A Riesz basis for Bargmann—Fock space related to sampling and interpolation

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Abstract. It is shown that the Bargmann—Fock spaces of entire functions, $\mathbf{A}^p(\mathbf{C})$, $p \geqq 1$ have a bounded unconditional basis of Wilson type [DJJ] which is closely related to the reproducing kernel. From this is derived a new sampling and interpolation result for these spaces.

0. Introduction

The Bargman—Fock spaces were used in [B] as the representation space for the canonical commutation rules in quantum mechanics. Since then they have appeared in many different contexts, e.g. in signal analysis, representation theory of nilpotent Lie groups [Fo] and as a class of symbols in the theory of Hankel and Toeplitz operators [JPR]. Various types of atomic decompositions have been obtained in [DG], [JPR].

The objective of this paper is the construction of a simple Riesz basis for these spaces, which is related to the reproducing kernel of these spaces.

Definition 0.1. We define the **Bargmann—Fock space** of entire functions, $\mathbf{A}^p(\mathbf{C})$, $1 \leqq p < \infty$ as follows.

$$\mathbf{A}^p(\mathbf{C}) = \left\{ f \textit{ entire}: \iint |f(z)|^p e^{-\pi p|z|^2/2}\, dx\, dy < \infty \right\}$$

where $z = x + iy$.

$A^\infty(\mathbf{C})$ is defined by

$$\mathbf{A}^\infty(\mathbf{C}) = \left\{ f \textit{ entire}: \sup_{z \in \mathbf{C}} |f(z)|\, e^{\pi|z|^2/2} < \infty \right\}.$$

$A^0(\mathbf{C})$ is defined by

$$\mathbf{A}^0(\mathbf{C}) = \left\{ f \textit{ entire}: \lim_{|z| \to \infty} |f(z)|\, e^{\pi|z|^2/2} = 0 \right\}.$$

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$\mathbf{A}^2(\mathbf{C})$ is a Hilbert space with respect to the obvious inner product and has the reproducing kernel $e_w(z)=e^{\pi\bar{w}z}\in\mathbf{A}^2(\mathbf{C})$, this means that $f(w)=\langle f, e_w\rangle$ for all $f\in\mathbf{A}^2(\mathbf{C})$.

The collection $\{z^m(\pi^m/m!)^{1/2},\ m\geqq 0\}$ is an orthonormal basis for the Hilbert space $\mathbf{A}^2(\mathbf{C})$. Thus, if $f(z)=\sum_{m=0}^{\infty} a_n z^n$ then $\|f\|_{A^2}^2=\sum_{m=0}^{\infty}|a_m|^2 m!/\pi^m$.

From the Taylor expansion of entire functions it is clear that the monomials z^m, $m\geqq 0$ form a Schauder basis for $\mathbf{A}^p(\mathbf{C})$, $p\neq 2$. However, no characterization of $\mathbf{A}^p$ in terms of Taylor coefficients seems to be known, and it is unlikely that a function $f\in\mathbf{A}^p(\mathbf{C})$ can be characterized by the magnitude of its Taylor coefficients alone.

The existence of an unconditional basis for $\mathbf{A}^p(\mathbf{C})$, which provides an isomorphism $\mathbf{A}^p(\mathbf{C})\leftrightarrow l^p$ has recently been observed in [FGW]. It is based on (1) an ingenious construction of an orthonormal basis for $L^2(\mathbf{R})$ [DJJ] where all basis functions are derived from a single function by the action of the Heisenberg group — a so-called **Wilson basis,** and (2) on the identification of the Bargmann—Fock spaces with certain spaces of distributions on $\mathbf{R}$, the so-called **modulation spaces.**

Despite the marvelous properties of the Wilson type bases, the underlying basic function appears to be quite complicated. It bears little connection with the reproducing kernel of the Bargmann—Fock spaces. It would be desirable to have an unconditional basis for the Bargmann—Fock spaces which has the same structure as the Wilson-type bases and is closely connected to the reproducing kernel.

The main result of this paper provides a surprisingly simple and *explicit* unconditional basis for the $\mathbf{A}^p(\mathbf{C})$ which satisfies all these requirements. In what follows, let $\phi(x)=2^{1/4}e^{-\pi x^2}$.

Theorem 1. *For* $\ell=0, 1, \ldots;\ n\in\mathbf{Z}$, *let* $w_{\ell n}=n/2+i\ell$. *The collection of vectors*

$$\{\Psi_{\ell,n}\colon \ell=0, 1, \ldots;\ n\in\mathbf{Z}\}$$

defined by

(1)
$$\Psi_{0,n}=e^{-\pi n^2/2}e^{\pi z n}$$
$$\Psi_{\ell,n}=2^{1/2}e^{-\pi|w_{\ell n}|^2/2}\left(x^{\pi z\overline{w_{\ell n}}}+(-1)^{\ell+n+\ell n}e^{\pi z w_{\ell n}}\right),$$

$\ell>0$, *is a bounded unconditional basis for the Bargmann—Fock spaces* $\mathbf{A}^p(\mathbf{C})$ *for* $1\leqq p<\infty$.

Every $F\in A^p(\mathbf{C})$ *has a unique expansion*

(2)
$$F=\sum_{\ell=0}^{\infty}\sum_{n\in\mathbf{Z}}\langle F, \tilde{\Psi}_{\ell,n}\rangle\Psi_{\ell,n}$$

where the biorthogonal basis $\{\tilde{\Psi}_{\ell,n}\}$,

$$\tilde{\Psi}_{0,n} = 2e^{-\pi n^2/2} e^{\pi z n} \Phi(z-n) \tag{3}$$
$$\tilde{\Psi}_{\ell,n} = 2^{3/2} e^{-\pi |w_{\ell n}|^2/2} \left(e^{\pi z \overline{w_{\ell n}}} \Phi(z-\overline{w_{\ell n}}) + (-1)^{\ell+n+\ell n} e^{\pi z w_{\ell n}} \Phi(z-w_{\ell n})\right)$$

has the same structure. The function $\Phi \in \mathbf{A}^p(\mathbf{C})$ *for each p and depends only on* ϕ.

In other words, the basis consists of simple linear combinations of the reproducing kernel e_w. Therefore, Theorem 1 implies immediately the following sampling and interpolation theorem.

Corollary 2. (a) *Sampling: Let* $F \in \mathbf{A}^p(\mathbf{C})$ *for some* $1 \leq p < \infty$. *Let*

$$a_{0,n} = F(n), \quad n \in \mathbf{Z}, \tag{4}$$
$$a_{\ell,n} = F(w_{\ell,n}) + (-1)^{\ell+n+\ell n} F(\overline{w_{\ell,n}}), \quad \ell = 1, 2, \ldots;\ n \in \mathbf{Z}.$$

Then

$$F(z) = \sum_{\ell=0}^{\infty} \sum_{n\in \mathbf{Z}} a_{\ell,n} e^{-\pi |w_{\ell n}|^2/2} \tilde{\Psi}_{\ell,n}(z) \tag{5}$$

where the series converges in $\mathbf{A}^p(\mathbf{C})$.

(b) *Interpolation: Let* $\{a_{\ell,n}\colon \ell = 0, 1, \ldots;\ n \in \mathbf{Z}\}$ *and* $1 \leq p < \infty$ *be given such that*

$$\sum_{\ell=0}^{\infty} \sum_{n\in \mathbf{Z}} |a_{\ell,n}|^p e^{-p\pi |w_{\ell n}|^2/2} < \infty.$$

Then there exists $F \in \mathbf{A}^p(\mathbf{C})$ *such that*

$$F(n) = a_{0,n}, \quad n \in \mathbf{Z},$$
$$F(w_{\ell,n}) + (-1)^{\ell+n+\ell n} F(\overline{w_{\ell,n}}) = a_{\ell,n} \quad \ell = 1, 2, \ldots;\ n \in \mathbf{Z}.$$

Specifically, F is given by (5).

By duality, the collection $\{\Psi_{\ell,n}\}$ is a weak basis for $A^\infty(\mathbf{C})$ and by a standard argument also an unconditional basis for $A^0(\mathbf{C})$.

This is a new type of sampling and uniqueness theorem. It is well-known that $F \in \mathbf{A}^2(\mathbf{C})$ is uniquely determined by its values at $n+il$, $n, l \in \mathbf{Z}$. On the other hand, the interpolation problem for this lattice is ill-posed: If $\sum_{\ell=0}^{\infty} \sum_{n\in\mathbf{Z}} |a_{\ell,n}|^2 e^{-\pi |w_{\ell n}|^2} < \infty$, the interpolating entire function F which satisfies $F(n+il) = a_{ln}$ is not necessarily in $\mathbf{A}^2(\mathbf{C})$ and the reconstruction of F from its sampled values is extremely unstable (see, e.g., [D], [DG]).

On the set $\mathbf{Z} \cup \{n/2 + il,\ n, l \in \mathbf{Z}\}$, however, F is overdetermined and the interpolation problem is unsolvable in general (cf. [DG]).

Corollary 2 shows that in order to achieve uniqueness and stable interpolation appropriate sums and differences of the samples have to be taken. We are not aware of any results of this type in the literature.

Recently, Seip and Wallstén [SW] have shown that whenever $ab<1$, the set $\lambda_{mn}=bm+ian$ is a set of sampling for the Bargmann—Fock spaces with $p>0$. That is, a function can be reconstructed from its sampled values in a stable way. The interpolation problem on such a set is not solvable in general. In this context, they ask whether there exists a set in $\mathbf{C}$ of both sampling and interpolation, or equivalently a Riesz basis consisting of reproducing kernel functions. Theorem 1 and Corollary 2 can be regarded as a possible solution to this problem.

The idea behind the proof of Theorem 1 comes from signal analysis and the theory of the Gabor transform [D], [HW]. The central fact is that the Bargmann—Fock spaces $A^p(\mathbf{C})$ are isometrically isomorphic to spaces of functions on $\mathbf{R}$ called coorbit spaces [FG 1], [FG 2]. The coorbit space $Co(L^p)$ is defined to be those functions on $\mathbf{R}$ whose Gabor transform lies in $L^p(\mathbf{R}\times\hat{\mathbf{R}})$ (see Section 1.4).

The technical part of the proof consists in showing that a certain operator — the Gabor frame operator (see Section 1.2) — is invertible on these coorbit spaces. Theorem 1 then follows by taking the Bargmann transform. A similar proof shows that Theorem 1 also holds true for even more general spaces of entire functions related to coorbits of spaces other than L^p.

The technical result about the invertibility of the frame operator on other spaces is of interest in its own right since it provides estimates for the lattice density for Gabor frames for spaces other than the Hilbert space $L^2(\mathbf{R})$. The Seip—Wallstén result provides the best possible such estimate when the analyzing function is a Gaussian. It says that for any $ab<1$, $\{E_{mb}T_{na}\phi\}$ is a Gabor frame for $Co(L^p)$, $p\geqq 1$ (see [Gr]).

It is very likely that a direct proof of Theorem 1 within the theory of entire functions will be found. The possibility of such a result, however, emerged from ideas in signal analysis and the theory of Wilson bases for distribution spaces on $\mathbf{R}$.

The paper is organized as follows:

Section 1 contains basic results on frames in Hilbert spaces and in particular Gabor frames on the Hilbert space $L^2(\mathbf{R})$. Some elementary properties of the Bargmann—Fock spaces and coorbit spaces are also included. Section 2 contains the statement of the main lemma mentioned above, namely that the Gabor frame operator is an isomorphism on the coorbit spaces in question. The proofs of Theorem 1 and Corollary 2 are presented in this section. Section 3 contains the proof of the main lemma, which relies on a characterization of the coorbit spaces $Co(L^p)$ by means of the Zak transform [AT], [Z]. Some basic properties of the Zak transform are presented.

1. An unconditional basis for Bargmann—Fock space

In this section we collect the necessary concepts and results on frames, Gabor frames and the Bargmann transform. For motivation and detailed proofs, we refer to the quoted references, in particular [DS], [D], [HW].

1.1 Frames.

A **frame** in a separable Hilbert space H is a collection of vectors $\{x_n\}$ with the property that for some constants c_1 and $c_2>0$

$$c_1\|x\|^2 \leqq \sum_n |\langle x, x_n\rangle|^2 \leqq c_2\|x\|^2 \tag{6}$$

for every $x\in H$. If $c_1=c_2$ then the frame is said to be **tight.**

In general frames are not bases and are overcomplete in the sense that any finite subset of the elements lies in the closed linear span of the other. However, like for an orthonormal basis, any vector can be recovered from its frame coefficients in a simple way.

Associated to each frame $\{x_n\}$ is an operator called the **frame operator,** denoted here by S and defined by $Sx=\sum_n \langle x, x_n\rangle x_n$. The sum defining Sx converges unconditionally in H for each x. S is a bounded, linear, positive operator (hence self-adjoint), and (6) implies that S is invertible on H. The **dual frame** associated to a frame $\{x_n\}$ is the collection $\{S^{-1}x_n\}$, denoted $\{\tilde{x}_n\}$. The dual frame is a frame in its own right with bounds c_2^{-1} and c_1^{-1}, and we have the following identity.

$$x = \sum_n \langle x, x_n\rangle \tilde{x}_n = \sum_n \langle x, \tilde{x}_n\rangle x_n \tag{7}$$

where the sums converge in H for each x. If $\{x_n\}$ were a tight frame $(c_1=c_2)$, then (7) becomes $x=c_1^{-1}\sum_n \langle x, x_n\rangle x_n$.

If $\{x_n\}$ is a frame in a Hilbert space H with frame operator S, then a simple computation shows that $\sum \langle x, S^{-1/2}x_n\rangle S^{-1/2}x_n=x$. Thus, $\{S^{-1/2}x_n\}$ is a tight frame with bound 1.

1.2 Gabor frames.

For fixed $a, b>0$, and $g\in L^2(\mathbf{R})$, a **Gabor system** is the collection of functions $\{g_{n,m}\}=\{e^{2\pi imbx}g(x-na)\}$. We will also use the notation $\{g_{n,m}\}=\{E_{mb}T_{na}g\}$ where $E_x g(t)=e^{2\pi itx}g(t)$ and $T_x g(t)=g(t-x)$. If $\{g_{n,m}\}$ is a frame for $L^2(\mathbf{R})$, then it is called a **Gabor frame.** To emphasize dependence on parameters, we say that (g, a, b) generates a frame if $\{E_{mb}T_{na}g\}$ is a frame for $L^2(\mathbf{R})$. It is easy to construct Gabor frames for $L^2(\mathbf{R})$.

Since the frame operator S associated to the Gabor frame $\{E_{mb}T_{na}g\}$ (the **Gabor frame operator**) commutes with the operators E_{mb} and T_{na}, the dual frame is also a Gabor frame, namely $\{E_{mb}T_{na}S^{-1}g\}$. This commutativity also implies that if $\{E_{mb}T_{na}g\}$ is a frame then $\{E_{mb}T_{na}S^{-1/2}g\}$ is a tight frame with frame bound 1. If for $a, b>0$, $ab\leq 1$, the Gabor system (g, a, b) generates a tight frame, then $c_1=c_2=\|g\|_2^2/ab$ (see [D]). This implies that for any Gabor frame, $\|S^{-1/2}g\|_2=ab$.

1.3 Wilson bases.

The following remarkable result has been proved in [DJJ].

Theorem 1.1. *Suppose that* $\psi\in\mathscr{S}(\mathbf{R})$ *is such that* $\|\psi\|_2=1$, $\hat{\psi}$ *is real-valued, and* $(\psi, 1/2, 1)$ *generates a tight frame for* $L^2(\mathbf{R})$. *Then the following collection is an orthonormal basis for* $L^2(\mathbf{R})$.

$$\begin{aligned} \psi_{0,n}(x) &= \psi(x-n) \\ \psi_{\ell,n}(x) &= 2^{-1/2}\left(e^{-2\pi i\ell x}\psi(x-n/2)+(-1)^{\ell+n}e^{2\pi i\ell x}\psi(x-n/2)\right) \end{aligned} \tag{8}$$

for $\ell=1, 2, \ldots$ *and* $n\in\mathbf{Z}$.

Corollary 1.2. *Let* $g\in\mathscr{S}(\mathbf{R})$ *be such that* $(g, 1/2, 1)$ *generates a frame for* $L^2(\mathbf{R})$ *and* $\hat{g}$ *is real-valued. Then the following collection is a bounded, unconditional basis for* $L^2(\mathbf{R})$.

$$\begin{aligned} g_{0,n}(x) &= g(x-n) \\ g_{\ell,n}(x) &= 2^{-1/2}\left(e^{-2\pi i\ell x}g(x-n/2)+(-1)^{\ell+n}e^{2\pi i\ell x}g(x-n/2)\right). \end{aligned} \tag{9}$$

The dual basis is given by

$$\begin{aligned} \tilde{g}_{0,n}(x) &= S^{-1}g(x-n) \\ \tilde{g}_{\ell,n}(x) &= 2^{-1/2}\left(e^{-2\pi i\ell x}S^{-1}g(x-n/2)+(-1)^{\ell+n}e^{2\pi i\ell x}S^{-1}g(x-n/2)\right). \end{aligned} \tag{10}$$

Proof. Given such a g, we know that $(S^{-1/2}g, 1/2, 1)$ generates a tight frame for $L^2(\mathbf{R})$. Moreover, $S^{-1/2}g\in\mathscr{S}(\mathbf{R})$ and $(S^{-1/2}g)^\wedge$ is real-valued (see Lemma 2.1b or [DJJ]). So by Theorem 1.1, the collection $\{\psi_{\ell,n}\}$ corresponding to this tight frame is an (unnormalized) orthogonal basis for $L^2(\mathbf{R})$. Applying the isomorphism $S^{1/2}$ to this collection gives the collection (9). Moreover, it is clear that the dual basis corresponding to $\{S^{1/2}\psi_{\ell,n}\}$ is $\{S^{-1/2}\psi_{\ell,n}\}=\{\tilde{g}_{\ell,n}\}$ which is (10). ∎

1.4. The Bargmann transform.

The **Bargmann transform** B is a unitary map from $L^2(\mathbf{R})$ onto $\mathbf{A}^2(\mathbf{C})$ given by [Fo]

$$Bf(z) = 2^{1/4}e^{-\pi z^2/2}\int e^{2\pi xz}e^{-\pi x^2}f(x)\,dx. \tag{11}$$

If $\phi(x)=2^{1/4}e^{-\pi x^2}$, the L^2 normalized Gaussian, then $B\phi(z)=1$.

The following formulas establish a fundamental relation between the theory of the Bargmann—Fock spaces and Gabor theory. Putting $w=p+iq$, we obtain

$$B(E_q T_p \phi)(z) = e^{\pi i pq}e^{-\pi|w|^2/2}e^{\pi z w}. \tag{12}$$

Thus, given $f\in L^2(\mathbf{R})$,

$$\langle f, E_q T_p \phi\rangle = \langle Bf, B(E_q T_p\phi)\rangle = e^{\pi i pq}e^{-\pi|w|^2/2}F(\overline{w}) \quad \text{where} \quad F = Bf. \tag{13}$$

It follows immediately that $\langle f, E_q T_p \phi\rangle\in L^p(\mathbf{R}^2)$ for some function f on $\mathbf{R}$ if and only if $F\in \mathbf{A}^p(\mathbf{C})$ for $1\leqq p<\infty$.

The images of $\mathbf{A}^p(\mathbf{C})$ under the inverse Bargmann transform are spaces of tempered distributions, which are interesting in their own right, the so called modulation spaces [F] or coorbit spaces, denoted $Co(L^p)$, of the Heisenberg group under the Schrödinger representation [FG 1], [FG 2].

Given $g\in\mathcal{S}(\mathbf{R})$ fixed and the Schrödinger representation $\pi(q,p,\tau)f=\tau T_q E_p f$ for $f\in L^2(\mathbf{R})$ of the Heisenberg group $\mathbf{H}=\mathbf{R}^2\times\mathbf{T}$, they are defined by

$$Co(L^p) = \{f\in\mathcal{S}'\colon \langle f, \pi(q,p,\tau)g\rangle\in L^p(\mathbf{H})\}$$

with norm $\|f\|_{Co\,L^p}=\|\langle f,\pi(q,p,\tau)g\rangle\|_{L^p(\mathbf{H})}$. The definition of $Co(L^p)$ is independent of the analyzing function g. The choice $g=\phi$ reveals that $Co(L^p)$ and $\mathbf{A}^p(\mathbf{C})$ are (isometrically) isomorphic Banach spaces under the Bargmann transform. See [FG 1], [FG 2] for more details on coorbit spaces.

Our interest in the spaces $Co(L^p)$ is that the Gabor frame operator is easier to handle than the corresponding operator on $\mathbf{A}^p(\mathbf{C})$.

2. Construction of the unconditional basis

The images of the bases (8) (resp. (9)) under the Bargmann transform are orthogonal (resp. unconditional) bases for $\mathbf{A}^2(\mathbf{C})$. For the orthogonal Wilson bases, it has been shown that they are also unconditional bases for the spaces $Co(L^p)$ and by the Bargmann transform one obtains unconditional bases for the $\mathbf{A}^p(\mathbf{C})$ [FGW]. We will use this fact in the proof of Theorem 1.

The proof of an analogous statement for the bases of Corollary 1.2 poses additional difficulties because one has to deal with the systems of functions derived from g and (for the dual basis) from $\tilde{g}=S^{-1}g$. The arguments of [FGW] carry over provided that S is also invertible on other coorbit spaces. For $g=\phi$, this is guaranteed by the following lemma.

Main Lemma 2.1. *Let* $\phi(x)=2^{1/4}e^{-\pi x^2}$, *and let S be the Gabor frame operator defined by*

$$Sf = \sum_n \sum_m \langle f, E_m T_{n/2}\phi\rangle E_m T_{n/2}\phi.$$

Then

(a) *S and $S^{1/2}$ are isomorphisms of the spaces $Co(L^p)$ for $1\leq p<\infty$.*

(b) *$S^{-1/2}\phi\in\mathscr{S}(\mathbf{R})$ and $(S^{-1/2}\phi)^\wedge$ is real-valued.*

Theorem 1 and Corollary 2 now follow.

Proof of Theorem 1. As in Corollary 1.2, the collection

$$(14)\qquad \begin{aligned} \psi_{0,n} &= T_n S^{-1/2}\phi \\ \psi_{\ell,n} &= 2^{1/2}(E_{-\ell}T_{n/2}S^{-1/2}\phi+(-1)^{\ell+n}E_\ell T_{n/2}S^{-1/2}\phi) \end{aligned}$$

is an orthonormal basis for $L^2(\mathbf{R})$ and by [FGW], Theorem 1, an unconditional basis for $Co(L^p)$, $1\leq p<\infty$. By Lemma 2.1 $S^{1/2}$ is an isomorphism of $Co(L^p)$ for $1\leq p<\infty$. Therefore $\{S^{1/2}\psi_{\ell,n}\}$ is also an unconditional basis for $Co(L^p)$. By the commutativity properties of the frame operator, this is given by

$$(15)\qquad \begin{aligned} S^{1/2}\psi_{0,n} &= T_n\phi \\ S^{1/2}\psi_{\ell,n} &= 2^{1/2}(E_{-\ell}T_{n/2}\phi+(-1)^{\ell+n}E_\ell T_{n/2}\phi). \end{aligned}$$

The dual basis corresponding to this system is $\{2S^{-1/2}\psi_{\ell,n}\}$ which is given by

$$(16)\qquad \begin{aligned} 2S^{-1/2}\psi_{0,n} &= 2T_n S^{-1}\phi \\ 2S^{-1/2}\psi_{\ell,n} &= 2^{3/2}(E_{-\ell}T_{n/2}S^{-1}\phi+(-1)^{\ell+n}E_\ell T_{n/2}S^{-1}\phi). \end{aligned}$$

Since the Bargmann transform is an isomorphism of $Co(L^p)$ onto $\mathbf{A}^p(\mathbf{C})$, the collection

$$\{e^{-\pi i n l/2}BS^{1/2}\psi_{\ell,n}\} = \{\Psi_{\ell,n}\}$$

is an unconditional basis for $\mathbf{A}^p(\mathbf{C})$ with dual basis

$$\{2e^{-\pi i n l/2}BS^{-1/2}\psi_{\ell,n}\} = \{\tilde{\Psi}_{\ell,n}\}.$$

The formulas (1) and (3) can be verified directly from (12) where $\Phi=BS^{-1}\phi$. ∎

Corollary 2 now follows directly from Theorem 1, (12), and (13).

3. Proof of the Main Lemma

For the proof we make use of the Zak transform, a useful tool for studying properties of Gabor expansions which has been used in this and other contexts. For more details on the Zak transform, see e.g., [AT], [Fo], [J 1], [J 2] etc. Since the following properties are of independent interest for Gabor expansions, we state them in a slightly more general form than is required for Lemma 2.1.

Definition 4.1. Given $a>0$, the **Zak transform**, or **Weil—Brezin map** denoted Z_a is a unitary mapping from $L^2(\mathbf{R})$ onto $L^2[0,1]^2$ given by

$$Z_a f(t,\omega) = a^{1/2} \textstyle\sum_k e^{2\pi ik\omega} f(a(t-k)).$$

$Z_a f$ satisfies the following **quasi-periodicity** relations

$$Z_a f(t+1,\omega) = e^{2\pi i\omega} Z_a f(t,\omega), \quad Z_a f(t,\omega+1) = Z_a f(t,\omega).$$

Note that $Z_a f(t,\omega)=Z_1(D_a f)(t,\omega)$ where D_a is the dilation operator $D_a f(x)= a^{1/2}f(at)$. From now on, Z_1 will be denoted simply by Z.

Also note that f is even if and only if $Z_a f(t,\omega)=Z_a f(-t,-\omega)$.

If $ab=1/N$ for some integer $N\geqq 1$ a simple calculation yields

$$Z_a(E_{mb}T_{na}g)(t,\omega) = e^{2\pi imn/N} e^{2\pi imt/N} e^{-2\pi in\omega} Z_a g(t,\omega-m/N). \tag{17}$$

Then next well-known lemma shows that the frame operator is equivalent to a multiplication operator by means of the Zak transform.

Lemma 4.2. *Let $g\in L^2(\mathbf{R})$, and suppose that $ab=1/N$ for some $a,b>0$, $N\in\mathbf{Z}$, $N>0$. Let $G(t,\omega)=\sum_{j=0}^{N-1} |Z_a g(t,\omega-j/N)|^2$. The Gabor frame operator S is given by $Sf=Z_a^{-1}GZ_a f$ for $f\in\mathcal{S}(\mathbf{R})$ where the operator G stands for multiplication by $G(t,\omega)$. Consequently, (g,a,b) generates a frame if and only if there exist constants $A,B>0$ such that*

$$A \leqq \textstyle\sum_{j=0}^{N-1} |Z_a g(t,\omega-j/N)|^2 \leqq B \tag{18}$$

for almost all $(t,\omega)\in[0,1]^2$.

Proof. Denote by $\langle\cdot,\cdot\rangle$ the usual inner product on $L^2[0,1]^2$, and let $G_j(t,\omega)= Z_a g(t,\omega-j/N)$, for $j=0,1,\dots,N-1$. Assume $f\in\mathcal{S}(\mathbf{R})$. Applying (17), the unitarity of Z, and Fourier series on $L^2[0, 1]^2$,

$$\begin{aligned} Z_a Sf &= \sum_{n,m} \langle Z_a f, Z_a(E_{mb}T_{na}g)\rangle Z_a(E_{mb}T_{na}g) \\ &= \sum_{n,m} \langle Z_a f\, \overline{Z_a g(t', \omega' - m/N)}, e^{2\pi i(mt'/N - n\omega')}\rangle\, e^{2\pi i(mt/N - n\omega)} Z_a g(t, \omega - m/N) \\ &= \sum_{j=0}^{N-1} e^{2\pi ijt/N} G_j(t, \omega) \sum_{n,k} \langle Z_a f\, e^{-2\pi ijt'/N} \overline{G_j}, e^{2\pi i(kt' - n\omega')}\rangle e^{2\pi i(kt - n\omega)} \\ &= \sum_{j=0}^{N-1} e^{2\pi ijt/N} G_j(t, \omega) Z_a f(t, \omega) e^{-2\pi ijt/N} \overline{G_j(t, \omega)} \\ &= Z_a f(t, \omega) \sum_{j=0}^{N-1} |Z_a g(t, \omega - j/N)|^2. \quad \blacksquare \end{aligned}$$

Lemma 4.3. *Let* $g, f \in \mathcal{S}(\mathbf{R})$. *Then*

$$\|f\|^p_{Co(L^p)} = \sum_{n,m} \int_{[0,1]^2} \left| \int_{[0,1]^2} e^{-2\pi iyt}\, \overline{Zg(t-x, \omega-y)}\, Zf(t, \omega)\, e^{2\pi i(nt - m\omega)}\, dt\, d\omega \right|^p dx\, dy. \tag{19}$$

Proof. First observe that $Z(E_yT_xg)(t, \omega) = e^{2\pi iyt} Zg(t-x, \omega-y)$. Since Z is unitary, we obtain

$$\begin{aligned} &\|f\|^p_{Co\,L^p} \\ &= \int_{\mathbf{R}^2} |\langle f, E_yT_xg\rangle|^p\, dx\, dy = \int_{\mathbf{R}^2} |\langle Zf, Z(E_yT_xg)\rangle|^p\, dx\, dy \\ &= \int_{\mathbf{R}^2} \left| \int_{[0,1]^2} Zf(t, \omega)\, \overline{Zg(t-x, \omega-y)}\, e^{-2\pi iyt}\, dt\, d\omega \right|^p dx\, dy \\ &= \sum_{n,m} \int_{[0,1]^2} \left| \int_{[0,1]^2} Zf(t, \omega)\, \overline{Zg(t-x+m, \omega-y+n)}\, e^{-2\pi i(y-n)t}\, dt\, d\omega \right|^p dx\, dy \\ &= \sum_{n,m} \int_{[0,1]^2} \left| \int_{[0,1]^2} Zf(t, \omega)\, e^{-2\pi iyt}\, \overline{Zg(t-x, \omega-y)}\, e^{2\pi i(nt - m\omega)}\, dt\, d\omega \right|^p dx\, dy. \quad \blacksquare \end{aligned}$$

Lemma 4.4. *Let* $M(t, \omega)$ *be an absolutely convergent Fourier series on* $[0, 1]^2$, *that is,* $M(t, \omega) = \sum_{n,m} \gamma_{n,m} e^{2\pi i(nt + m\omega)}$ *with* $\|M\|_A = \sum_{n,m} |\gamma_{n,m}| < \infty$.

Then for any $a > 0$, *the operator* $S_M = Z_a^{-1} M Z_a$ *is continuous on* $Co(L^p)$, $1 \leq p < \infty$ *and*

$$\|S_M f\|_{Co\,L^p} \leq \|M\|_A \|f\|_{Co\,L^p}. \tag{20}$$

Proof. Note first that since $Co(L^p)$ is invariant under dilation, it suffices to consider only the case $a = 1$. Next observe that for each fixed $(x, y) \in [0, 1]^2$, the function

$$e^{-2\pi iyt}\, \overline{Zg(t-x, \omega-y)}\, Zf(t, \omega)$$

is periodic in (t, ω) with period 1. Thus for each (x, y) the inner integral in (19) is just the Fourier transform of this function at $(-n, m)$.

We first show (20) for $f\in\mathcal{S}(\mathbf{R})$. We show that for each $(x,y)\in[0,1]^2$

$$(21)\qquad \sum_{n,m}\left|\int_{[0,1]^2} e^{-2\pi iyt}\,\overline{Zg(t-x,\omega-y)}\,M(t,\omega)\,Zf(t,\omega)\,e^{2\pi i(nt-m\omega)}\,dt\,d\omega\right|^p$$
$$\leqq \|M\|_A^p \sum_{n,m}\left|\int_{[0,1]^2} e^{-2\pi iyt}\,\overline{Zg(t-x,\omega-y)}\,Zf(t,\omega)\,e^{2\pi i(nt-m\omega)}\,dt\,d\omega\right|^p.$$

This estimate holds because the left hand side is just the p-th power of the ℓ^p norm of the Fourier coefficients of the function

$$M(t,\omega)\,e^{-2\pi iyt}\,\overline{Zg(t-x,\omega-y)}\,Zf(t,\omega)$$

which is nothing more than the convolution of $\gamma_{n,m}$ with the Fourier coefficients of

$$e^{-2\pi iyt}\,\overline{Zg(t-x,\omega-y)}\,Zf(t,\omega).$$

By standard properties of convolutions, inequality (21) holds for each fixed $(x,y)\in[0,1]^2$.

Integrating both sides of (21) over $(x,y)\in[0,1]^2$ gives (20) for $f\in\mathcal{S}(\mathbf{R})$. By the density of $\mathcal{S}(\mathbf{R})$ in $Co(L^p)$ [F], (20) follows. ∎

Proof of Lemma 2.1. The function $Z_{1/2}\phi$ is smooth and in fact [DG]

$$Z_{1/2}\phi(t,\omega) = \frac{1}{\sqrt{2}}\,e^{-\pi/4t^2}\,\theta_3\left(\omega-\frac{i}{4}\,t\,\middle|\,\frac{i}{4}\right)$$

where θ_3 is the Jacobi theta function defined by

$$\theta_3(z|\tau) = 1+2\sum_{k=1}^{\infty}\cos(2\pi kz)\,e^{i\pi\tau k^2}.$$

It is known that $\theta_3(z|\tau)$ has only one zero at $z=\frac{1}{2}+\frac{1}{2}\tau$. Thus, $Z_{1/2}\phi$ vanishes at $(1/2, 1/2)$ and nowhere else in $[0,1]^2$ ([R], p. 314ff.). Thus, it is clear that the function

$$G(t,\omega) = |Z_{1/2}\phi(t,\omega)|^2+|Z_{1/2}\phi(t,\omega-1/2)|^2$$

is bounded above and away from zero. Thus, by Lemma 4.2, $(\phi, 1/2, 1)$ generates a frame for $L^2(\mathbf{R})$ and the frame operator is given by $Sf=Z_{1/2}^{-1}GZ_{1/2}f$.

Since S is equivalent to multiplication by G, $S^{1/2}f=Z_{1/2}^{-1}G^{1/2}Z_{1/2}f$. The boundedness of S and $S^{1/2}$ on $Co(L^p)$ will follow from Lemma 4.4 and the fact that G and $G^{1/2}$ are absolutely convergent Fourier series. This is true since $\phi\in\mathcal{S}(\mathbf{R})$ implies that $G\in C^\infty[0,1]^2$. Since G is bounded below, $G^{1/2}\in C^\infty[0,1]^2$. The invertibility of S and $S^{1/2}$ amounts to the assertion that both G^{-1} and $G^{-1/2}$ are absolutely convergent Fourier series. This follows from the smoothness of G and classical results of Wiener and Levy ([Zy], Sec. VI.5) on absolutely convergent Fourier series. This completes the proof of part (a) of Lemma 2.1.

Since $f \in \mathscr{S}$ if and only if $Zf \in C^\infty[0,1]^2$, it is clear from the above that $S^{-1/2}\phi = Z_{1/2}^{-1} G^{-1/2} Z_{1/2}\phi \in \mathscr{S}(\mathbf{R})$. To see that $(S^{-1/2}\phi)^\wedge$ is real-valued, we prove that $S^{-1/2}\phi$ is even. Now, $Z_{1/2}S^{-1/2}\phi = G^{-1/2}Z_{1/2}\phi$. Since ϕ is even, $Z_{1/2}\phi(-t,-\omega) = Z_{1/2}\phi(t,\omega)$. Also, $G(-t,-\omega) = G(t,\omega)$ since ϕ is even and since the Zak transform of any function is 1-periodic in ω. This completes the proof of Lemma 2.1. ∎

It is clear that we can extend Theorem 1 and Corollary 2 to a more general class of entire functions on **C**, e.g., to the spaces

$$A_w^{p,q}(\mathbf{C}) = \left\{F \text{ entiere: } \left(\int \left(\int |F(x+iy)|^p w^p(x,y) e^{-\pi p(x^2+y^2)/2} dx\right)^{q/p} dy\right)^{1/q} < \infty\right\}$$

where w is for example a weight of polynomial growth. These spaces are isomorphic to coorbit spaces of weighted mixed norm spaces. To prove the corresponding version of Theorem 1, it has to be shown that powers of the frame operator are bounded on appropriate coorbit spaces. The only change occurs in Lemma 4.4 where the multiplier must now satisfy $M(t,\omega) = \sum_{n,m} \gamma_{n,m} e^{2\pi i(nt+m\omega)}$ with $\sum_{n,m} |\gamma_{n,m}| \omega(n,m) < \infty$ and weighted convolution inequalities must now be used.

As long as the weight ω is a Beurling—Domar weight, e.g., any weight of polynomial growth, the theorem of Wiener—Levy is still applicable and a version of Lemma 2.1 holds. The exact details are left to the reader. For completeness, we state the result: *The collection* $\{\Psi_{\ell,n}\}$ *of* (1) *is an unconditional basis for the weighted spaces* $A_\omega^{p,q}(\mathbf{C})$.

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