On some L^p-estimates for hyperbolic equations

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Introduction

The aim of this work is to show some L^p-estimates for the operators which are used to represent the solutions of Cauchy problems for hyperbolic equations with constant coefficients.

Now, we set for $d>0$, $k\in\mathbf{R}$, $\xi\in\mathbf{R}^n$

$$m_{k,d}(\xi) = e^{i|\xi|^d}a_k(\xi), \tag{1}$$

where $a_k(\xi)\in S^{-k}$ and $0\notin\operatorname{supp} a_k$. ($S^{-k}$ is the class of smooth functions which satisfy the estimates $|D^\alpha a(\xi)|\leqq C_\alpha(1+|\xi|)^{-k-|\alpha|}$ for all derivations D^α.) The operators $M_{k,d}(D)=F^{-1}m_{k,d}F$ (F denotes the Fourier transform, and F^{-1} its inverse) are used to represent the fundamental solutions of Cauchy problems for Schrödinger equations ($d=1$) and wave equations ($d=2$). It was not so easy to obtain L^p-estimates for them because we cannot apply the famous Marcinkiewicz theorem: "*If* $|D^\alpha m(\xi)|\leqq C_\alpha|\xi|^{-|\alpha|}$ *for* $|\alpha|\leqq[n/2]+1$, *the operator* $M(D)=F^{-1}mF$ *is* L^p-*bounded* ($1<p<\infty$)." (See, for example, Stein [15].) But many authors such as Hirshmann [5], Wainger [17], Fefferman—Stein [3], Sjöstrand [14], Miyachi [10], [11], and Peral [12] contributed to it and gave

Theorem A. (i) *In the case* $d\neq 1$, *the operator* $M_{k,d}(D)$ *is* L^p-*bounded if and only if* $k\geqq nd\left|\frac{1}{p}-\frac{1}{2}\right|$. (ii) *In the case* $d=1$, *the operator* $M_{k,d}(D)$ *is* L^p-*bounded if and only if* $k\geqq(n-1)\left|\frac{1}{p}-\frac{1}{2}\right|$.

We remark that theorem A in the case (i) is valid even if we replace the phase function $|\xi|^d$ in the symbol (1) by any real function $\varphi(\xi)$ which is homogeneous of degree d (that is, $\varphi(\varrho\xi)=\varrho^d\varphi(\xi)$ for $\varrho>0$), while the case (ii) does not have such a generalization. (See Miyachi [10].) This fact, as well as the difference of the critical order for k, expresses the specialty of the case (ii). In this paper, we shall consider to what extent we can generalize the phase function in the case (ii).

This problem is deeply related to the Cauchy problem for hyperbolic equations. Let $p(\tau, \xi)$ be a strictly hyperbolic polynomial of degree m, that is, $p(\tau, \xi)=(\tau-\varphi_1(\xi))\dots(\tau-\varphi_m(\xi))$, where the characteristic roots $\{\varphi_l\}_{l=1}^m$ are homogeneous of degree 1 and are ordered as $\varphi_1(\xi)>\dots>\varphi_m(\xi)$ $(\xi\neq 0)$. Then the solution of the following Cauchy problem

$$(2)\qquad \begin{cases} P(D_t, D_x)u(t, x) = 0, \\ D_t^j u(0, x) = g_j(x); \quad j = 0, 1, 2, \dots, m-1, \end{cases}$$

associated with the polynomial $p(\tau, \xi)$ is represented as

$$u(t, x) = \sum_{l=1}^m \sum_{j=0}^{m-1} M_t^{l,j}(D)g_j$$

up to a regular term. Here the symbol of the operator $M_t^{l,j}(D)$ is of the form

$$m_t^{l,j}(\xi) = e^{it\varphi_l(\xi)}a_{l,j}(\xi),$$

where $a_{l,j}(\xi)\in S^{-j}$ and $0\notin \operatorname{supp} a_{l,j}$. So, in order to obtain L^p-estimates for them, we need to extend Theorem A in the case (ii), replacing the phase function $|\xi|^d$ in the symbol (1) by such functions as characteristic roots of strictly hyperbolic polynomials.

Now, we shall show some properties of characteristic roots $\{\varphi_l\}_{l=1}^\infty$, especially of φ_1 (accordingly of $\varphi_m(\xi)=-\varphi_1(-\xi)$). If $p(\tau, \xi)$ is complete, that is, depends on essentially all variables τ and ξ, we can write $\varphi_1(\xi)=\varphi(\xi)+\alpha(\xi)$ with some homogeneous function $\varphi(\xi)>0$ of degree 1 which is real analytic at $\xi\neq 0$, and with some polynomial $\alpha(\xi)$ of degree 1. Furthermore, the hypersurface $\Sigma=\{\xi\in\mathbf{R}^n;\ \varphi(\xi)=1\}$ is strictly convex, that is, every tangent plane of Σ never lies on Σ except for the tangent point. Particularly, in case of $m=2$, the Gaussian curvature of Σ never vanishes. On the other hand, the factor $e^{it\alpha(\xi)}$ corresponds to the translation of variables, so it is negligible for L^p-estimates. About the properties of strictly hyperbolic polynomials described here, consult Beals [1, Section 5] and the papers cited therein.

Then we set a symbol on $\mathbf{R}^n$ as

$$(3)\qquad m_k(\xi) = e^{i\varphi(\xi)}a_k(\xi),$$

where $\varphi(\xi)>0$ is homogeneous of degree 1 and real analytic at $\xi\neq 0$, and $a_k(\xi)\in S^{-k}$, $0\notin \operatorname{supp} a_k$. When is the operator $M_k(D)=F^{-1}m_kF$ L^p-bounded? And what effect does the hypersurface $\Sigma=\{\xi\in\mathbf{R}^n;\ \varphi(\xi)=1\}$ have on this problem? There have already been the following answers to this given by Miyachi [11] and Beals [1].

Theorem B. (i) *In the case that the Gaussian curvature of Σ never vanishes, the operator $M_k(D)$ is L^p-bounded if $k \geqq (n-1)\left|\frac{1}{p}-\frac{1}{2}\right|$.* (ii) *In the case that Σ is strictly convex, the operator $M_k(D)$ is L^p-bounded if $k > (n-1)\left|\frac{1}{p}-\frac{1}{2}\right|$.*

The condition that the Gaussian curvature of Σ never vanishes implies that Σ is strictly convex, but not vice versa. See, for example, Kobayashi—Nomizu [7, Chapter 7]. The following is our main theorem which claims that theorem B in the case (ii) is valid for $k=(n-1)\left|\frac{1}{p}-\frac{1}{2}\right|$ as well.

Main Theorem. *If the hypersurface Σ is strictly convex and $k \geqq (n-1)\left|\frac{1}{p}-\frac{1}{2}\right|$, the operator $M_k(D)$ is bounded on the Sobolev spaces $H_p^s(\mathbf{R}^n)$ $(1<p<\infty,\ s\in\mathbf{R})$ and the Besov spaces $B_{p,q}^s(\mathbf{R}^n)$ $(1\leqq p,\ q\leqq\infty,\ s\in\mathbf{R})$.*

Here the Besov spaces are a generalization of classes of Hölder continuous functions. For instance, $B_{\infty,\infty}^s\,(s>0)$ is "almost" the same as the class of functions which are $[s]$-times differentiable and whose derivatives are Hölder continuous of order $s-[s]$. So the boundedness theorems on these spaces are useful to discuss the regularity of the solution of problem (2) in a classical sense. For more information about Sobolev spaces and Besov spaces, see Bergh—Löfström [2] and Triebel [16].

We cannot show any results about the case when the hypersurface Σ is not necessary convex. Although Marshall [9] treated this case and gave some estimates, our results are not included in his results. Recently, without any assumptions on the hypersurface Σ, Seeger—Sogge—Stein [13] have shown some results which contains our main theorem.

The proof of our main theorem is based on two theories. One is the analysis of Fourier integrals with degenerating phase functions, and the other is the theory of Hardy spaces. The following sections are devoted to the details of them.

Finally, we remark on notation. Throughout this paper, the capital "C" (with some indices) in estimates always denotes a constant (depending on the indices) which may be different in each occasion.

1. Fourier integral

As we will see later in the following sections, we need estimates for the convolution kernel $K(x)=F^{-1}m_k(x)$ to prove the main theorem, and we can reduce it to the analysis of a Fourier integral which we shall describe hereafter.

Let $U\subset\mathbf{R}^n$ be an open neighbourhood of the origin, and let $h\colon U\to\mathbf{R}$ be a real analytic function which is convex or concave, that is, the Hessian matrix h'' is semi-definite (not necessary definite). Then we set for $t\in\mathbf{R}$ and $z\in U$,

$$I(t;\,z) = \int_U e^{itE(y;z)}g(y)\,dy. \tag{1.1}$$

Here the phase function $E(y; z)$ is defined by

$$E(y;\ z) = h(y) - h(z) - h'(z)\cdot(y-z),$$

and the amplitude function $g(y)\in C_0^\infty(U)$.

We shall investigate the asymptotic behavior of the function $I(t; z)$ with respect to the variable t at infinity. In this case, the parameter $z\in U$ denotes the critical point of the function $y\mapsto E(y; z)$, but there is a possibility that the Hessian matrix $E''_{yy}(y; z)=h''(y)$ degenerates there. So we cannot use the stationary phase method as used in Hörmander [6], but need more precise discussions. For our aim, we shall split the integral (1.1) into two parts, the part near the critical point z and the other part away from it:

$$I_1(t;\ z) = t^{-n/3}\int e^{itE(t^{-1/3}y+z;\ z)} g(t^{-1/3}y+z)\chi(|y|)\,dy, \tag{1.2}$$

$$I_2(t;\ z) = t^{-n/3}\int e^{itE(t^{-1/3}y+z;\ z)} g(t^{-1/3}y+z)(1-\chi)(|y|)\,dy. \tag{1.3}$$

Here $\chi(\varrho)\in C_0^\infty(\mathbf{R})$ and $\chi\equiv 1$ near the origin. Then for the functions I_j and $I'_j=dI_j/dt$ $(j=1, 2)$ we have the following proposition.

Proposition 1.1. *If* $\delta>0$ *is sufficiently small, then for* $t>0$, $z\in B_\delta=\{x\in U; |x|\leqq\delta\}$, $g\in C_0^\infty(B_\delta)$, *and* $l\geqq n/2$

$$|I_1(t;\ z)| \leqq Ct^{-n/2}|\det h''(z)|^{-1/2}, \tag{1.4}$$

$$|tI'_1(t;\ z)| \leqq Ct^{-n/2}(|\det h''(z)|^{-1/2} + t^{-1/3}|\det h''(z)|^{-3/2}), \tag{1.5}$$

$$|I_2(t;\ z)|,\ |tI'_2(t;\ z)| \leqq C_l t^{-n/2}\zeta_l(t)|\det h''(z)|^{-(2l-n+1)/2}. \tag{1.6}$$

Here the constants C *and* C_l *are independent of the variables* t *and* z, *and*

$$\zeta_l(t) = \begin{cases} t^{(2n+1-4l)/6} & \text{if } \frac{n}{2}\leqq l<\frac{n+1}{2}, \\ t^{(n-2l)/6}|\log t|^{(n+2-2l)/2} & \text{if } \frac{n+1}{2}\leqq l<\frac{n}{2}+1, \\ t^{(n-2l)/6} & \text{if } l\geqq\frac{n}{2}+1. \end{cases}$$

We remark that the estimates (1.6) with $l=n/2$ and $l=(n+1)/2$ give respectively

$$|I_2(t;\ z)|,\ |tI'_2(t;\ z)| \leqq C_l t^{-n/2+1/6}|\det h''(z)|^{-1/2}, \tag{1.6'}$$

$$|I_2(t;\ z)|,\ |tI'_2(t;\ z)| \leqq C_l t^{-n/2-1/6}|\log t|^{1/2}|\det h''(z)|^{-1}. \tag{1.6''}$$

The proofs of estimates (1.4) and (1.5) are carried out by the usual stationary phase method. If we set

$$\varphi(y;\ t, z) = \sum_{|\alpha|=3}\frac{y^\alpha}{\alpha!}\int_0^1(1-\theta)^2\frac{\partial^\alpha h}{\partial x^\alpha}(\theta t^{-1/3}y+z)\,d\theta$$

and

$$f(y;\ t, z) = e^{i\varphi(y;\ t, z)}g(t^{-1/3}y+z)\chi(|y|),$$

equality (1.2) is transformed as

$$I_1(t;\, z) = t^{-n/3}\int e^{(i/2)t^{1/3}\langle h''(z)y,\, y\rangle} f(y:\, t, z)\, dy$$

$$= \left(\frac{2\pi}{t}\right)^{n/2} \frac{e^{i(\pi/4)\operatorname{sgn} h''(z)}}{|\det h''(z)|^{1/2}} \int e^{-(i/2)t^{-1/3}\langle h''(z)^{-1}\eta,\, \eta\rangle} \hat{f}(\eta;\, t, z)\, d\eta,$$

where $\hat{f}(\eta; t, z)$ is the Fourier transform of the function $y\mapsto f(y; t, z)$. Here we have used Taylor's formula, Fourier's inversion formula and Fresnel's integral. (See Hörmander [6, p. 145].) Estimates (1.4) and (1.5) are easily obtained from this if we notice the estimates $\| |\eta|^k \hat{f}\|_{L^1(\mathbf{R}^n)} \leqq C \|f\|_{H_2^s(\mathbf{R}^n)}$ $(k\geqq 0,\ s>n/2+k)$.

In order to obtain estimate (1.6), we rewrite equality (1.3) with polar coordinates as

$$I_2(t;\, z) = t^{-n/3}\int_{S^{n-1}} G(t;\, z, \omega)\, d\omega, \tag{1.7}$$

where

$$G(t;\, z, \omega) = \int_0^\infty e^{itF(t^{-1/3}\varrho;\, z, w)}\beta(\varrho, t;\, z, \omega)\, d\varrho,$$

$$F(\varrho;\, z, \omega) = h(\varrho\omega+z)-h(z)-\varrho h'(z)\cdot\omega,$$

$$\beta(\varrho, t;\, z, \omega) = g(t^{-1/3}\varrho\omega+z)(1-\chi)(\varrho)\varrho^{n-1}.$$

For the sake of simplicity, we shall often omit the parameters z and ω.

Now, for $l=0, 1, 2, \dots$, integration by parts yields

$$G(t) = \int_0^\infty e^{itF(t^{-1/3}\varrho)}(L^*)^l \beta(\varrho, t)\, d\varrho. \tag{1.8}$$

Here

$$L = \frac{1}{it^{2/3}F'(t^{-1/3}\varrho)}\frac{\partial}{\partial\varrho}$$

and L^* is the transpose of L. By induction, we easily have

$$(L^*)^l = \left(\frac{i}{t}\right)^l \sum t^{r/3}\frac{F^{(1+s_1)}\dots F^{(1+s_p)}}{(F')^q}(t^{-1/3}\varrho)\frac{\partial^r}{\partial\varrho^r}, \tag{1.9}$$

where the summation $\sum$ is a finite sum of the cases $s_1+\dots+s_p+r=l$, $q-p=l$. By the analyticity and the convexity (concavity) of the function h, the derivatives $F^{(m)}$ satisfy the following lemma.

Lemma 1.2. *If $\delta>0$ is sufficiently small and if $|\varrho|, |z|\leqq\delta$, then for $m=0, 1, 2, \dots$*

$$|F'(\varrho)| \geqq C\varrho|\langle h''(z)\omega, \omega\rangle|,$$

$$|F^{(m)}(\varrho)| \leqq C_m\varrho^{1-m}|F'(\varrho)|.$$

Here the constants C and C_m are independent of the variables ϱ, z and ω.

Proof. See Beals [1, lemma 3.2, 3.3].

We may assume that $t^{-1/3}\varrho$ is sufficiently small and t is sufficiently large. Then if we use lemma 1.2, we obtain from equalities (1.8) and (1.9)

$$|G(t)| \leqq C_l t^{-l} \sum \int_0^\infty \left| t^{r/3} \frac{F^{(1+s_1)} \dots F^{(1+s_p)}}{(F')^q} (t^{-1/3}\varrho) \frac{\partial^r \beta}{\partial \varrho^r}(\varrho, t) \right| d\varrho$$

$$\leqq \frac{C_l t^{-l/3}}{|\langle h''(z)\omega, \omega\rangle|^l} \sum_{r\leqq l} \int_0^\infty \left| \frac{\partial^r \beta}{\partial \varrho^r}(\varrho, t) \right| \frac{d\varrho}{\varrho^{2l-r}}$$

$$\leqq \frac{C_l t^{-l/3}}{|\langle h''(z)\omega, \omega\rangle|^l} \begin{cases} t^{(n-2l)/3} & \text{if } l < \frac{n}{2}, \\ \log t & \text{if } l = \frac{n}{2}, \\ 1 & \text{if } l > \frac{n}{2}. \end{cases}$$

For $l \geqq n/2$, which is not necessarily integer, we have by interpolation

$$|G(t)| \leqq \frac{C_l t^{-n/6}}{|\langle h''(z)\omega, \omega\rangle|^l} \zeta_l(t).$$

From this and equality (1.7), we can obtain estimate (1.6) with the function $I_2(t; z)$ if we use the following lemma.

Lemma 1.3. *Let Q be a positive-definite quadratic form on $\mathbf{R}^n$, and let λ be the maximal eigenvalue of Q. Then, if $l \geqq n/2$,*

$$\int_{S^{n-1}} \frac{d\omega}{\langle Q\omega, \omega\rangle^l} \leqq C_{n,l} \frac{\lambda^{((2l-n)(n-1))/2}}{(\det Q)^{(2l-n+1)/2}}.$$

Proof. Let $\{\lambda_j\}_{j=1}^n$ be the eigenvalues of the quadratic form Q. Then we have

$$\int_{S^{n-1}} \frac{d\omega}{\langle Q\omega, \omega\rangle^l} = \frac{2}{\Gamma(l)} \int_{S^{n-1}} d\omega \cdot \int_0^\infty e^{-\varrho^2 \langle Q\omega, \omega\rangle} \varrho^{2l-1} d\varrho$$

$$= \frac{2}{\Gamma(l)} \int_{\mathbf{R}^n} e^{-(\lambda_1 x_1^2 + \dots + \lambda_n x_n^2)} |x|^{2l-n} dx_1 \dots dx_n$$

$$\leqq C_{n,l} \sum_{j=1}^n \int_{\mathbf{R}^n} e^{-(\lambda_1 x_1^2 + \dots + \lambda_n x_n^2)} |x_j|^{2l-n} dx_1 \dots dx_n$$

$$= C_{n,l} \sum_{j=1}^n \int_{\mathbf{R}^n} e^{-|x|^2} \left(\frac{|x_j|}{\sqrt{\lambda_j}} \right)^{2l-n} \frac{dx_1 \dots dx_n}{\sqrt{\lambda_1 \dots \lambda_n}}$$

$$\leqq C_{n,l} \frac{\lambda^{((2l-n)(n-1))/2}}{(\det Q)^{(2l-n+1)/2}}.$$

This yields the lemma.

Estimate (1.6) with the derivative $I_2'(t; z)$ is obtained in a similar way if we notice the equality

$$G'(t) = i\int_0^\infty e^{itF(t^{-1/3}\varrho)}F(t^{-1/3}\varrho)\beta(\varrho, t)\,d\varrho$$

$$-\frac{i}{3t^{1/3}}\int_0^\infty e^{itF(t^{-1/3}\varrho)}F'(t^{-1/3}\varrho)\varrho\beta(\varrho, t)\,d\varrho+\int_0^\infty e^{itF(t^{-1/3}\varrho)}\frac{\partial\beta}{\partial t}(\varrho, t)\,d\varrho$$

$$= i\int_0^\infty e^{itF(t^{-1/3}\varrho)}(L^*)^{l+1}\big(F(t^{-1/3}\varrho)\beta(\varrho, t)\big)\,d\varrho$$

$$+\frac{1}{3t}\int_0^\infty e^{itF(t^{-1/3}\varrho)}(L^*)^l\frac{\partial(\varrho\beta)}{\partial\varrho}(\varrho, t)\,d\varrho+\int_0^\infty e^{itF(t^{-1/3}\varrho)}(L^*)^l\frac{\partial\beta}{\partial t}(\varrho, t)\,d\varrho.$$

We shall omit the details.

2. Convolution kernel

In this section, we shall investigate the properties of the convolution kernel

$$K(x) = F^{-1}m_k(x) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} e^{i\{x\cdot\xi+\varphi(\xi)\}}a_k(\xi)\,d\xi, \tag{2.1}$$

especially the relationship with the geometrical properties of the hypersurface $\Sigma=\{\xi\in\mathbf{R}^n;\ \varphi(\xi)=1\}$. We may assume, without loss of generality, that $a_k(\xi)$ is homogeneous of degree $-k$ for large $|\xi|$ and vanishes near the origin. Since the main theorem in the case $n=1$ is trivial, we shall assume $n\geqq 2$ throughout this section.

Let the Gauss map ν of the hypersurface Σ be

$$\nu\colon\ \Sigma\ni p\mapsto\frac{\nabla\varphi(p)}{|\nabla\varphi(p)|}\in S^{n-1},$$

and let $\varkappa(p)$ be the Gaussian curvature at the point $p\in\Sigma$ with respect to the Gauss map ν. By the strict convexity of the hypersurface Σ, the Gauss map ν is homeomorphism. The following proposition says that the kernel $K(x)$ has a singularity on the hypersurface

$$\Sigma^* = \{-\nabla\varphi(\xi);\ \xi\in\Sigma\} = \{x;\ H(x) = 0\},$$

where

$$H(x) = |x|-\big|\nabla\varphi\big(\nu^{-1}(-x/|x|)\big)\big|.$$

Proposition 2.1. (i) *The kernel $K(x)$ is smooth in $\mathbf{R}^n\setminus\Sigma^*$ and we have*

$$\left(\frac{\partial}{\partial x}\right)^\beta K(x) = O(|x|^{-M})\quad\text{as}\quad |x|\to\infty \tag{2.2}$$

for every derivative $\left(\frac{\partial}{\partial x}\right)^{\beta} K(x)$ *and for every* $M>0$. (ii) *There exists a decomposition* $K(x)=\sum_{j=1}^{\infty} K_j(x)$ *such that, for sufficiently small* $\varepsilon, \eta \geqq 0$, *every term* $K_j(x)$ *has the estimate*

$$(2.3) \qquad \left\| \varkappa\left(\nu^{-1}(-x/|x|)\right)^{-\eta} H(x)^{\varepsilon} \left(\frac{\partial}{\partial x}\right)^{\beta} K_j(x) \right\|_{L^1} \leqq C_{\beta,\varepsilon,\eta} 2^{j((n-1)/2-k+|\beta|-\varepsilon)}.$$

Here the constant $C_{\beta,\varepsilon,\eta}$ *is independent of the number* j.

The expression (2.1) of the kernel $K(x)$ is in the sense of an oscillatory integral, therefore we can rewrite it as

$$(2.4) \qquad K(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{i\{x\cdot\xi+\varphi(\xi)\}} (L^*)^M a_k(\xi)\, d\xi$$

for all positive integers M. Here

$$L = \frac{(x+\nabla\varphi)\cdot\nabla_{\xi}}{i|x+\nabla\varphi|^2}$$

and L^* is the transpose of L. From this we can easily obtain the property (2.2).

In order to prove estimates (2.3), we shall (micro)localize the problem. That is, by the compactness of the sphere S^{n-1} and the rotation invariance of the geometrical properties, we may assume that the function $a_k(\xi)$ in equality (2.1) is supported in a sufficiently small open conic neighbourhood Γ of the point $e_n=(0, \dots, 0,1)\in S^{n-1}$. Then by equality (2.4) again, we can see that we have only to pay attention to x near the point $-\nabla\varphi(e_n)\in\Sigma^*$. Since Euler's identity $\varphi(\xi)=\xi\cdot\nabla\varphi(\xi)$ yields $\varphi'_{x_n}(e_n)=\varphi(e_n)>0$, the hypersurface Σ can be expressed locally as

$$\Sigma\cap\Gamma = \{(y, h(y));\ y\in U\}$$

by the implicit function theorem. Here $U\subset\mathbf{R}^{n-1}$ is a sufficiently small open neighbourhood of the origin and h: $U\to\mathbf{R}$ is a real analytic function. The strict convexity of the hypersurface Σ implies that the function h is concave and the map h': $U\to h'(U)\subset\mathbf{R}^{n-1}$ is homeomorphism.

In this situation, we shall rewrite estimate (2.3) in terms of the function h. For the point x near $-\nabla\varphi(e_n)\in\Sigma^*$, we define the point $z\in U$ by $\Sigma\ni(z, h(z))=\nu^{-1}(-x/|x|)$. If we write $x=(x', x_n)$, $x'=(x_1, \dots, x_{n-1})$, this is equivalent to the equality

$$(2.5) \qquad h'(z) = -\frac{x'}{x_n}$$

because of the trivial equality

$$(2.6) \qquad \frac{-x}{|x|} = \frac{\nabla\varphi}{|\nabla\varphi|}(z, h(z))$$

and of the fact that the vector $(-h'(z), 1)$ is normal to the hypersurface Σ at the point $(z, h(z))$.

Then we know that the Gaussian curvature $\varkappa$ is represented as

$$\varkappa(\nu^{-1}(-x/|x|)) = \frac{(-1)^{n-1}\det h''(z)}{\{1+|\nabla h(z)|^2\}^{(n+1)/2}}. \tag{2.7}$$

See, for example, Flanders [4, p. 126]. On the other hand, by Euler's identity $(z, h(z))\cdot\nabla\varphi(z, h(z))=1$ and by equalities (2.5) and (2.6), we have

$$H(x) = -x_n\left|\nabla\varphi\left(\nu^{-1}(-x/|x|)\right)\right|\left(x_n^{-1}+h(z)-h'(z)\cdot z\right). \tag{2.8}$$

Besides, we set $K_j(x)$ as

$$K_j(x) = F_\xi^{-1}\left[m_k(\xi)\Phi_j(x_n\varphi(\xi))\right](x)$$

$$= \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} e^{i\{x\cdot\xi+\varphi(\xi)\}}a_k(\xi)\Phi_j\left(x_n\varphi(\xi)\right)d\xi.$$

Here $\{\Phi_j(t)\}_{j=0}^\infty$ is a partition of unity of Littlewood—Paley, that is,

$$\Phi(t)\in C_0^\infty(\{t;\ t>0\}),\quad \Phi_j(t) = \Phi(|t|/2^j)\quad (j\geqq 1)\quad\text{and}\quad \textstyle\sum_{j=1}^\infty \Phi_j(t) = 1.$$

By the change of variables $\xi\mapsto(ty, th(y))$ and $t\mapsto x_n^{-1}t$ $(t>0,\ y\in U)$, we have for large j

$$K_j(x) = \frac{(x_n)^{k-n}}{(2\pi)^n}\int_0^\infty\int_U e^{it(x_n^{-1}+h(y)-h'(z)\cdot y)}t^{n-1-k}\Phi(t2^{-j})g(y)\,dt\,dy. \tag{2.9}$$

Here we have used equality (2.5) again, and $g\in C_0^\infty(U)$ is a function which is supported in a sufficiently small neighbourhood of the origin.

Then, by equalities (2.5), (2.7), (2.8) and (2.9), the proof of estimate (2.3) in the case $\beta=0$ is reduced to the estimate

$$\left\||\det h''(z)|^{1-\eta}\tau^\varepsilon F_t^{-1}[I(t;\ z)t^{n-1-k}\Phi(t2^{-j})](\tau)\right\|_{L^1(\mathbf{R}\times U)} \leqq C_{\varepsilon,\eta}2^{j((n-1)/2-k-\varepsilon)}, \tag{2.10}$$

$(\tau\in\mathbf{R},\ z\in U)$ if we change variables in the order $x'\mapsto x_nx'$, $x_n\mapsto\tau^{-1}$, $x'\mapsto -h'(z)$ and $\tau\mapsto\tau-h(z)+h'(z)\cdot z$. Here the function

$$I(t;\ z) = \int_U e^{it\{h(y)-h(z)-h'(z)\cdot(y-z)\}}g(y)\,dy$$

is the same as in equality (1.1) with n replaced by $n-1$. Furthermore, if we notice the equality

$$\|\tau^\varepsilon F_t^{-1}[I(t;\ z)t^{n-1-k}\Phi(t2^{-j})](\tau)\|_{L^1(\mathbf{R})}$$

$$= 2^{j(n-1-k-\varepsilon)}\|\tau^\varepsilon F_t^{-1}[I(2^jt;\ z)t^{n-1-k}\Phi(t)](\tau)\|_{L^1(\mathbf{R})},$$

and notice that the function $|\det h''(z)|^{-(\varepsilon+\eta)}$ is locally integrable for sufficiently small $\varepsilon, \eta \geqq 0$ by the Weierstrass preparation theorem, we can see that estimate (2.10) is obtained from the estimate

$$\|\tau^\varepsilon F_t^{-1}[I(2^j t;\, z)\Phi(t)](\tau)\|_{L^1(\mathbf{R})} \leqq \frac{C}{2^{j(n-1)/2}|\det h''(z)|^{1+\varepsilon}} \tag{2.11}$$

($z \in U$). Here we have replaced $t^{n-1-k}\Phi(t)$ by $\Phi(t)$ again.

To obtain estimate (2.11), we shall use the following lemma.

Lemma 2.2. *If* $0 \leqq \varepsilon < \frac{1}{2}$, *there exists a constant* C *such that the estimate*

$$\|\tau^\varepsilon \hat{f}(\tau)\|_{L^1(\mathbf{R})} \leqq C(\|f\|_{L^2(\mathbf{R})})^{1/2-\varepsilon}(\|f'\|_{L^2(\mathbf{R})})^{1/2+\varepsilon}$$

holds for all functions f *on* $\mathbf{R}$.

Proof. From the Schwarz inequality and the Plancherel theorem, we obtain

$$\int_{|\tau|>t} |\tau^\varepsilon \hat{f}(\tau)|\, d\tau = \int_{|\tau|>t} |\tau^{\varepsilon-1}\cdot \tau\hat{f}(\tau)|\, d\tau \leqq Ct^{\varepsilon-1/2}\|f'\|_{L^2}.$$

Similarly we have

$$\int_{|\tau|<t} |\tau^\varepsilon \hat{f}(\tau)|\, d\tau \leqq Ct^{\varepsilon+1/2}\|f\|_{L^2}.$$

Choosing $t=\|f'\|_{L^2}/\|f\|_{L^2}$, we have the lemma.

From this lemma, we obtain

$$\begin{aligned}&\|\tau^\varepsilon F_t^{-1}[I(2^j t)\Phi(t)](\tau)\|_{L^1(\mathbf{R})}\\ &\leqq C\left\{\|I(2^j t)\|_{L^\infty(A)} + (\|I(2^j t)\|_{L^\infty(A)})^{1/2-\varepsilon}(\|2^j I'(2^j t)\|_{L^\infty(A)})^{1/2+\varepsilon}\right\},\end{aligned}$$

where $A=\operatorname{supp}\Phi \subset \{t;\, t>0\}$ and we omit the parameter z. On the other hand, if we use proposition 1.1 with n replaced by $n-1$, we have easily by estimates (1.4) and (1.6″)

$$\|I(2^j t)\|_{L^\infty(A)},\ (\|I_1(2^j t)\|_{L^\infty(A)})^{1/2-\varepsilon}(\|2^j I_2'(2^j t)\|_{L^\infty(A)})^{1/2+\varepsilon},$$

$$(\|I_2(2^j t)\|_{L^\infty(A)})^{1/2-\varepsilon}(\|2^j I_2'(2^j t)\|_{L^\infty(A)})^{1/2+\varepsilon} \leqq \frac{C}{2^{j(n-1)/2}|\det h''(z)|},$$

and by estimates (1.4) and (1.5),

$$(\|I_1(2^j t)\|_{L^\infty(A)})^{1/2-\varepsilon}(\|2^j I_1'(2^j t)\|_{L^\infty(A)})^{1/2+\varepsilon} \leqq \frac{C}{2^{j(n-1)/2}|\det h''(z)|^{1+\varepsilon}}.$$

Furthermore, by estimates (1.5), (1.6′) and (1.6″), we have

$$\begin{aligned}&(\|I_2(2^j t)\|_{L^\infty(A)})^{1/2-\varepsilon}(\|2^j I_1'(2^j t)\|_{L^\infty(A)})^{1/2+\varepsilon}\\ &\leqq C2^{-j(n-1)/2}\left\{(2^{-j/6}j^{1/2}|\det h''(z)|^{-1})^{1/2-\varepsilon}(|\det h''(z)|^{-1/2})^{1/2+\varepsilon}\right.\\ &\left.+(2^{j/6}|\det h''(z)|^{-1/2})^{1/2-\varepsilon}(2^{-j/3}|\det h''(z)|^{-3/2})^{1/2+\varepsilon}\right\} \leqq \frac{C}{2^{j(n-1)/2}|\det h''(z)|^{1+\varepsilon}}.\end{aligned}$$

From them all, we obtain estimate (2.11), therefore estimate (2.3) in the case $\beta=0$. Since estimates (2.3) with the derivatives $\left(\frac{\partial}{\partial x}\right)^\beta K_j(x)$ are obtained from the one in the case $\beta=0$, the proposition has been proved.

Finally, we shall show a property of the function $H(x)$.

Proposition 2.3. *There exist a neighbourhood Δ of the hypersurface Σ^* and a constant C such that the estimate*

$$|H(y)-H(z)| \leqq C \frac{|y-x|}{\left|\varkappa\left(\nu^{-1}(-x/|x|)\right)\right|}$$

holds for all $x, y \in \Delta$.

Proof. We may assume that the points x and y are near the point $-\nabla\varphi(e_n)\in\Sigma^*$. Then all we have to show is the estimate

$$\left|\nu^{-1}(-y/|y|)-\nu^{-1}(-x/|x|)\right| \leqq C \frac{|y-x|}{\left|\varkappa\left(\nu^{-1}(-x/|x|)\right)\right|}.$$

If we set $(z, h(z))=\nu^{-1}(-x/|x|)$ and $(w, h(w))=\nu^{-1}(-y/|y|)$, we can see that it is reduced to the estimate

$$|w-z| \leqq C \frac{|h'(w)-h'(z)|}{|\det h''(z)|}$$

by equalities (2.5) and (2.7). On the other hand, lemma 1.2 yields

$$\left|\left\langle h'(w)-h'(z), \frac{w-z}{|w-z|}\right\rangle\right| \geqq C|w-z|\left|\left\langle h''(z)\frac{w-z}{|w-z|}, \frac{w-z}{|w-z|}\right\rangle\right|.$$

From this, we can easily obtain the proposition.

3. Hardy space

In this section, we shall give a proof of the main theorem. We have only to show it for $k=k(p)$, where

$$k(p) = (n-1)\left|\frac{1}{p}-\frac{1}{2}\right|.$$

The boundedness of the operator $M_{k(p)}(D)$ on the Sobolev spaces H^s_p is obtained from the L^p-estimate $(1<p<\infty)$, and on the Besov spaces $B^s_{p,q}$ from the uniform L^p-estimate $(1\leqq p\leqq\infty)$ for the operator $M_{k(p)}(D)\Phi_j(D)$ with respect to the number j. Here $\{\Phi_j\}_{j=1}^\infty$ is a partition of unity of Littlewood—Paley which is used to construct the theory of Besov spaces. (See, for example, Bergh—Löfström [2] or Triebel [16].)

In order to obtain these estimates, we shall use the theory of Hardy spaces $H^p(\mathbf{R}^n)$ introduced by Fefferman—Stein [3]. Since the estimate in the case $p=2$

is trivial, all we have to show is the boundedness of the operator $M_{k(p)}(D)$ from H^p to L^p for some $0<p<1$. In fact, then we can have the required L^p-estimates for $1\leqq p\leqq\infty$ by the interpolation theorem (see the proof of theorem 1 in Miyachi [10] and the papers cited therein) and the duality argument. Here we note the fact $H^p=L^p$ $(1<p<\infty)$ and the characterization of H^1 by the Riesz transform (see Stein [15, pp. 220—221]).

Furthermore, we shall use the characterization of H^p by the atom decomposition proved by Latter [8] (see also [10, theorem A]). That is, for $0<p\leqq 1$, any $f\in H^p(\mathbf{R}^n)$ can be represented as

$$f=\sum_{j=1}^{\infty}\lambda_j g_j,\quad \lambda_j\in\mathbf{C},\quad g_j\colon p-\text{atom},$$

and the norm $\|f\|_{H^p}$ is equivalent to the l^p-norm $(\sum_{j=0}^{\infty}|\lambda_j|^p)^{1/p}$. Here we call a function g on $\mathbf{R}^n$ a p-atom if there is a ball $B=B_g\subset\mathbf{R}^n$ such that $\operatorname{supp} g\subset B$, $\|g\|_{L^\infty}\leqq|B|^{-1/p}$ ($|B|$ is the Lebesgue measure of the ball B) and $\int g(x)x^\alpha dx=0$ for $|\alpha|\leqq[n/p-n]$. From this, all we have to show is reduced to the estimate

$$\|M_{k(p)}(D)f\|_{L^p(\mathbf{R}^n)}\leqq C,\quad f\in\mathcal{A}_{r,p} \tag{3.1}$$

for some $0<p<1$ and some constant C which is independent of $0<r<\infty$. Here $\mathcal{A}_{r,p}$ is the set of all functions f on $\mathbf{R}^n$ such that

$$\operatorname{supp} f\subset\{x;\ |x|\leqq r\},\quad \|f\|_{L^\infty}\leqq r^{-n/p},$$

and

$$\int f(x)x^\alpha dx=0\quad\text{for}\quad |\alpha|\leqq N=\left[\frac{n}{p}-n\right].$$

In the first place, suppose $f\in\mathcal{A}_{r,p}$ with $r\geqq b$ and $0<p<1$, where $b>0$ will be chosen later. Then we split $\mathbf{R}^n\diagdown 0$ into the following two parts:

$$\Delta_1=\{x;\ |H(x)|\leqq Ar\},$$
$$\Delta_2=\{x;\ Ar\leqq|H(x)|\},$$

where

$$H(x)=|x|-\left|\nabla\varphi\left(v^{-1}(-x/|x|)\right)\right|$$

(see section 2) and $A=3b^{-1}\|\nabla\varphi\|_{L^\infty}+1$.

For the part Δ_1, we have by Hölder's inequality $(p^{-1}=q^{-1}+2^{-1})$ and Plancherel's theorem

$$\begin{aligned}\|M_{k(p)}(D)f\|_{L^p(\Delta_1)}&\leqq\|1\|_{L^q(\Delta_1)}\|M_{k(p)}(D)f\|_{L^2}\\&\leqq Cr^{n/q}\|f\|_{L^2}\leqq Cr^{n/q+n/2-n/p}\leqq C.\end{aligned}$$

In order to obtain the estimate for the part Δ_2, we write

$$M_{k(p)}(D)f(x)=\int K(x-y)f(y)\,dy, \tag{3.2}$$

where

$$K(x)=F^{-1}m_{k(p)}(x).$$

Since the conditions $x \in \Delta_2$ and $|y| \leqq r$ imply $x-y \in \Delta = \{x;\ |H(x)| \geqq \|\nabla\varphi\|_{L^\infty}\}$ and $|H(x)| \leqq C|H(x-y)|$, we obtain from equality (3.2)

$$\|M_{k(p)}(D)f\|_{L^p(\Delta_2)} \leqq C\|H(x)^{-M}\|_{L^p(\Delta_2)}\|H(x)^M K(x)\|_{L^\infty(\Delta)}\|f\|_{L^1} \leqq Cr^{n-n/p} \leqq C,$$

if $M>0$ is sufficiently large. Here we have used proposition 2.1 with (i) and the relation $n-n/p<0$.

In the second place, suppose $f \in \mathscr{A}_{r,p}$ with $r \leqq b$ and $0<p<1$. Then we split $\mathbf{R}^n \setminus 0$ into the following three parts:

$$\Xi_1 = \{x;\ |H(x)| \leqq r\},$$
$$\Xi_2 = \{x;\ r \leqq |H(x)| \leqq B\},$$
$$\Xi_3 = \{x;\ |H(x)| \geqq B\},$$

where $B>0$ will be chosen later.

In order to obtain the estimate for the part Ξ_1, we notice the estimates

$$|\hat{f}(\xi)| \leqq C|\xi|^{N+1}\big\||x|^{N+1}f\big\|_{L^1} \leqq C|\xi|^{N+1}r^{N+1+n-(n/p)}$$

and $\|f\|_{L^2} \leqq Cr^{n/2-n/p}$. From them and Plancherel's theorem, we obtain

$$\|M_{k(p)}(D)f\|_{L^2}^2 \leqq C\Big(\int_0^{1/r}+\int_{1/r}^\infty\Big)|\xi|^{-2k(p)}|\hat{f}(\xi)|^2\,d\xi$$

$$\leqq C\Big(r^{2(N+1+n-n/p)}\int_0^{1/r}|\xi|^{-2k(p)+2(N+1)}\,d\xi + r^{2k(p)}\|f\|_{L^2}^2\Big) \leqq Cr^{1-2/p}.$$

Then by Hölder's inequality $(p^{-1}=q^{-1}+2^{-1})$ we have

$$\|M_{k(p)}(D)f\|_{L^p(\Xi_1)} \leqq \|1\|_{L^q(\Xi_1)}\|M_{k(p)}(D)f\|_{L^2} \leqq Cr^{1/q+1/2-1/p} \leqq C.$$

For the part Ξ_2, we shall decompose the kernel $K(x)=\sum_{j=0}^\infty K_j(x)$ as in proposition 2.1 with (ii), and take the numbers B, $b>0$ so small that the estimate in proposition 2.3 holds with the neighbourhood $\{x-y;\ x \in \Xi_2,\ |y| \leqq b\}$ of the hypersurface $\Sigma^* = \{x;\ H(x)=0\}$. Then by Hölder's inequality $(p^{-1}=1+q^{-1})$, we have

$$\Big\|\int K_j(x-y)f(y)\,dy\Big\|_{L^p(\Xi_2)} \tag{3.3}$$

$$\leqq C\|H(x)^{-\varepsilon}\|_{L^q(\Xi_2)}\Big\|\Big(|H(x)|^\varepsilon + \frac{r^\varepsilon}{|\kappa(\nu^{-1}(-x/|x|))|^\varepsilon}\Big)K_j(x)\Big\|_{L^1}\|f\|_{L^1}$$

$$\leqq Cr^{1/q-\varepsilon}\big(2^{j((n-1)/2-k(p)-\varepsilon)}+r^\varepsilon 2^{j((n-1)/2-k(p))}\big)r^{n-n/p}$$

$$= C\{(2^j r)^{(n-1)(1-1/p)-\varepsilon}+(2^j r)^{(n-1)(1-1/p)}\}$$

if $\varepsilon>0$ is sufficiently small and $\varepsilon q>1$ (equivalently $p>(1+\varepsilon)^{-1}$). On the other hand, if we use the equality

$$(3.4)\qquad \int K_j(x-y)f(y)\,dy$$

$$=\int\left\{K_j(x-y)-\sum_{|\beta|\le N}\left(\frac{\partial}{\partial x}\right)^{\beta}K_j(x)\frac{(-y)^{\beta}}{\beta!}\right\}f(y)\,dy$$

$$=(N+1)\sum_{|\beta|=N+1}\int_0^1\int_{|y|\le r}(1-\theta)^N\left(\frac{\partial}{\partial x}\right)^{\beta}K_j(x-\theta y)\frac{(-y)^{\beta}}{\beta!}f(y)\,d\theta\,dy,$$

we have similarly

$$(3.5)\qquad \left\|\int K_j(x-y)f(y)\,dy\right\|_{L^p(\Xi_2)}\le C\|H(x)^{-\varepsilon}\|_{L^q(\Xi_2)}$$

$$\times\sup_{|\beta|=N+1}\left\|\left(|H(x)|^{\varepsilon}+\frac{r^{\varepsilon}}{|\varkappa(\nu^{-1}(-x/|x|))|^{\varepsilon}}\right)\left(\frac{\partial}{\partial x}\right)^{\beta}K_j(x)\right\|_{L^1}\||y|^{N+1}f\|_{L^1}$$

$$\le C\{(2^j r)^{(n-1)(1-1/p)+N+1-\varepsilon}+(2^j r)^{(n-1)(1-1/p)+N+1}\}$$

for the same ε, p and q as in estimate (3.3). Then taking the integer j_0 such that $2^{j_0}r\le 1<2^{j_0+1}r$, we have for $(1+\varepsilon)^{-1}<p<1$

$$\|M_{k(p)}(D)f\|_{L^p(\Xi_2)}^p\le\sum_{j=1}^{\infty}\left\|\int K_j(x-y)f(y)\,dy\right\|_{L^p(\Xi_2)}^p$$

$$\le C\sum_{j=1}^{j_0}(2^j r)^{(n-1)(p-1)+(N+1-\varepsilon)p}+C\sum_{j=j_0+1}^{\infty}(2^j r)^{(n-1)(p-1)}\le C,$$

by equality (3.2) and estimates (3.3), (3.5). Here we have used the relations $(n-1)(p-1)+(N+1-\varepsilon)p>0$ and $(n-1)(p-1)<0$ which are possible for sufficiently small $\varepsilon>0$.

Finally, we shall show the estimate for the part Ξ_3. Let the number $b>0$ be chosen so small that the neighbourhood $\Xi=\{x-y;\ x\in\Xi_3,\ |y|\le b\}$ of the set Ξ_3 is away from the hypersurface $\Sigma^*=\{x;\ H(x)=0\}$. Then from equality (3.2) and equality (3.4) with K_j replaced by K, we obtain

$$\|M_{k(p)}(D)f\|_{L^p(\Xi_3)}\le C\left\|(1+|x|)^{-M}\right\|_{L^p(\Xi_3)}$$

$$\times\sup_{|\beta|=N+1}\left\|(1+|x|)^M\left(\frac{\partial}{\partial x}\right)^{\beta}K(x)\right\|_{L^{\infty}(\Xi)}\left\||y|^{N+1}f\right\|_{L^1}\le Cr^{n-n/p+N+1}\le C.$$

Here we have used proposition 2.1 with (i) and the relation $n-n/p+N+1>0$. Thus we have obtained estimate (3.1) and finished the proof of the main theorem

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