A linear system on a threefold

Dong-Kwan Shin

Introduction

Throughout this paper, we are working on the complex number field **C**. The aim of this paper is to study the rational map associated to the adjoint linear system on the smooth projective threefold X. The adjoint linear system is the linear system of the form $|K_X+D|$ where D is a divisor on X. Recently, Reider introduced a nice technique to understand an adjoint linear system on a smooth projective surface. (See Reider [5].) We couldn't generalize Reider's theorem to the threefold. But we are able to have some results on a threefold by using Reider's theorem. Here is our main result:

Theorem 5. *Let X be a smooth projective threefold and let D be a nef and big divisor on X. Assume that $h^0(X, \mathcal{O}_X(mD)) \geqq 2$ for some positive integer m. Then $\Phi_{|K_X+nD|}$ is birational for a positive integer $n \geqq m+4$ such that $h^0(X, \mathcal{O}_X((n-m)D)) \geqq 1$.*

1. Preliminaries

Let X be a smooth projective variety of dimension n.

We denote a linear equivalence by $\sim$ and a numerical equivalence by $\equiv$. Denote by $\mathrm{Div}(X)$ a free abelian group generated by the divisors on X. Denote the canonical divisor of X by K_X.

Then we say that $D \in \mathrm{Div}(X)$ is *nef* if $D \cdot C \geqq 0$ for any curve C on X, and *big* if $\varkappa(D, X) = \dim X$, where $\varkappa(D, X)$ is the Kodaira dimension of D on X.

For $D \in \mathrm{Div}(X)$, $\Phi_{|D|}$ denotes the rational map associated with the complete linear system $|D|$ if $h^0(X, \mathcal{O}_X(D)) \neq 0$.

Theorem 1. (Kawamata—Viehweg vanishing theorem.) *Let X be a nonsingular projective variety and $D \in \mathrm{Div}(X)$. If D is nef and big, then $H^i(X, \mathcal{O}_X(K_X+D)) = 0$ for all $i > 0$.*

For a proof, see Kawamata [3].

Lemma 2. *Let S be a nonsingular projective surface and let R be a nef divisor with $R^2>0$. Given a positive integer n, let A_n be the set of effective divisors E on S such that $R\cdot E=0$ and $E^2\geqq -n$. Then A_n is a finite set.*

For a proof, see Benveniste [1] or Matsuki [4].

Theorem 3. *Let S be a nonsingular surface and let L be a nef divisor.*

(a) *If $L^2\geqq 5$ and p is a base point of $|K_S+L|$, then there exists an effective divisor E passing through p such that*

$$\text{either (i)}\quad L\cdot E=0,\quad E^2=-1 \text{ or } -2,$$

$$\text{or (ii)}\quad L\cdot E=1,\quad E^2=0.$$

(b) *If $L^2\geqq 10$ and the points p, q are not separated by $|K+L|$, then there exists an effective divisor E on S passing through p and q such that*

$$\text{either (i)}\quad L\cdot E=0,\quad E^2=-1 \text{ or } -2,$$

$$\text{or (ii)}\quad L\cdot E=1,\quad E^2=-1 \text{ or } 0,$$

$$\text{or (iii)}\quad L\cdot E=2,\quad E^2=0.$$

For a proof, see Reider [5].

Lemma 4. *Let $|S|$ be a complete linear system on the nonsingular projective variety X, and D a divisor on X with $|D|\neq\emptyset$. Assume that $|S|$ has no base points and is not composed with a pencil. If $\Phi_{|S+D|}$ is not a birational map, then for a general member Y of $|S|$, $\Phi_{|S+D|}$ is not birational when restricted to Y.*

Proof. We may assume that D is effective because $|D|\neq\emptyset$. Choose a section $t_D\in H^0(X,\mathcal{O}_X(D))$ which determines D.

Since $\Phi_{|S+D|}$ is not birational, there exists a nonempty Zariski open set U in X such that $U\cap D=\emptyset$ and the base locus of $|S+D|$ is disjoint from U and such that for any $x\in U$, there is some $y\in U$ distinct from x with $\Phi_{|S+D|}(x)=\Phi_{|S+D|}(y)$. We may also assume that $U\cap Y\neq\emptyset$ because Y is a general member of $|S|$. So take a section $s_Y\in H^0(X,\mathcal{O}_X(S))$ which determines Y.

For any $x\in U\cap Y$, there exists $y\in U$ distinct from x with $\Phi_{|S+D|}(x)=\Phi_{|S+D|}(y)$. Since $t_Ds_Y\in H^0(X,\mathcal{O}_X(S+D))$, there exists $\alpha\in\mathbf{C}\setminus\{0\}$ such that $t_D(x)s_Y(x)=\alpha\, t_D(y)s_Y(y)$. Since we have $U\cap D=\emptyset$, we have $t_D(y)\neq 0$ and $s_Y(x)=0$. Therefore $s_Y(y)=0$. It means $y\in Y$. In other words, $\Phi_{|S+D|}|_Y$ is not birational. □

2. Main Theorem

Theorem 5. *Let X be a smooth projective threefold and let D be a nef and big divisor on X. Assume that $h^0(mD)\geqq 2$ for some positive integer m. Then $\Phi_{|K_X+nD|}$ is birational for a positive integer $n\geqq m+4$ such that $h^0(X, \mathcal{O}_X(K_X+(n-m)D))\geqq 1$.*

Proof. We shorten $\Phi_{|mD|}$ to Φ_m and let W_m be an image of Φ_m. Then we have the following two case: case (1) $\dim W_m\geqq 2$ and case (2) $\dim W_m=1$.

Let $|mD|=|M|+Z$, where $|M|$ is the moving part of $|mD|$ and Z is the fixed part of $|mD|$.

Consider the following commutative diagram:

$$\begin{array}{ccc} X' & \xrightarrow{h} & W'_m \\ {\scriptstyle f}\downarrow & \searrow^{g} & \downarrow{\scriptstyle k} \\ X & \xrightarrow{\Phi_m} & W_m, \end{array}$$

where f is a succession of the blow-ups with nonsingular centers such that

(a) $g:=\Phi_m\circ f$ are a morphism and

(b) $g=k\circ h$ is the Stein factorization.

Let H_m be a hyperplane section of W_m in $\mathbf{P}^N$, where $N=h^0(mD)-1$.

We set $K_{X'}\sim f^*K_X+E$, where E is the ramification divisor of f.

$$\begin{aligned} f^*(K_X+nD)+E &\sim K_{X'}+(n-m)f^*D+f^*(mD) \\ &\sim K_{X'}+(n-m)f^*D+g^*H_m+f^*Z+E', \end{aligned}$$

where E' is an effective divisor supported on the exceptional locus of f. If the map $\Phi_{|K_{X'}+(n-m)f^*D+g^*H_m|}$ is birational, then so is $\Phi_{|K_X+nD|}$.

Case (1). $\dim W_m\geqq 2$.

Let S_m be a general member of $|g^*H_m|$. Lemma 4 implies that if ${}_{|K_{X'}+(n-m)f^*D+S_m|}$ is birational for a general $S_m\in|g^*H_m|$, then $\Phi_{|K_{X'}+(m-m)f^*D+S_m|}$ is birational.

Consider the following exact sequence:

$$\begin{aligned} 0 \to \mathcal{O}_{X'}(K_{X'}+(n-m)f^*D) &\to \mathcal{O}_{X'}(K_{X'}+(n-m)f^*D+S_m) \\ &\to \mathcal{O}_{S_m}(K_{S_m}+(n-m)R_m) \to 0, \end{aligned}$$

where $R_m=f^*D|_{S_m}$ and K_{S_m} is a canonical divisor of S_m.

Since f^*D is nef and big, $h^1(X', \mathcal{O}_{X'}(K_{X'}+(n-m)f^*D))=0$ by the Kawamata—Viehweg vanishing theorem. So the restriction to S_m gives $\Phi_{|K_{X'}+(m-m)f^*D+S_m|}|_{S_m}=$

$\Phi_{|K_{S_m}+(n-m)R_m|}$ on S_m. $R_m^2=(f^*D)^2\cdot S_m\geqq 1$ since f^*D is nef and big and since $|S_m|$ is base point free and $S_m\neq 0$. $(n-m)\geqq 4$ by hypothesis. Hence lemma 2 and theorem 3 imply that $\Phi_{|K_{S_m}+(n-m)R_m|}$ is birational on S_m.

Case (2). $\dim W_m=1$.

Let $a=\deg W_m$ in $\mathbf{P}^{h^0(mD)-1}$ and let $b=\deg k$.

Then g^*H_m is the disjoint union of $S_{m,i}$, $1\leqq i\leqq ab$, each of which is a general fiber of h. Denote $R_{m,i}:=f^*D|_{S_{m,i}}$ and the canonical divisor of $S_{m,i}$ by $K_{m,i}$.

Consider the following exact sequence:

$$0\to \mathcal{O}_{X'}(K_{X'}+(n-m)f^*D)\to \mathcal{O}_{X'}(K_{X'}+(n-m)f^*D+g^*H_m)$$

$$\to \bigoplus_{i=1}^{ab}\mathcal{O}_{S_{m,i}}(K_{m,i}+(n-m)R_{m,i})\to 0.$$

By Kawamata—Viehweg vanishing theorem, $h^1(X',\mathcal{O}_{X'}(K_{X'}+(n-m)f^*D))=0$. So the restriction map

$$H^0(X',\mathcal{O}_{X'}(K_{X'}+(n-m)f^*D+g^*H_m))$$

$$\xrightarrow{r}\bigoplus_{i=1}^{ab}H^0(S_{m,i},\mathcal{O}_{S_{m,i}}(K_{m,i}+(n-m)R_{m,i}))$$

is surjective.

$R_{m,i}^2=(f^*D)^2\cdot S_{m,i}\geqq 1$ since D is nef and big, and since $S_{m,i}$ is nef and $\neq 0$. Hence we have $\Phi_{|K_{m,i}+(n-m)R_{m,i}|}$ on $S_{m,i}$ with $R_{m,i}^2\geqq 1$. By theorem 3, $\Phi_{|K_{m,i}+(n-m)R_{m,i}|}$ is birational on $S_{m,i}$.

Since the restriction map r is surjective, $\Phi_{|K_{X'}+(n-m)f^*D+g^*H_m|}$ separates the fibers of g and the components of a fiber at least on some nonempty Zariski open subset of X.

Therefore $\Phi_{|K_{X'}+(n-m)f^*D+g^*H_m|}$ is birational. □

Remark. We can easily notice that the condition $h^0(X,\mathcal{O}_X(K_X+(n-m)D))\geqq 1$ holds for sufficiently large n by computing Euler characteristic $\chi(K_X+(n-m)D)$. For example, in case of smooth threefold of general type with K_X nef, $h^0(X,\mathcal{O}_X(nD))\geqq 4$ for $n\geqq 2$. Hence by theorem 5, $\Phi_{|7K_X|}$ is birational. This result is already known, but theorem 5 gives a short proof of it. (See Matsuki [4].)

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Dong-Kwan Shin
Department of Mathematics
King ejong University
Seoul 13—747
South-Korea