On Hille—Tamarkin operators and Schatten classes

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Abstract. We determine the smallest Schatten class containing all integral operators with kernels in $L_p(L_{p',q})^{\text{symm}}$, where $2<p<\infty$ and $1\leqq q\leqq\infty$. In particular, we give a negative answer to a problem posed by Arazy, Fisher, Janson and Peetre in [1].

1. Setting of the problem

Let (Ω, μ) be a σ-finite measure space and let K be a $\mu\times\mu$-measurable kernel defined on $\Omega\times\Omega$. The integral operator associated to K is given by

$$T_K f(x) = \int_\Omega K(x,y) f(y)\, d\mu(y), \quad x\in\Omega.$$

We shall consider T_K as a bounded operator in $L_2(\Omega, \mu)$, for this purpose we shall impose certain summability conditions on the kernel K.

Let $1<p<\infty$, $1/p+1/p'=1$, $1\leqq q\leqq\infty$ and let $L_{p',q}$ be the Lorentz function space (see, e.g., [2]). We say that $K\in L_p(L_{p',q})$ if

$$\|K\|_{L_p(L_{p',q})} = \left(\int_\Omega \|K(x,\,.)\|^p_{L_{p',q}}\, d\mu(x)\right)^{1/p} < \infty.$$

Similarly, we say that $K\in(L_{p',q})L_p$ if

$$\|K\|_{(L_{p',q})L_p} = \left(\int_\Omega \|K(.\,,y)\|^p_{L_{p',q}}\, d\mu(y)\right)^{1/p} < \infty.$$

When

$$K\in L_p(L_{p',q})\cap(L_{p',q})L_p$$

we write

$$K\in L_p(L_{p',q})^{\text{symm}}.$$

Integral operators generated by kernels satisfying summability conditions of the type mentioned above are called Hille—Tamarkin operators. They often arise

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in functional analysis (see, e.g., [5], [6]). We are interested in the relationship between summability properties of K and the degree of compactness of T_K on $L_2(\Omega, \mu)$. For this we need the Schatten classes.

Recall that given any compact (linear) operator T in $L_2(\Omega, \mu)$, the singular numbers of T are defined by $s_n(T)=\lambda_n(|T|)$, $n\in \mathbf{N}$, where $|T|=(T^*T)^{1/2}$ and the λ_n's are the non-zero eigenvalues of $|T|$, arranged in non-increasing order and repeated according to their algebraic multiplicities. In the special case when T is compact and self-adjoint, we have $s_n(T)=|\lambda_n(T)|$, $n\in \mathbf{N}$.

The Schatten—Lorentz class $S_{p,q}$ consists of all compact operators T on $L_2(\Omega, \mu)$ having finite quasi-norm

$$\|T\|_{p,q} = \left(\sum_{n=1}^{\infty} (n^{1/p} s_n(T))^q n^{-1}\right)^{1/q}.$$

These classes are lexicographically ordered, i.e. $S_{p_0,q_0} \subsetneqq S_{p_1,q_1}$ if $p_0<p_1$ and $1\leqq q_0, q_1\leqq \infty$, or $p_0=p_1$ and $q_0<q_1$. The space $S_{p,p}$ is just the Schatten—von Neumann p-class S_p. For more details on singular numbers and Schatten classes see, e.g., [3], [5] or [6].

The following result is due to Russo [7].

Theorem 1. *If* $2<p<\infty$ *and* $K\in L_p(L_{p'})^{\text{symm}}$, *then* $T_K\in S_p$.

Russo's theorem has been recently improved by Arazy, Fisher, Janson and Peetre [1].

Theorem 2. *Let* $2<p<\infty$, $p\leqq q\leqq\infty$ *and let* $K\in L_p(L_{p',q})^{\text{symm}}$. *Then* $T_K\in S_{p,q}$.

As a matter of fact, the case $q=\infty$ in Theorem 2 was established by Janson and Wolff [4].

The methods developed by Arazy, Fisher, Janson and Peetre in [1] do not apply to the case $1\leqq q<p$. They left as an open problem the following question:

Problem. Does Theorem 2 hold for $1\leqq q<p$? In particular, can Russo's theorem be improved to the effect that if $K\in L_p(L_{p'})^{\text{symm}}$ then it follows that $T_K\in S_{p,p'}$?

In this note we show that the answer to this problem is "no". Moreover, we give examples showing that Theorem 2 is optimal.

2. The counter-example

Our results can be formulated as follows:

Theorem 3. *Let* $2<p<\infty$ *and* $1\leqq q\leqq\infty$.

(i) *Given any σ-finite measure space* (Ω, μ), *if* K *is a kernel over* $\Omega\times\Omega$ *such that* $K\in L_p(L_{p',q})^{\text{symm}}$, *then* $T_K\in S_{p,\max(p,q)}$.

(ii) *Let* $\Omega=[0, 1]$ *with Lebesgue measure. There is a kernel* K *over* $[0, 1]\times[0, 1]$ *such that* $K\in L_p(L_{p',q})^{\mathrm{symm}}$ *but* $T_K\notin S_{p,r}$ *for every* $r<\max(p, q)$.

Proof. Statement (i) is a trivial consequence of Theorem 2, since for $q<p$ it holds $L_{p',q}\subset L_{p',p}$.

To prove (ii), we distinguish two cases. Assume first $q\leq p$ and denote by $l_{p,r}$ the Lorentz sequence space. Choose a sequence of positive numbers (α_n) such that

$$(\alpha_n)\in l_p\setminus\bigcup_{r<p} l_{p,r},$$

and consider the kernel

$$K(x, y) = \sum_{n=1}^{\infty} 2^n\alpha_n\chi_n(x)\chi_n(y), \quad x, y\in[0, 1],$$

where χ_n is the characteristic function of the interval $I_n=(2^{-n}, 2^{-n+1})$. For $x\in I_n$, we have

$$\|K(x, .)\|_{L_{p',q}} = 2^n\alpha_n\|\chi_n\|_{L_{p',q}} = c2^n\alpha_n|I_n|^{1/p'},$$

where $c=(p'/q)^{1/q}$. Hence, since $|I_n|=2^{-n}$,

$$\|K\|_{L_p(L_{p',q})} = c\left(\sum_{n=1}^{\infty}(2^n\alpha_n|I_n|^{1/p'})^p\int_{I_n}dx\right)^{1/p} = c\,\|(\alpha_n)\|_{l_p}<\infty.$$

The same estimate holds for $\|K\|_{(L_{p',q})L_p}$, therefore $K\in L_p(L_{p',q})^{\mathrm{symm}}$.

The operator T_K generated by K is self-adjoint because $K(x, y)$ is real and symmetric. Moreover, T_K is given by

$$T_Kf(x) = \sum_{n=1}^{\infty} 2^n\alpha_n\left(\int_0^1\chi_n(y)f(y)\,dy\right)\chi_n(x).$$

Thus

$$T_K\chi_n = 2^n\alpha_n|I_n|\,\chi_n = \alpha_n\chi_n.$$

It follows that

$$s_n(T_K) = |\lambda_n(T_K)| = \alpha_n$$

and consequently

$$\|T_K\|_{p,r} = \|(\alpha_n)\|_{l_{p,r}} = \infty \quad \text{for every} \quad r<p.$$

Now we treat the case $q>p$. Take any $\gamma>1/q$ and consider the function

$$f(x) = \sum_{n=1}^{\infty}\alpha_n e^{2\pi inx}, \quad x\in\mathbf{R},$$

where

$$\alpha_n = n^{-1/p}(\log(1+n))^{-1/q}(\log\log(2+n))^{-\gamma}.$$

By [8], V.2.6, $|f(x)|$ behaves like

$$g(x) = |x|^{-1/p'}\left(\log\frac{1}{|x|}\right)^{-1/q}\left(\log\log\frac{1}{|x|}\right)^{-\gamma}$$

as $|x|\to 0$. Since $\gamma q>1$, g belongs to $L_{p',q}([0, 1], dx)$, and therefore the same holds for f. Consider next the kernel of convolution with f,

$$K(x, y) = f(x-y), \quad x, y \in [0, 1].$$

For every $x, y \in [0, 1]$, we get

$$\|K(x, .)\|_{L_{p',q}} = \|K(., y)\|_{L_{p',q}} = \|f\|_{L_{p',q}}$$

Hence $K \in L_p(L_{p',q})^{\text{symm}}$. On the other hand, it is well-known that the eigenvalues of T_K coincide with the Fourier coefficients of f. Besides, T_K is self-adjoint because $K(x, y) = \overline{K(y, x)}$. Consequently,

$$s_n(T_K) = |\lambda_n(T_K)| = |\hat{f}(n)| = \alpha_n, \quad n \in \mathbf{N}.$$

Taking into account that $(\alpha_n) \notin l_{p,r}$ for every $r < q$, we obtain that

$$T_K \notin \bigcup_{r<q} S_{p,r}.$$

The proof is complete.

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