A new type of Littlewood-Paley partition

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Abstract. We define a partition of $\mathbf{Z}$ into intervals $\{I_j\}$ and prove the Littlewood—Paley inequality $\|f\|_p \leqq C_p \|Sf\|_p$, $2 \leqq p < \infty$. Here f is a function on $[0, 2\pi)$ and $Sf = (\Sigma |\Delta_j|^2)^{1/2}$, $\hat{\Delta}_j = \hat{f}\chi_{I_j}$. This is a new example of a partition having the Littlewood—Paley property since the $\{I_j\}$ are not of the type obtained by iterating lacunary partitions finitely many times.

Introduction. In this paper we define a certain partition of the integers into intervals, and prove that it has the Littlewood—Paley property (1.1). The other inequality ($\geqq$) in (1.1) was proved by Rubio de Francia [4] for arbitrary intervals.

All previously known interval partitions satisfying (1.1) were obtained by iterating lacunary partitions, as far as we are aware [1], [2], [3], [5]. The present partition cannot be obtained in this way, as was shown in [3]. Briefly, the argument there was that (finitely) iterated lacunary partitions do not contain arbitrarily large "trees" of intervals, whereas the present partition is itself an infinite "tree".

There is, however, a different relationship between lacunarity and the present partition. This is that the set of *lengths* of its intervals is a finite union of lacunary sequences. This fact is not used in our proofs directly, but it suggests a way to generalize our partition. We discuss this at the end of the paper.

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Definitions and statement of result. Let E and F be finite nonempty subsets of $\mathbf{Z}$. Define

$$l(E) = \max_{x, y \in E} |x-y|,$$

$$\delta(E) = \min_{\substack{x, y \in E \\ x \neq y}} |x-y|,$$

$$d(E, F) = \min_{x \in E, y \in F} |x-y|.$$

Fix $\lambda>0$ and a sequence $\lambda_k \geqq \lambda$, $k=1, 2, \ldots$, and define sets E_k by

$$E_1 = \{0, 1\},$$
$$E_k^* = m_k E_k + t_k,$$
$$E_{k+1} = \{a_k, b_k\} \cup E_k \cup E_k^*,$$

using sequences m_k, $t_k \in \mathbf{N}$, a_k, $b_k \in \mathbf{Z}$, such that

1. $\delta(E_k^*) \geqq \lambda_k l(E_k)$,
2. $d(E_k, E_k^*) \geqq \lambda_k [l(E_k) + l(E_k^*)]$,
3. $a_k < E_k < E_k^* < b_k$, and
   $d(a_k, E_k)$, $d(E_k^*, b_k) \geqq \lambda_k l(E_k \cup E_k^*)$.

(A glance at Fig. 1 shows that all of these conditions can be satisfied very easily.)

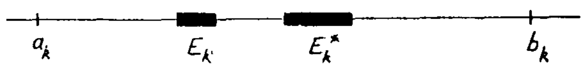

Fig. 1. Construction of E_{k+1}.

Define $E_\infty = \bigcup_{k=1}^\infty E_k$. Clearly $E_k \uparrow E_\infty$, and each E_k has a natural binary tree structure. We can represent this as in Fig. 2, which shows a subtree E of E_∞, its first node Δ, and its left and right subtrees F and G.

In Fig. 2 and later, we make the convention that names of subsets of E_∞ such as E, F, G also denote the collections of intervals between their points. So we think of Δ as the difference:

$$\Delta = E \setminus (F \cup G) = \{\Delta_1, \Delta_2, \Delta_3\},$$

consisting of the 3 intervals shown in Fig. 2.

Clearly the following properties hold at each node Δ:

(E1) $\delta(G) \geqq \lambda l(F)$,
(E2) $l(\Delta_3) \equiv d(F, G) \geqq \lambda [l(F) + l(G)]$,
(E3) $l(\Delta_1), l(\Delta_2) \geqq \lambda l(F \cup G)$.

For consistency, the intervals Δ_j should be thought of as real intervals $[x, y)$ with length $l(\Delta_j) = y - x$. If a trigonometric polynomial $\mathscr{F}$ is given, Δ, F, G, etc. will also denote partial sums of $\mathscr{F}$ in the natural sense (we refer the reader to Figure 2 again):

$$\hat{\Delta}_j = \chi_{\Delta_j} \hat{\mathscr{F}},$$
$$\Delta = \Delta_1 + \Delta_2 + \Delta_3,$$
$$\hat{F} = \hat{\mathscr{F}} \chi_F = \sum_{\Delta_j \in F} \hat{\Delta}_j, \quad \hat{G} = \hat{\mathscr{F}} \chi_G = \sum_{\Delta_j \in G} \hat{\Delta}_j.$$

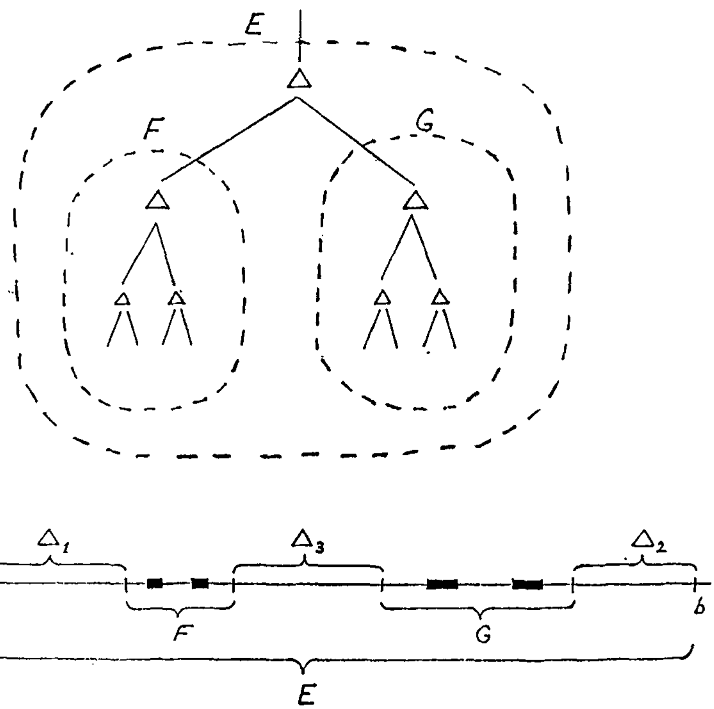

Fig. 2. Tree structure. We define $\Delta = \{\Delta_1, \Delta_2, \Delta_3\}$.

We also have

$$\mathscr{F} = \sum_{\Delta_j \in E_\infty} \Delta_j,$$

since in fact the intervals of E_∞ cover $\mathbf{Z}$, as seen from step 3 of the construction.

For a set of integers $A = \{n_j\}$ (where $n_j < n_{j+1}$) and trigonometric polynomial f define $\hat{f}_j = \hat{f}\chi_{[n_j, n_{j+1})}$ and

$$S_A(f) = (\sum |f_j|^2)^{1/2}.$$

We will simply write S for S_{E_∞}, so with the notation above we have

$$S(\mathscr{F}) = (\sum_{\Delta_j \in E_\infty} |\Delta_j|^2)^{1/2}.$$

Theorem 1. *If $\lambda_k \to \infty$, then there exist $C_p > 0$ such that for all $\mathscr{F}$,*

$$\|\mathscr{F}\|_p \leqq C_p \|S(\mathscr{F})\|_p, \quad 2 \leqq p < \infty. \tag{1.1}$$

We prove this by taking $p = 2n$, $n = 1, 2, \ldots$. For n fixed, we prove (1.1) for any tree satisfying (E1)—(E3) with a sufficiently large λ (depending on n). Then, a routine diagonal argument followed by norm interpolation proves the theorem.

The $p=2n$ proof uses induction on n, with an induction hypothesis involving weighted norms which is stronger than (1.1).

We also need finite refinements of the given partitions of the integers. For $A=\{n_j\}$, call S_1 an m-refinement of S_A if each interval $[n_j, n_{j+1})$ is partitioned into at most m subintervals, A' consists of the endpoints of these intervals, and $S_1=S_{A'}$. Clearly

$$S_A(\mathscr{F}) \leqq \sqrt{m} S_1(\mathscr{F}), \tag{1.2}$$

and it is well-known [1] that also

$$\|S_1(\mathscr{F})\|_p \leqq C(p, m) \|S_A(\mathscr{F})\|_p, \quad 1 < p < \infty. \tag{1.3}$$

Define the degree of a trigonometric polynomial w, denoted $\deg w$, to be the least integer l such that

$$\operatorname{supp} \hat{w} \subset [-l, l].$$

The notation $\int f$ denotes the integral over the circle $[0, 2\pi)$ with respect to standard Lebesgue measure. Our induction on n occurs in the proof of the following theorem.

Theorem 2. *For each integer $n \geqq 1$, there exist $C > 0$ and $m \in \mathbf{N}$ such that, if $\lambda \geqq 3n$ in* (E1)—(E3), *then for some m-refinement S_1 of S we have:*

$$\int w |\mathscr{F}|^{2n} \leqq C \int w S_1(\mathscr{F})^{2n}, \tag{2.1}$$

whenever $w \geqq 0$ and $\mathscr{F}$ are trigonometric polynomials satisfying

$$\mathscr{F} = \sum_{\Delta_j \in E} \Delta_j \quad \text{and} \quad \deg w < \delta(E) \equiv \min_{\Delta_j \in E} l(\Delta_j)$$

for some subtree E of E_∞ (i.e. $\operatorname{supp} \hat{\mathscr{F}} \subset \bigcup_{\Delta_j \in E} \Delta_j$).

Lemma 1. *Let $I_1, J_1, I_2, J_2, I_3, \ldots, J_N, I_{N+1}$ be adjacent intervals in $\mathbf{Z}$ (see Fig. 3). Let f be a trigonometric polynomial with $\operatorname{supp} \hat{f} \subset I_1 \cup J_1 \cup I_2 \cup \ldots \cup J_N \cup I_{N+1}$, and write $f = F_1 + \Delta_1 + F_2 + \ldots + \Delta_N + F_{N+1}$, where $\hat{F}_k = \hat{f} \chi_{I_k}$, $\hat{\Delta}_k = \hat{f} \chi_{J_k}$.*

Let $v \geqq 0$ be a trigonometric polynomial with

$$\deg v < \min_k l(J_k).$$

Then

$$\int v \cdot \sum_{k=1}^{N+1} |F_k|^2 \leqq 2 \int v |f|^2 + 70 \int v \cdot \sum_{k=1}^{N} |\Delta_k|^2. \tag{*}$$

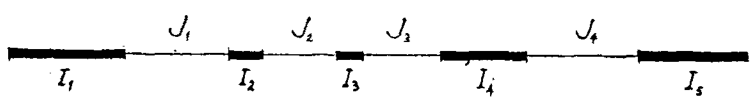

Fig. 3. The case $N=4$.

Proof.

$$\int v|f|^2 = \int vf\bar{f} = \int v \sum (|F_k|^2 + |\Delta_k|^2) + 2\,\mathrm{Re} \int v \cdot \sum (F_{k+1}\bar{\Delta}_k + \Delta_k \bar{F}_k + \Delta_k \bar{\Delta}_{k+1}),$$

(since all other terms in the expansion of $f\bar{f}$ are orthogonal to v).

But

$$|F_{k+1}\bar{\Delta}_k| \leqq \varepsilon |F_{k+1}|^2 + \frac{1}{\varepsilon}|\Delta_k|^2,$$

$$|\Delta_k \bar{F}_k| \leqq \varepsilon |F_k|^2 + \frac{1}{\varepsilon}|\Delta_k|^2,$$

$$|\Delta_k \bar{\Delta}_{k+1}| \leqq |\Delta_k|^2 + |\Delta_{k+1}|^2,$$

for any $\varepsilon > 0$ Therefore,

$$\int v|f|^2 \geqq (1-4\varepsilon)\int v \sum |F_k|^2 + \left(1 - \frac{4}{\varepsilon} - 4\right)\int v \sum |\Delta_k|^2.$$

So choosing $\varepsilon = \frac{1}{8}$ gives

$$\int v|f|^2 \geqq \frac{1}{2}\int v \sum |F_k|^2 - 35 \int v \sum |\Delta_k|^2,$$

which gives $(*)$.

Proof of Theorem 2. For $n=1$ write $\mathscr{F} = \mathscr{F}_1 + \mathscr{F}_2$ where $\mathscr{F}_1$ and $\mathscr{F}_2$ consist of alternating Δ_j (i.e. moving from left to right, every second Δ_j is 0). These alternating Δ_j are orthogonal with respect to the weight w, so we have

$$\int w|\mathscr{F}|^2 \leqq 2\int w|\mathscr{F}_1|^2 + 2\int w|\mathscr{F}_2|^2 = 2\int wS(\mathscr{F}_1)^2 + 2\int wS(\mathscr{F}_2)^2, = 2\int wS(\mathscr{F})^2.$$

Now take $n \geqq 2$ and assume that theorem 2 is true for all integers y, $1 \leqq y \leqq n-1$. Fix a version of E_∞ satisfying (E1)—(E3) with $\lambda \geqq 3n$. We wish to replace (E3) by

(E3′) $$2nl(F \cup G) \geqq l(\Delta_1),\ l(\Delta_2) \geqq nl(F \cup G).$$

This involves taking a 2-refinement and then re-defining Δ, F, and G: Introduce points a', b' at a distance of $nl(F \cup G)$ to the left and right of each F and G respectively, as shown in Fig. 4. When this has been carried out at all (except terminal)

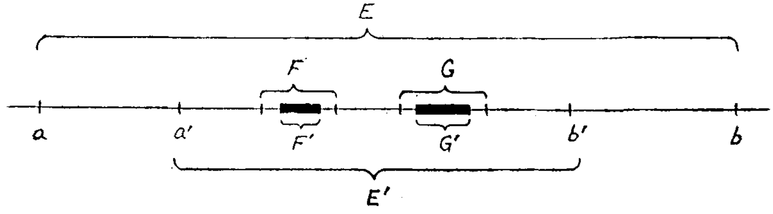

Fig. 4.

nodes, let E'_∞ consist of all the a' and b', with the obvious tree structure, inherited from E_∞. For a subtree E' define

$$\Delta' = E' \setminus (F' \cup G'),$$

and Δ'_1, Δ'_2, Δ'_3 analogous to Fig. 2. Properties (E1), (E2), (E3′) are easily verified for the Δ'_j, F', G'. If an m-refinement of S is further 2-refined by using the new points a', b' then a $2m$-refinement of $S_{E'_\infty}$ is produced. Similarly, if the points a, b are used to 2-refine an m-refinement of $S_{E'_\infty}$ then a $2m$-refinement of S is produced. So by (1.2) and the induction hypothesis, we may assume that inequality (2.1) holds for $1 \leq y \leq n-1$, with S_1 replaced by a $2m$-refinement of $S_{E'_\infty}$ (and a different constant C). Conversely, the induction step will be established if we can prove that (2.1) holds for the integer n, with S_1 replaced by some m-refinement of $S_{E'_\infty}$. This will be our objective.

To simplify notation we return to the original prime-free notation, and assume (E3′) instead of (E3). The upper bound in (E3′) is used below for (3.2) and (3.3) in Lemma 4. To begin the proof of (2.1), write $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$, where $\mathcal{F}_1$ and $\mathcal{F}_2$ consist of alternate levels of E, as shown in Fig. 5.

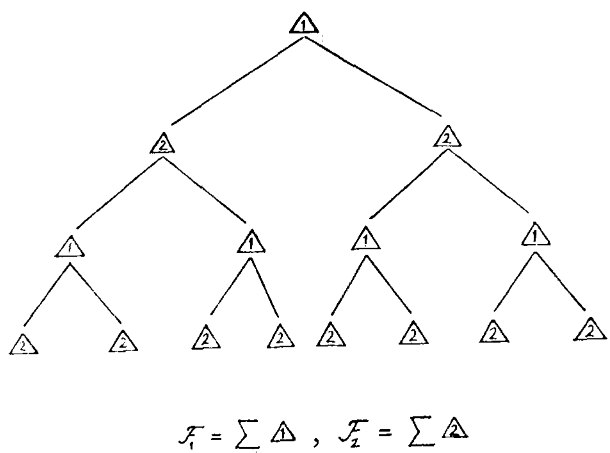

Fig. 5. The splitting $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$.

Without loss of generality, we assume that $\mathcal{F} = \mathcal{F}_1$ or $\mathcal{F} = \mathcal{F}_2$. We then have the property that for every node Δ with subnodes Δ_F, Δ_G:

$$\Delta = 0 \quad \text{or} \quad \Delta_F = \Delta_G = 0. \tag{2.2}$$

Now for every (non-terminal) Δ define P_Δ by

$$|F + \Delta + G|^{2n} = |F|^{2n} + |G|^{2n} + P_\Delta.$$

Iterating this equation, we obtain

$$(2.3)\qquad |\mathscr{F}|^{2n} = \sum_{\Delta\in E} P_\Delta,$$

if for the terminal Δ we set $P_\Delta = |\Delta|^{2n}$.

For the non-terminal Δ we have

$$(2.4)\qquad P_\Delta = \sum K F^a \Delta^b G^c \bar{F}^\alpha \bar{\Delta}^\beta \bar{G}^\gamma,$$

where $K=K(a, b, c, \alpha, \beta, \gamma)$, and the sum is over integers $a, b, c, \alpha, \beta, \gamma \geqq 0$ with $a+b+c=\alpha+\beta+\gamma=n$, $a+\alpha<2n$, $c+\gamma<2n$.

Lemma 3.

$$\left|\int wP_\Delta\right| \leqq C_n \int w\big((3.1)+(3.2)+(3.3)+(3.4)\big)$$

where (3.1)—(3.4) *denote the following terms*

(3.1) $|F|^{2x}|G|^{2y}$, *for integers* $x, y \geqq 1$, $x+y=n$,
(3.2) $|F|^{2n-2}|\Delta|^2$,
(3.3) $|G|^{2n-2}|\Delta|^2$,
(3.4) $|\Delta|^{2n}$.

Proof. This is clear for the terminal Δ since then $P_\Delta = |\Delta|^{2n} = (3.4)$. For non-terminal Δ we have the following 3 types of terms in (2.4).

1. Suppose $b+\beta=0$. Then $a+\alpha \geqq 1$ and $c+\gamma \geqq 1$. Suppose $a+\alpha=1$ with say $a=1$, $\alpha=0$. Then we have the term

$$\int wFG^{n-1}\bar{G}^n = \int w(G\bar{G})^{n-1}(F\bar{G}) = 0$$

since $\deg w+(n-1)l(G)<d(F, G)$ by (E2). The case $a=0$, $\alpha=1$ is just the complex conjugate. Similarly, if $c+\gamma=1$ with $c=1$, $\gamma=0$ we have

$$\int wF^{n-1}G\bar{F}^n = \int w(F\bar{F})^{n-1}(G\bar{F}) = 0$$

and the conjugate $c=0$, $\gamma=1$. We can therefore assume $a+\alpha \geqq 2$, $c+\gamma \geqq 2$, and so

$$|F^a\Delta^b G^c \bar{F}^\alpha \bar{\Delta}^\beta \bar{G}^\gamma| = |F|^{a+\alpha}|G|^{c+\gamma}$$

is majorized by type (3.1) terms. This is clear if $a+\alpha$ is even, because then $c+\gamma$ is also even. If it is odd, so is $c+\gamma$, and we substitute $|F||G| \leqq |F|^2+|G|^2$, noting that $a+\alpha \geqq 3$, $c+\gamma \geqq 3$ in this case.

2. Suppose $b+\beta=1$. Suppose $a+\alpha=0$. Then we have $(b=1, \beta=0)$

$$\int w\Delta G^{n-1}\bar{G}^n = \int w(G\bar{G})^{n-1}(\bar{G}\Delta) = 0,$$

or its conjugate ($b=0$, $\beta=1$). To see that it is 0 consider (2.2). If $\Delta=0$ it is clear. Otherwise $\Delta_G=0$, so that, for the polynomials, we have

$$G = F_1+\Delta_G+G_1 = F_1+G_1,$$

where F_1 and G_1 denote the left and right subtrees of G, as polynomials. In terms of the trees we state this as

$$G = F_1 \cup G_1,$$

and therefore

$$l(G) = l(F_1 \cup G_1),$$

$$d(G, \Delta) = d(F_1 \cup G_1, \Delta).$$

Now by (E3′),

$$d(F_1 \cup G_1, \Delta) \geqq nl(F_1 \cup G_1),$$

so clearly $\deg w+(n-1)l(G)<d(G, \Delta)$, which implies that the polynomial $w(G\bar{G})^{n-1}$ is orthogonal to the polynomial $(\bar{G}\Delta)$.

Suppose $c+\gamma=0$. Then similarly we have

$$\int wF^{n-1}\Delta\bar{F}^n = \int w(F\bar{F})^{n-1}(\bar{F}\Delta) = 0,$$

or its conjugate.

We can therefore assume $a+\alpha\equiv p\geqq 1$ and $c+\gamma\equiv q\geqq 1$. Then

$$|F^{\alpha}\Delta^b G^c \bar{F}^{\alpha}\bar{\Delta}^{\beta}\bar{G}^{\gamma}| = |F|^p|G|^q|\Delta|,$$

and either p is odd and q is even or vice versa, since $p+q+1=2n$. Say $p\equiv 2r+1$ is odd and $q\equiv 2y$ is even. Then

$$\begin{aligned} |F|^p|G|^q|\Delta| &= |F|^{2r}|G|^{2y}|\Delta|\,|F| \\ &\leqq |F|^{2r+2}|G|^{2y}+|F|^{2r}|G|^{2y}|\Delta|^2 \\ &\leqq |F|^{2r+2}|G|^{2y}+|F|^{2n-2}|\Delta|^2+|G|^{2n-2}|\Delta|^2, \end{aligned}$$

which are of type (3.1), (3.2), (3.3) respectively.

3. Suppose $b+\beta>1$. Then

$$|F^a\Delta^b G^c\bar{F}^{\alpha}\bar{\Delta}^{\beta}\bar{G}^{\gamma}| = |F|^p|G|^q|\Delta|^{r+2}$$

for $p, q, r\geqq 0$, $p+q+r=2n-2$. This is at most

$$|F|^{2n-2}|\Delta|^2+|G|^{2n-2}|\Delta|^2+|\Delta|^{2n-2}|\Delta|^2,$$

which are of type (3.2), (3.3) and (3.4).

Lemma 4. *If* t_Δ *is any of the terms* (3.1)—(3.4), *then*

$$\sum_{\Delta\in E}\int wt_\Delta$$

is majorized by the sum of

(4.1) $$C\int w|\mathcal{F}|^{2x}S_1(\mathcal{F})^{2y}$$

over integers $0\leqq x\leqq n-1$, $x+y=n$. *Here* S_1 *is some m-refinement with m depending only on n, and C is a constant depending only on n.*

Proof.

(3.4) $t_\Delta=|\Delta|^{2n}$. Recall that $\Delta=\Delta_1+\Delta_2+\Delta_3$ as in Fig. 2. So

$$\sum_{\Delta\in E}|\Delta|^{2n}\leqq C\sum_{\Delta_j\in E}|\Delta_j|^{2n}\leqq C\left(\sum_{\Delta_j\in E}|\Delta_j|^2\right)^n=CS(\mathcal{F})^{2n}.$$

S is of course a 1-refinement of S. We will use several different refinements below, so the S_1 in (4.1) can simply be defined as their common refinement, by inequality (1.2).

(3.2) $$t_\Delta=|F|^{2n-2}|\Delta|^2\leqq 3|F|^{2n-2}(|\Delta_1|^2+|\Delta_2|^2+|\Delta_3|^2).$$

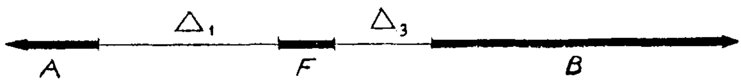

Fig. 6.

Consider Fig. 6 as the $N=2$ case of Lemma 1. Here $\mathcal{F}=A+\Delta_1+F+\Delta_3+B$ with A defined as everything left of Δ_1, and B everything right of Δ_3. By (E3′), we can partition Δ_1, Δ_2, Δ_3 into $10n$ or fewer intervals δ with

$$\deg w+(n-2)\,l(F)+l(\delta)<l(\Delta_3).$$

So if $v=w|F|^{2n-4}|\delta|^2$, then

$$\deg v<l(\Delta_3)<l(\Delta_1).$$

For each δ, Lemma 1 gives

$$\int w|F|^{2n-2}|\delta|^2=\int v|F|^2\leqq\int v(|A|^2+|F|^2+|B|^2)$$
$$\leqq 2\int v|\mathcal{F}|^2+70\int v(|\Delta_1|^2+|\Delta_3|^2)$$
$$\leqq 2\int v|\mathcal{F}|^2+70\int vS(\mathcal{F})^2.$$

Substituting v and using the inequality $a^{n-2}b\leqq\varepsilon a^{n-1}+c(\varepsilon)b^{n-1}$ taking $a=|F|^2$, $b=|\mathcal{F}|^2$ or $S(\mathcal{F})^2$, and ε small, it follows that

$$\int w|F|^{2n-2}|\delta|^2\leqq C\int w\left(|\mathcal{F}|^{2n-2}+S(\mathcal{F})^{2n-2}\right)|\delta|^2.$$

Summing the latter over $\Delta\in E$ and the $10n$ δ's we obtain

$$\sum_{\Delta\in E}\int wt_\Delta\leqq C\int w\left(|\mathcal{F}|^{2n-2}+S(\mathcal{F})^{2n-2}\right)S_1(\mathcal{F})^2$$

where S_1 is the refinement determined by the δ's. The latter is clearly majorized by 2 terms of type (4.1).

(3.3) $t_\Delta = |G|^{2n-2}|\Delta|^2$. This is similar to the previous term and will be omitted.

(3.1) $t_\Delta = |F|^{2x}|G|^{2y}$, $x, y \geqq 1$, $x+y=n$.

Let $\tilde{w} = w|F|^{2x}$. Then by property (E1),

$$\deg \tilde{w} \leqq \deg w + xl(F) < \delta(G).$$

Since $1 \leqq y \leqq n-1$, the induction hypothesis gives $\int \tilde{w}|G|^{2y} \leqq C \int \tilde{w} S_1(G)^{2y}$ for some S_1 depending on y.

Write $S_1(G)$ in terms of intervals $\delta \in G$,

$$S_1(G)^2 = \sum_{\delta \in G} |\delta|^2.$$

we have

$$\sum_{\Delta \in E} |F|^{2x} S_1(G)^{2y} = \sum_{\Delta \in E} |F|^{2x} \sum_{\delta_1, \ldots, \delta_y \in G} |\delta_1|^2 \ldots |\delta_y|^2$$
$$= \sum_{\delta_1, \ldots, \delta_y \in E} |\delta_1|^2 \ldots |\delta_y|^2 \sum_{G \ni \delta_1, \ldots, \delta_y} |F|^{2x}.$$

Here $\delta_1, \ldots, \delta_y \in E$ means intervals of the refinement S_1 (which is defined on all of E_∞) lying in E. Fix such $\delta_1, \ldots, \delta_y$. Then $G \ni \delta_1, \ldots, \delta_y$ means all subtrees G containing $\{\delta_1, \ldots, \delta_y\}$. These G must be of the form

$$G_1 \supset G_2 \supset \ldots \supset G_N \supset \{\delta_1, \ldots, \delta_y\}.$$

Let $F_1, \Delta^{(1)}, F_2, \Delta^{(2)}, \ldots, F_N, \Delta^{(N)}$ be the corresponding F's and nodes Δ. Recall that each Δ has 3 intervals

$$\Delta = \{\Delta_1, \Delta_2, \Delta_3\}$$

as in Fig. 2. From this we deduce Fig. 7.

In Fig. 7 the H_k are by definition the spaces between $\Delta_3^{(k)}$ and $\Delta_1^{(k+1)}$. We also define A to be everything left of $\Delta_1^{(1)}$, and B everything right of $\Delta_3^{(N)}$, so that

$$\mathscr{F} = A + \Delta_1^{(1)} + F_1 + \Delta_3^{(1)} + H_1 + \ldots + \Delta_1^{(N)} + F_N + \Delta_3^{(N)} + B.$$

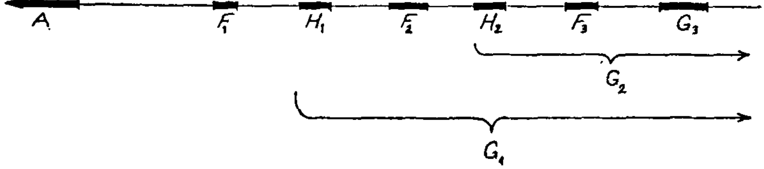

Fig. 7. The $N=3$ case. Lengths are not to scale.

We note that

$$l(\Delta_1^{(1)}) > l(\Delta_3^{(1)}) > l(\Delta_1^{(2)}) > \ldots > l(\Delta_3^{(N)}) \geqq \lambda l(G_N),$$

$$l(F_1) < l(F_2) < \ldots < l(F_N) \leqq \frac{1}{\lambda} l(\Delta_3^{(N)}),$$

by properties (E1), (E2), (E3′).

Setting

$$v = w\,|\delta_1|^2 \ldots |\delta_y|^2 \left(\textstyle\sum_{k=1}^N |F_k|^2\right)^{x-1},$$

we have

$$\deg v \leqq \deg w + y l(G_N) + (x-1)\, l(F_N) < l(\Delta_3^{(N)}).$$

So by Lemma 1,

$$\begin{aligned}\int w\,|\delta_1|^2 \ldots |\delta_y|^2 \textstyle\sum_{k=1}^N |F_k|^{2x} &\leqq \int w\,|\delta_1|^2 \ldots |\delta_y|^2 \left(\sum |F_k|^2\right)^x \\ = \int v \textstyle\sum |F_k|^2 &\leqq \int v\left(|A|^2 + \sum |F_k|^2 + \sum |H_k|^2 + |B|^2\right) \\ &\leqq 2\int v\,|\mathcal{F}|^2 + 70 \int v \textstyle\sum \left(|\Delta_1^{(k)}|^2 + |\Delta_3^{(k)}|^2\right) \\ &\leqq 2\int v\,|\mathcal{F}|^2 + 70\int v S(\mathcal{F})^2.\end{aligned}$$

Substituting v it follows that

$$\int w\,|\delta_1|^2 \ldots |\delta_y|^2 \textstyle\sum |F_k|^{2x} \leqq c \int w\,|\delta_1|^2 \ldots |\delta_y|^2 \left(|\mathcal{F}|^{2x} + S(\mathcal{F})^{2x}\right).$$

Summing over $\delta_1, \ldots, \delta_y \in E$, the result is

$$\textstyle\sum_{\Delta\in E} \int w\,|F|^{2x} |G|^{2y} \leqq c \int w S_1(\mathcal{F})^{2y} \left(|\mathcal{F}|^{2x} + S(\mathcal{F})^{2x}\right),$$

and this again reduces to (4.1) type terms.

This completes the proof of Lemma 4.

Completion of proof of Theorem 2. Combining Lemmas 3 and 4, we have proved

$$\int w\,|\mathcal{F}|^{2n} \leqq c \textstyle\sum_{x=0}^{n-1} \int w\,|\mathcal{F}|^{2x} S_1(\mathcal{F})^{2(n-x)}.$$

Again using inequalities of the form

$$a^x b^y \leqq \varepsilon a^n + c(\varepsilon)\, b^n$$

$(x+y=n,\ x \leqq n-1)$ for small $\varepsilon > 0$, we obtain (2.1).

Proof of Theorem 1. Taking $w=1$ in Theorem 2 and using (1.3) gives

$$\|\mathcal{F}\|_{2n} \leqq c_n \|S(\mathcal{F})\|_{2n}$$

whenever $\lambda_k \geqq 3n$ in the construction of E_∞. If $\lambda_k \to \infty$ then $\lambda_k \geqq 3n$ eventually. Let S_n be the square function of the tree whose non-terminal nodes coincide with those of E_∞ above the first level where $\lambda \leqq 3n$ in (E1)—(E3), and whose terminal nodes are the intervals containing the left and right subtrees below this level. Then

$$\|\mathscr{F}\|_{2n} \leqq c_n \|S_n(\mathscr{F})\|_{2n} \leqq C'_n \|S(\mathscr{F})\|_{2n}$$

since S is a certain m_n-refinement of S_n. By interpolation, we obtain (1.1) for all $2 \leqq p < \infty$.

Further Remarks.

1. The proof of (1.1) also works when S is a classical lacunary square function; $\Delta_j = [n_j, n_{j+1}) \cup (-n_{j+1}, -n_j]$, where $n_{j+1}/n_j \geqq \lambda > 1$. In fact it becomes much simpler, since the terms $|F|^{2x}|G|^{2y}$ of Lemma 3 do not arise. One just iterates the equation
$$|F_{j+1}|^{2n} = |F_j + \Delta_j|^{2n} = |F_j|^{2n} + P_{\Delta_j}.$$
2. In paper [3] we defined sets E_∞ without using the a_k and b_k of the present paper, i.e. just by
$$E_{k+1} = E_k \cup E_k^*.$$
It turns out that the present E_∞ is a "lacunary" refinement of the former, and therefore both satisfy (1.1), by the classical vector-valued Littlewood—Paley inequality.
3. In Theorem 1, it probably suffices if $\lambda_k \geqq \lambda$ for some $\lambda > 0$. This would require more splitting of $\mathscr{F}$ and a more detailed iteration expansion than in Lemma 3. Our purpose here was just to show the existence of some version of E_∞ satisfying (1.1).
4. One generalization of E_∞ is the following: We observe that if we rearrange the intervals of E_∞ in order of increasing length, we obtain a classical lacunary partition (by properties (E1)—(E3)). So a natural conjecture is that any partition which is a rearrangement of a lacunary partition has the Littlewood—Paley property. A proof of this conjecture may be possible along the lines of the present paper. Such a rearrangement has a natural binary tree structure, obtained by always choosing the longest interval as the node. However property (E1) does not hold in general. This probably means that the terms $|F|^{2x}|G|^{2y}$ of Lemma 3 must be estimated directly (without the induction hypothesis) by further expansion.
5. It would be very nice to find a less computational proof of Theorem 1.

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