from this and the condition $2s-1\geqq 1$. ∎

We are now ready to complete the proof of Theorem 2.11. Fix a positive $\varepsilon<\sigma-1$. For every $\lambda_n\in\Lambda$ we choose s_n as the root of the equation

$$d_{\lambda_n}^{s_n}=d_{\lambda_n}\log^{-1-\varepsilon}1/d_{\lambda_n}, \tag{2.28}$$

or

$$s_n=1+(1+\varepsilon)\frac{\log\log(1/d_{\lambda_n})}{\log(1/d_{\lambda_n})}. \tag{2.29}$$

First, note that condition (2.12) implies the convergence of the series

$$\sum d_\lambda\log^{-1-\varepsilon}(1/d_\lambda)<\infty \tag{2.30}$$

for every positive ε. Indeed, this sum is equal to

$$\int_0^1 \log^{-1-\varepsilon} \frac{1}{1-r}\, d\varphi(r).$$

Integration by parts and an application of Hölder's inequality then lead to (2.30).

Combining (2.30) and (2.28) we see that the product (2.15) converges for every $\omega \in \Omega$.

The independence of $\{\theta_n\}_{n=1}^{\infty}$ implies

$$\int_{\Omega} |B_{\Lambda_\omega}^{(s)}(z)|^p\, d\omega = \prod_{n=1}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} |b_{\lambda_n e^{i\theta_n}}^{(s_n)}(z)|^p\, d\theta_n.$$

Lemma 2.18 and the inequality $1+x \leqq \exp(x)$ result in the following estimate for the function g in (2.16):

(2.31) $$g(z) \leqq \exp\left(\tfrac{p}{2} \sum_{n=1}^{\infty} h(s_n)\, d_{\lambda_n}^{2s_n} (1-|\lambda|^2 r^2)^{1-2s_n}\right).$$

For a fixed $r=|z|$ we split the sum in (2.31) into two parts: the sum over $|\lambda| \geqq r_1$ and the sum over $|\lambda| < r_1$. Let r_1 be defined as in Proposition 2.23. The sum over $|\lambda| \geqq r_1$ is bounded by the finite sum

(2.32) $$c \sum_{|\lambda| \geqq r_1} d_\lambda \log^{-1-\varepsilon}(1/d_\lambda) \leqq \text{const}.$$

(Here we used (2.25) and (2.30)). For the sum over $|\lambda| < r_1$ (*i.e.* $\lambda \in \Lambda_{r_1}$) we have

(2.33) $$\begin{aligned} \sum_{\Lambda_{r_1}} h(s_n)\, d_{\lambda_n} &= \sum_{\Lambda_{r_1}} d_{\lambda_n} + \sum_{\Lambda_{r_1}} (h(s_n)-1)\, d_{\lambda_n} \\ &\leqq \varphi_2(r_1) + 2 \sum (s_n-1)^2 d_{\lambda_n} \leqq 2\varphi(r_1) + c, \end{aligned}$$

where the first inequality follows from Proposition 2.22 and the last one follows from (2.5), (2.29) and (2.30). Combining (2.32) and (2.33) we obtain

$$g(z) \leqq c \exp(p\varphi(r_1)),$$

where r_1 depends on $r=|z|$ as in (2.24). Using (2.12) and the change of variable $r_1 = r_1(r)$ we obtain

$$\int_{\mathrm{D}} g(z)\, dA(z) \leqq c \int_0^1 \exp(p\varphi(r_1))\, dr \leqq c \int_0^1 \exp(p\varphi(r)) \log^{\sigma} \frac{1}{1-r}\, dr < \infty.$$

Hence $B_{\Lambda_\omega}^{(s)}$ belongs to the space $L^p(\Omega, A^p)$ and the proof of Theorem 2.11 is complete. The sharpness of this result will be discussed in Section 3. ▮

3. Concluding remarks and conjectures

3.1. First we discuss how far the condition in Theorem 2.11 is from being necessary. Let Λ be an A^p-zero set, i.e. there exists a nonzero function $f\in A^p$ with $f|_\Lambda\equiv 0$. Without loss of generality we can assume $f(0)\neq 0$. Jensen's formula and Hölder's inequality give (see [Hor])

$$|f(0)|^p \exp\big(p\varphi_1(r)\big) \leqq \frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p\,d\theta \overset{\text{def}}{=} M_p^p(f,r) \tag{3.1}$$

(for the definition of φ_1 see (2.2)). For all functions $f\in A^p$ we have

$$2\int_0^1 rM_p^p(f,r)\,dr = \|f\|_p^p < \infty,$$

$$M_p^p(f,r) \leqq \frac{\|f\|_p^p}{1-r},$$

$$M_p^p(f,r) = o\big((1-r)^{-1}\big).$$

Thus, inequality (3.1) implies the two (generally speaking, not equivalent) necessary conditions for A^p-zero sets:

(1) The L^p-type condition

$$\int_0^1 \exp\big(p\varphi_1(r)\big)\,dr < \infty; \tag{3.2}$$

(2) The L^∞-type condition

$$\exp\big(p\varphi_1(r)\big) = o\big((1-r)^{-1}\big). \tag{3.3}$$

Lemma 2.4 shows that if the set Λ satisfies the following growth condition for the function $n(r)=\operatorname{card}(\Lambda_r)$:

$$n(r) = O\left(\frac{1}{1-r}\right), \tag{3.4}$$

then the function φ_1 in (3.2) and (3.3) can be replaced by the function φ. In this case the necessary condition (3.2) differs from the probabilistic condition (2.12) in Theorem 2.11 by a logarithmic factor. In particular we see that the constant $1/p$ in Theorem 1 is sharp, i.e. cannot be replaced by any larger one.

Conjecture 1. Let Λ be a discrete subset of the disk $\mathbf{D}$ that satisfies (3.2) and (3.3). Then for almost all $\omega\in\Omega$ the set Λ_ω is an A^p-zero set.

The Zero-One law guarantees that for every set Λ either Λ_ω is almost surely (a.s.) an A^p-zero set, or for almost all $\omega\in\Omega$ the set Λ_ω is a set of uniqueness (i.e. not a zero set) for A^p. If Conjecture 1 is true, we have the following refinement of the Zero-One law:

Conjecture 2. The following alternative holds: Either Λ_ω is almost surely an A^p-zero set, or Λ_ω is *never* an A^p-zero set.

In order to justify this conjecture we prove its analog for the class $A^{-\infty}=\bigcup_{p>0} A^p$ of all functions analytic in $\mathbf{D}$, satisfying the growth condition

$$|f(z)| \leqq C_f(1-|z|)^{-n_f} \quad (z\in\mathbf{D}, n_f > 0).$$

This class was studied in detail in [Kor1].

Theorem 3.5. *Let $\Lambda=\{\lambda\}_{\lambda\in\Lambda}$ be a discrete subset of the disk $\mathbf{D}$. The following alternative holds: Either Λ_ω is an $A^{-\infty}$-zero set for almost all $\omega\in\Omega$, or Λ_ω is* never *an $A^{-\infty}$-zero set.*

Proof. If for an $\omega\in\Omega$ the set Λ_ω is an $A^{-\infty}$-zero set, then

$$\sum_{\lambda\in\Lambda_r}(1-|\lambda|) \leqq C\log\frac{1}{1-r} \tag{3.5}$$

with some $C>0$ (see (3.1.4) in [Kor1], or it can be deduced from (3.1)). We can split the set Λ into the union of at most $m=[2C]+2$ disjoint sets: $\Lambda=\bigcup_{k=1}^m \Lambda^k$ so that for every k the set $\Lambda^k=\{\mu\}$ satisfies

$$\limsup_{\varepsilon\to 0}\frac{\sum_{|\mu|<1-\varepsilon}(1-|\mu|)}{\log 1/\varepsilon} < \frac{1}{2}.$$

Corollary 2.13 asserts that the random Blaschke type product $B_{\Lambda^k_\omega}(z)$ with zeros on Λ^k_ω is in A^2 for almost all $\omega\in\Omega^{(k)}$. The product $B_{\Lambda_\omega}=\prod_{k=1}^m B_{\Lambda^k_\omega}$ defined on $\Omega=\prod\Omega^{(k)}$ will be in $A^{2/m}$ for almost all $\omega\in\Omega$, hence Λ_ω is almost surely an $A^{-\infty}$-zero set. ∎

Note that Theorem 3.5, in particular, implies that (3.5) completely characterizes the moduli of $A^{-\infty}$-zero sets, a result first established in [Kor1].

3.2. It is possible to rewrite formulas (3.2), (3.3), and (2.12) in terms of the sequence

$$t_n = \prod_{k=1}^n \frac{1}{r_k},$$

where $\{r_n\}_{n=1}^\infty$ is the nondecreasing rearrangement of the set $\{|\lambda|:\ \lambda\in\Lambda\}$.

Theorem 3.6. *Let Λ be a discrete subset of $\mathbf{D}$. Define the sequence $\{t_n\}$ as above. Then*

1) *If Λ is an A^p-zero set, then*

$$t_n = O(n^{1/p}), \tag{3.7}$$

$$\sum\frac{t_n^p}{n^2} < \infty; \tag{3.8}$$

2) *If* $1 \leqq p \leqq 2$ *and* $\sum t_n^p (r_n - r_{n-1}) \log^\sigma \frac{1}{1-r_n} < \infty$ *with some* $\sigma > 1$, *then the random set* Λ_ω *is almost surely an* A^p*-zero set.*

Proof. (3.7) was proved in [Hor]. The integral in (3.2) is comparable to the sum

$$\sum_{n=1}^{\infty} \frac{t_n^p r_n^{np}}{n^2}.$$

Lemma 2 ([Kah], p. 151) says that this series converges if and only if

$$\sum_{n=1}^{\infty} \frac{t_n^p}{n^2} < \infty,$$

which proves (3.8).

The second part of the theorem can be derived directly from Theorem 2.11 if the integral in (2.12) is replaced by the infinite sum. ∎

3.3. The proof of Theorem 2.11 and Corollary 2.13 can be extended with minor changes to the case $0 < p < 1$. Unfortunately the author was unable to obtain a sharp result for the case $p > 2$. The following result establishes the right rate of growth for the function $\varphi(r)$, but the constant is not sharp.

Theorem 3.9. *Let* $p > 2$ *and* $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ *be a discrete subset of the unit disk* **D** *such that*

$$\limsup_{\varepsilon \to 0} \frac{\sum_{|\lambda_n| < 1-\varepsilon} (1 - |\lambda_n|)}{\log (1/\varepsilon)} < 2^{1-2p}. \tag{3.10}$$

Then the random set Λ_ω *is almost surely an* A^p*-zero set.*

Proof. The key point in the proof of Theorem 2.11 was the estimate for a single Blaschke-type factor $b_\lambda^{(s)}$ obtained in Lemma 2.18. For the case $p > 2$ we need the following estimate:

Lemma 3.11. *Let* $p > 2$, $\lambda \in \mathbf{D}$, *and* $s \geqq 1$. *Then*

$$\frac{1}{2\pi} \int_0^{2\pi} |b_\lambda^{(s)}(re^{i\theta})|^p \, d\theta \leqq 1 + p d_\lambda^s + 2^{2p-2} d_\lambda^{2s} (1 - |\lambda|^2 r^2)^{1-2s}. \tag{3.12}$$

Proof. First consider the case $p = 2n$ with n an integer. Define the function

$$q(e^{i\theta}) = -(1 - |\lambda|^2)^s (1 - \bar{\lambda} r e^{i\theta})^{-s},$$

so that $b_\lambda^{(s)} = 1 + q$ and the integral in (3.12) can be written as

$$\langle (1+q)^n, (1+q)^n \rangle_{H^2} = \sum_{k,l=0}^{n} C_n^k C_n^l \langle q^k, q^l \rangle_{H^2}. \tag{3.13}$$

It is easy to see that $\langle 1, q \rangle_{H^2} = \langle q, 1 \rangle_{H^2} = d_\lambda^s$.

For each $k \geqq 1$ we have the following Taylor expansion for q^k:

$$q^k(z) = (-d_\lambda^s)^k \sum_{m=0}^{\infty} \frac{\Gamma(m+ks)}{m!\,\Gamma(ks)} (\bar{\lambda} r z)^m.$$

Hence for $k, l \geqq 1$ the scalar product of q^k and q^l in H^2 admits the estimate

$$|\langle q^k, q^l \rangle_{H^2}| \leqq d_\lambda^{s(k+l)} \sum_{m=0}^{\infty} \frac{\Gamma(m+ks)\Gamma(m+ls)}{(m!)^2 \Gamma(ks)\Gamma(ls)} |\lambda r|^{2m}.$$

Since

$$\Gamma(m+ks)\Gamma(m+ls) \leqq \Gamma\big(m+(k+l)s-1\big)\Gamma(m+1),$$

we obtain

$$|\langle q^k, q^l \rangle_{H^2}| \leqq h_{k,l}(s)\, d_\lambda^{s(k+l)} \sum_{m=0}^{\infty} \frac{\Gamma\big(m+(k+l)s-1\big)}{m!\,\Gamma\big((k+l)s-1\big)} |\lambda r|^{2m} \tag{3.14}$$

$$= h_{k,l}(s)\, d_\lambda^{s(k+l)} (1-|\lambda r|^2)^{1-(k+l)s},$$

where $h_{k,l}(s) = \Gamma\big((k+l)s-1\big)/\big(\Gamma(ks)\Gamma(ls)\big)$. Combining (3.13) and (3.14) we see that

$$\|1+q\|_{2n}^{2n} \leqq 1 + 2nd_\lambda^s + \sum_{k,l=1}^{n} C_n^k C_n^l h_{k,l}(s)\, d_\lambda^{s(k+l)} (1-|\lambda r|^2)^{1-(k+l)s}.$$

It is not hard to see that

$$\sum_{k,l=1}^{n} C_n^k C_n^l h_{k,l}(s) \leqq 2^{4n-2}$$

for all s sufficiently close to 1. On the other hand, $d_\lambda(1-|\lambda r|^2)^{-1} \leqq 1$ for all $r<1$. Hence we conclude that

$$\|1+q\|_{2n}^{2n} \leqq 1 + 2nd_\lambda^s + 2^{4n-2} d_\lambda^{2s} (1-|\lambda r|^2)^{1-2s},$$

and the proof of the lemma is thus complete for the case $p=2n$.

For an arbitrary p we take $n=[p/2]+1$, apply the previous case, and use the inequality $\|b_\lambda^{(s)}\|_{H^p} \leqq \|b_\lambda^{(s)}\|_{H^{2n}}$. ▮

In order to complete the proof of Theorem 3.9 we mimic the pattern of the proof of Theorem 2.11. We will omit the details here. ▮

3.4. The Blaschke-type product which appeared in the factorization (1.7) of a function $f \in A^2$ is a special case of a more general construction (2.9), with $s(\lambda)=2$. The following theorem describes the probabilistic properties of this product; it can be proved in the same way as Theorem 2.11:

Theorem 3.15. *Let* $f \in A^2$ *and* $\Lambda = \{\lambda \in \mathbf{D}: f(\lambda)=0\}$. *Then the random product*

$$B_{\Lambda_\omega}(z) = \prod_{\lambda_n \in \Lambda} f_{\lambda_n e^{i\theta_n}}(z) \tag{3.16}$$

belongs almost surely to the space $\bigcap_{p<1} A^p$.

If

$$\limsup_{\varepsilon\to 0}\frac{\sum_{|\lambda|<1-\varepsilon}(1-|\lambda|)}{\log 1/\varepsilon}<\frac{1}{4},$$

then the product (3.16) *belongs almost surely to* A^2 *(cf. Corollary* 2.13*).*

3.5. Finally we consider a generalization of Theorem 2.11 to a wide collection of spaces.

Let $k(r)$ $(0\leqq r<1)$ be a nondecreasing function satisfying the following conditions:

(3.17) $$(1)\quad k(r)\to+\infty \quad\text{as}\quad r\to 1,$$

(3.18) $$(2)\quad (1-r)k(r)\to 0 \quad\text{as}\quad r\to 1,$$

(3.19) $$(3)\quad \int_r^1 k(t)\,dt = O((1-r)k(r)) \quad\text{as}\quad r\to 1.$$

Define $A^{(k)}$ as a set of all holomorphic in $\mathbf{D}$ functions satisfying

(3.20) $$\log|f(z)|\leqq a_f k(|z|)+b_f,$$

where constants a_f and b_f depend on f. $A_0^{(k)}$ will denote those functions in $A^{(k)}$ whose constant a_f in (3.20) can be chosen arbitrarily small: $f\in A_0^{(k)}$ if for every $\varepsilon>0$ there exists $C_\varepsilon>0$ such that

$$\log|f(z)|\leqq \varepsilon k(|z|)+C_\varepsilon.$$

When $k(r)=\log(1/(1-r))$, the space $A^{(k)}$ coincides with $A^{-\infty}$ discussed above (see Theorem 3.5).

Theorem 3.21. *Let* $k(r)$ *satisfy conditions* (3.17)—(3.19).

1) *If* $\Lambda=\{\lambda_n\}_{n=1}^\infty\subset\mathbf{D}$ *is an* $A^{(k)}$*-zero set, then*

(3.22) $$\limsup_{r\to 1}\frac{\sum_{|\lambda_n|<r}(1-|\lambda_n|)}{k(r)}=c<\infty.$$

2) *If* $\Lambda\subset\mathbf{D}$ *satisfies* (3.22), *then for almost all independent choices of* $\{\theta_n\}_{n=1}^\infty$ *the set* $\Lambda_\omega=\{\lambda_n e^{i\theta_n}\}$ *is an* $A^{(k)}$*-zero set.*

Both statements remain true if we replace $A^{(k)}$ by $A_0^{(k)}$ and the constant c in (3.22) by 0.

Outline of the proof. The first statement follows easily from Jensen's inequality and condition (3.20). To prove the second statement in the case $k(r)=(1-r)^{-\alpha}$, $0<\alpha<1$, one replaces the sequence $\{s_n\}$ in (2.29) by $s_n=1+\alpha+(1+\varepsilon)/\log(1/d_{\lambda_n})$ and then repeats arguments of the proof of Theorem 2.11. The detailed proof for the general spaces $A^{(k)}$ will appear elsewhere. ∎

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Received Aug. 8, 1990

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