Boundedness of oscillatory singular integrals on Hardy spaces

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1. Introduction

Let $x\in\mathbf{R}^n$, $P(x)$ be a real-valued polynomial, and $K(x)$ be a Calderón—Zygmund kernel. Define T:

$$Tf(x) = \text{p.v.} \int_{\mathbf{R}^n} e^{iP(x-y)}K(x-y)f(y)\,dy. \tag{1.1}$$

In this paper we prove the boundedness of such operators on the Hardy space H^1. Our method is general enough to even allow us to treat the weighted Hardy space H^1_w, when $w\in A_1$.

L^p estimates for such operators were established by F. Ricci and E. M. Stein ([11]). In fact, the operators they treated are more general, in the sense that they are not necessarily of convolution type. Later S. Chanillo and M. Christ proved that these operators are also of weak-type (1, 1) ([2]).

The study of oscillatory singular integral operators on Hardy spaces began with the investigation on operators with bilinear phase functions by D. H. Phong and E. M. Stein ([10]). They introduced some variants of the standard H^1 space, H^1_E (which is closely related to the given bilinear form), and proved that such operators are bounded from H^1_E to L^1. This result was used to prove the L^p boundedness by interpolating between L^2 and L^∞. Results of this form, but for operators with polynomial phase functions, were obtained by the second author in [9].

Weighted norm estimates (L^p, weak (1, 1)) for convolution operators with oscillating kernels were obtained by S. Chanillo, D. Kurtz and G. Sampson in [1], [3] and [4], where the phase function in the oscillatory factor is of the form $i|x|^a$. More recently, Y. Hu ([6], [7]) proved some weighted norm estimates for operators with polynomial phase functions. In particular, it was proved in [7] that, when the dimension is 1, the operator given in (1.1) is bounded on H^1_w, for $w\in A_1$.

The main part of this paper is to show that this is true in all dimensions. We state our theorem as follows.

Theorem 1. *Let* $x \in \mathbf{R}^n$, $P(x)$ *be a polynomial which satisfies* $\nabla P(0)=0$, *and* T *be defined as in* (1.1), $w \in A_1$. *Then there is a constant* C, *which depends only on the* A_1 *constant of* w *and the degree of* $P(x)$ *(not its coefficients), such that*

$$\|Tf\|_{H^1_w} \leqq C \|f\|_{H^1_w},$$

for all $f \in H^1_w$.

Some relevant definitions will be given in Section 2.

This theorem differs from Theorem 1 in [9] in two aspects: (1) The Hardy spaces in [9] do not involve A_1 weights; (2) More importantly, the H^1 spaces in this paper is different from those in [9], even when $w \equiv 1$. In fact, when $w \equiv 1$, the spaces here are exactly the usual H^1 spaces, while the spaces in [9], [10] are closely related to the phase functions of the operators, even when one considers only convolution operators.

To prove our theorem, we further develop the techniques used in [9] and [10]. The method used in [7] relies heavily on knowledge of roots of a polynomial, which makes it difficult to apply to a higher dimensional situation.

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2. Notations and definitions

In this section we recall some definitions and results that are relevant to this article.

Definition 2.1. A function $K(x)$ in $C^1(\mathbf{R}^n \setminus \{0\})$ is a Calderón—Zygmund kernel if there is a constant $A>0$ such that

$$|K(x)| \leqq A|x|^{-n}, \quad |\nabla K(x)| \leqq A|x|^{-n-1},$$

and

$$\int_{a<|x|<b} K(x)\,dx = 0$$

holds for $b>a>0$.

Definition 2.2. Let $w(x)$ be a nonnegative, locally integrable function in $\mathbf{R}^n$. We say that $w \in A_1$ if

$$\frac{1}{|Q|}\int_Q w(x)\,dx \leqq C \operatorname*{ess\,inf}_{x \in Q} w(x) \tag{2.1}$$

holds for all cubes Q in $\mathbf{R}^n$. Let $C(w)$ denote the smallest constant for (2.1) to be true, which we call the A_1 constant of w.

For a cube $Q\subset\mathbf{R}^n$, $w\in A_1$, let $w(Q)=\int_Q w(x)\,dx$. If we let Q^* be the cube which has the same center as Q, but twice the sidelength, we have

$$w(Q^*) \leqq 2^n C(w)\,w(Q). \tag{2.2}$$

It is well-known that the necessary and sufficient condition for the Hardy—Littlewood maximal function to be bounded from L^1_w to $L^{1,\infty}_w$ is that $w\in A_1$ ([5], [8]).

Now we give the definition of the weighted Hardy space H^1_w. Let ψ belong to the Schwartz class $\mathscr{S}$, $\int_{\mathbf{R}^n}\psi(x)\,dx\neq 0$. For each $f\in\mathscr{S}'(\mathbf{R}^n)$, set

$$f^*(x) = \sup_{t>0} |(f*\psi_t)(x)|, \quad x\in\mathbf{R}^n,$$

where $\psi_t(x)=t^{-n}\psi(x/t)$. We have

Definition 2.3. A locally integrable function f is in the space H^1_w if

$$\|f^*\|_{L^1_w} = \int_{\mathbf{R}^n} f^*(x)\,w(x)\,dx < \infty,$$

and we define $\|f\|_{H^1_w}=\|f^*\|_{L^1_w}$.

We shall need the atomic decomposition for functions in H^1_w. First we recall the definition of a H^1_w atom ([14]).

Definition 2.4. Let $w\in A_1$. A real valued function $a(x)$ is a H^1_w atom if

(1) $a(x)$ is supported in a cube $Q\subset\mathbf{R}^n$,

(2) $\int_Q a(x)\,dx=0$,

(3) $\|b\|_\infty\leqq\frac{1}{w(Q)}$, where $w(Q)=\int_Q w(x)\,dx$.

The following theorem is from [14].

Theorem 2.1. *For each* $f\in H^1_w$, *there exist atoms* $\{a_j\}$ *and coefficients* $\{\lambda_j\}$ *such that*

$$f(x) = \sum_j \lambda_j a_j(x), \tag{2.3}$$

and $\sum_j|\lambda_j|\leqq C\|f\|_{H^1_w}$, *where C depends only on* $C(w)$. *The sum in* (2.3) *is both in the sense of distributions and in the* H^1_w *norm.*

We would like to point out that the restriction $\nabla P(0)=0$ in Theorem 1 is essential. For example, we take $n=1$, $w\equiv 1$, $a(x)$ a nonzero atom which is supported in $I=(-\frac{1}{2},\frac{1}{2})$, and $K(x)=\frac{1}{x}$. If Theorem 1 were true for $P(x)=kx$, $k\neq 0$, it would imply that

$$\int e^{ikx}a(x)\,dx = 0,$$

for all k, which cannot be true. See also [7].

3. Some reductions

Let $w\in A_1$, and $a(x)$ be a H^1_w atom, which is supported in a cube $Q\subset\mathbf{R}^n$ and satisfies

(1) $\operatorname{supp}(a)\subset Q$,

(2) $\int_Q a(x)\,dx=0$,

(3) $\|a\|_\infty\leqq w(Q)^{-1}$.

Let x_0 be the center of Q, and δ be its sidelength, Q_0 be the cube which is centered at the origin, with sidelength 1, and $w_0=w(x_0+\delta x)$. It is easy to see that $w_0\in A_1$, and $C(w_0)=C(w)$. Set

$$b(x) = \delta^n a(x_0+\delta x).$$

We see that $b(x)$ is a $H^1_{w_0}$ atom, and it satisfies

$$\operatorname{supp}(b) \subset Q_0, \tag{3.1}$$

$$\int_{Q_0} b(x)\,dx = 0, \tag{3.2}$$

$$\|b(x)\|_\infty \leqq \frac{1}{w_0(Q_0)}. \tag{3.3}$$

We also have

$$(Ta)(x_0+\delta x) = \delta^{-n} T_1(b)(x),$$

where T_1 is given by

$$T_1 f(x) = \text{p.v.} \int_{\mathbf{R}^n} e^{iP(\delta x-\delta y)} K(x-y) f(y)\,dy$$

which leads to $\|Ta\|_{L^1_w}=\|T_1 b\|_{L^1_{w_0}}$.

To prove Theorem 1, we first prove the following:

Proposition 3.1. *Let $P(x)$ be a polynomial with $\nabla P(0)=0$, $w\in A_1$. Then for any H^1_w atom $a(x)$ we have*

$$\|T(a)\|_{L^1_w} \leqq C, \tag{3.4}$$

where C is a constant, depending only on the degree of $P(x)$ and the A_1 constant of w.

The preceding argument shows that it is sufficient to prove Proposition 3.1 for H^1_w atoms which satisfy (3.1)—(3.3) (with $w(x)$ replaced by $w_0(x)$).

4. Proof of Proposition 3.1

We list a few lemmas that are needed in the proof.

Lemma 4.1 (Ricci—Stein, [11]). *Let* $Q(x)=\sum_{|\alpha|\le d} q_\alpha x^\alpha$ *be a polynomial in* x, $x\in\mathbf{R}^n$, *with degree* d. *Suppose* $\varepsilon<1/d$, *then*

$$\int_{|x|\le 1} |Q(x)|^{-\varepsilon}\,dx \le A_\varepsilon\Big(\sum_{|\alpha|\le d}|q_\alpha|\Big)^{-\varepsilon}.$$

The bound A_ε *depends on* n *and* ε, *but not on the coefficients* $\{q_\alpha\}$.

The second lemma is of van der Corput type.

Lemma 4.2. *Suppose* $\phi(x)=\sum_{|\alpha|\le k} a_\alpha x^\alpha$ *is a real-valued polynomial in* $\mathbf{R}^n$ *of degree* k, *and* $\psi\in C_0^\infty$. *Then for any* α, $|\alpha|=k$, $a_\alpha\ne 0$, *we have*

$$\Big|\int_{\mathbf{R}^n} e^{i\phi(x)}\psi(x)\,dx\Big| \le C(\psi)\,|a_\alpha|^{-1/k}.$$

Proof of Lemma 4.2. Since $\partial^\alpha\phi/\partial x^\alpha(x)=\alpha!\,a_\alpha$, there exists a unit vector ξ such that

$$|(\xi\cdot\nabla)^k\phi(x)| \ge c\,|a_\alpha|.$$

By making a rotation, if necessary, we may assume that $\xi=(1,0,\dots,0)$. Therefore we have

$$\left|\frac{\partial^k\phi(y)}{\partial y_1^k}\right| \ge c\,|a_\alpha|.$$

The lemma follows by invoking the one-dimensional van der Corput's lemma. See also [12].

Lemma 4.3. *Suppose that* $P(x)$ *is a polynomial of degree* m, $m\ge 2$, *and* $P(x)=\sum_{|\alpha|\le m} a_\alpha x^\alpha$. *Let* φ *and* ψ *be two functions in* $C_0^\infty(\mathbf{R}^n)$. *Define* T_j *by*

$$(T_j f)(x) = \psi(2^{-j}x)\int_{\mathbf{R}^n} e^{iP(x-y)}\varphi(y)f(y)\,dy.$$

Then we have

$$\|T_j f\|_{L^2(\mathbf{R}^n,dx)} \le C2^{jn/2}\big(|a_\alpha|\,2^{j(m-1)}\big)^{-(1/4(m-1))}\,\|f\|_{L^2(\mathbf{R}^n,dx)},$$

for any α *with* $|\alpha|=m$.

Proof of Lemma 4.3. We fix α_0 with $|\alpha_0|=m$ and $a_{\alpha_0}\ne 0$. Consider the operator $T_j^*T_j$, which is given by

$$T_j^*T_j f(x) = \int_{\mathbf{R}^n} L_j(x,y)f(y)\,dy,$$

where

$$L_j(x, y) = \varphi(x)\varphi(y)\int_{\mathbf{R}^n} e^{i(P(z-y)-P(z-x))}\psi^2(2^{-j}z)\,dz.$$
$$= 2^{jn}\varphi(x)\varphi(y)\int_{\mathbf{R}^n} e^{i(P(2^jz-y)-P(2^jz-x))}\psi^2(z)\,dz.$$

Now we can write

$$P(2^jz-y)-P(2^jz-x)$$
$$= \sum_{|\alpha|=m} a_\alpha \sum_{\beta+\gamma=\alpha,\,|\beta|=m-1} C_{\beta\gamma}2^{j(m-1)}z^\beta(x^\gamma-y^\gamma)+R(x, y, z)$$
$$= 2^{j(m-1)}\sum_{|\beta|=m-1} z^\beta \sum_{|\gamma|=1} C_{\beta\gamma}a_{\beta+\gamma}(x^\gamma-y^\gamma)+R(x, y, z),$$

where $C_{\beta\gamma}$ are nonzero constants and $R(x, y, z)$ is a polynomial whose degree in z is strictly less than $m-1$. Take β_0, γ_0 such that $|\beta_0|=m-1$, $|\gamma_0|=1$ and $\beta_0+\gamma_0=\alpha_0$. We note that

$$\left(\frac{\partial}{\partial z}\right)^{\beta_0}(P(2^jz-y)-P(2^jz-x)) = 2^{j(m-1)}\sum_{|\gamma|=1}(\beta_0!)C_{\beta_0\gamma}a_{\beta_0+\gamma}(x^\gamma-y^\gamma).$$

By Lemma 4.2, we have

$$(4.1)\qquad |L_j(x, y)| \leqq C2^{jn}2^{-j}\left|\sum_{|\gamma|=1}C_{\beta_0\gamma}a_{\beta_0+\gamma}(x^\gamma-y^\gamma)\right|^{-(1/(m-1))}|\varphi(x)\varphi(y)|.$$

On the other hand, we have the following trivial estimate:

$$(4.2)\qquad |L_j(x, y)| \leqq \int_{\mathbf{R}^n}\psi^2(2^{-j}z)\,dz \leqq C2^{jn}.$$

Combining (4.1) and (4.2) we get

$$|L_j(x, y)| \leqq C2^{jn}2^{-j/2}\left|\sum_{|\gamma|=1}C_{\beta_0\gamma}a_{\beta_0+\gamma}(x^\gamma-y^\gamma)\right|^{-(1/2(m-1))}|\varphi(x)\varphi(y)|^{1/2}.$$

Applying Lemma 4.1, we obtain

$$(4.3)\qquad \sup_y\int_{\mathbf{R}^n}|L_j(x, y)|\,dx \leqq C2^{jn}2^{-j/2}|a_{\alpha_0}|^{-(1/2(m-1))},$$

where we used the fact that $\frac{1}{2(m-1)}<1$. Similarly we have

$$(4.4)\qquad \sup_x\int_{\mathbf{R}^n}|L_j(x, y)|\,dy \leqq C2^{jn}2^{-j/2}|a_{\alpha_0}|^{-(1/2(m-1))}.$$

Inequalities (4.3) and (4.4) imply that

$$\|T_j\|_{L^2\to L^2} \leqq C2^{jn/2}(2^{j(m-1)}|a_{\alpha_0}|)^{-(1/4(m-1))},$$

where the L^2 norm is the usual L^2 norm with Lebesgue measure. This proves the lemma.

Remark. Estimates that are similar to Lemma 4.3 were used in [9], where $T_jT_j^*$ was considered. The approach taken here is to consider $T_j^*T_j$ instead, thus producing the sharp bound needed in our problem.

Lemma 4.4 ([5]). *Let* $w\in A_1$. *Then there exists an* $\varepsilon>0$, *such that* $w^{1+\varepsilon}\in A_1$. ε *and the* A_1 *constant of* $w^{1+\varepsilon}$ *depend only on the* A_1 *constant of* w, *not* w *itself.*

We now begin our proof of Proposition 3.1. Assume that a is a H^1_w atom that satisfies (3.1)—(3.3). We shall prove (3.4) by using induction on m, the degree of $P(x)$.

When $m=0$, the phase function in T is identically zero. So T is the usual Calderón—Zygmund singular integral and (3.4) holds ([14]). We now assume that (3.4) is true for $\deg(P)\leqq m-1$.

To prove that (3.4) is true when the degree of P is m $(m\geqq 2)$, we write

$$P(x-y) = \sum_{|\alpha|=m} a_\alpha (x-y)^\alpha + P_{m-1}(x-y), \tag{4.5}$$

where $\deg(P_{m-1})\leqq m-1$. Take α_0 with $|\alpha_0|=m$ such that

$$|a_{\alpha_0}| = \max_{|\alpha|=m} |a_\alpha|.$$

Let $b=\max\{|a_{\alpha_0}|^{-1/(m-1)}, 2\}$. We break the integral into two parts:

$$\|T(a)\|_{L^1_w} \leqq \left|\int_{|x|\leqq b} T(a)(x)\,w(x)\,dx\right| + \left|\int_{|x|\geqq b} T(a)(x)\,w(x)\,dx\right| = I_1+I_2. \tag{4.6}$$

The first step is to show that $I_1\leqq C$. If $b=2$, the estimate follows from a standard argument:

$$\begin{aligned} I_1 &= \left|\int_{|x|\leqq 2} T(a)(x)\,w(x)\,dx\right| \leqq \|T(a)\|_{L^2_w}\Big(\int_{|x|\leqq 2} w(x)\,dx\Big)^{1/2} \\ &\leqq C\,\|a\|_{L^2_w}\,w(Q_0)^{1/2} \leqq C, \end{aligned} \tag{4.7}$$

where we used (2.2), (3.3) and the weighted L^p estimate for T ([6]).

Now assuming that $b=|\alpha_{\alpha_0}|^{-1/(m-1)}$, we have

$$\begin{aligned} |I_1| &\leqq C + \left|\int_{2\leqq|x|\leqq b} T(a)(x)\,w(x)\,dx\right| \\ &\leqq C + \int_{2\leqq|x|\leqq b}\left|\int_{\mathbf{R}^n} e^{iP_{m-1}(x-y)} K(x-y)\,a(y)\,dy\right| w(x)\,dx \\ &+ \int_{2\leqq|x|\leqq b}\int_{\mathbf{R}^n}\left|e^{i(P(x-y)-P_{m-1}(x-y)-\sum_{|\alpha|=m} a_\alpha x^\alpha)}-1\right| |K(x-y)\,a(y)|\,dy\,w(x)\,dx. \end{aligned}$$

The first integral is bounded, by our inductive hypothesis, while the second integral is bounded by

$$\begin{aligned} & C\sum_{|\alpha|=m}|a_\alpha|\int_{2\leqq|x|\leqq b}\int_{Q_0}\frac{|(x-y)^\alpha-x^\alpha|}{|x-y|^n}|a(y)|\,dy\,w(x)\,dx \\ &\leqq C\sum_{|\alpha|=m}|a_\alpha|\Big(\int_{2\leqq|x|\leqq b}|x|^{m-n-1}w(x)\,dx\Big)\Big(\int_{Q_0}|a(y)|\,dy\Big) \\ &\leqq C\sum_{|\alpha|=m}|a_\alpha|\sum_{j\geqq 1,\,2^j\leqq b}2^{j(m-1)}w(Q_0)\,w(Q_0)^{-1} \leqq C\,|a_{\alpha_0}|\,b^{m-1} = C. \end{aligned} \tag{4.8}$$

Now we prove that $I_2 \leqq C$. Assume that $2^{j_0} \leqq b \leqq 2^{j_0+1}$. Let $\varphi \in C_0^\infty(\mathbf{R}^n)$, and $\varphi = 1$ on Q_0. We also choose $\psi \in C_0^\infty(\mathbf{R}^n)$ such that

$$\operatorname{supp}(\psi) \subset \left\{\tfrac{1}{4} < |x| < 4\right\}, \quad \psi \geqq 0,$$

and

$$\psi(x) = 1, \quad \text{for} \quad 1 \leqq |x| \leqq 2.$$

We have

$$
\begin{aligned}
(4.9) \qquad I_2 &\leqq \int_{|x| \geqq b} \int_{\mathbf{R}^n} |K(x-y) - K(x)|\, |a(y)|\, dy w(x)\, dx \\
&+ \int_{|x| \geqq b} \left| \int_{\mathbf{R}^n} e^{iP(x-y)} a(y)\, dy \right| \frac{w(x)}{|x|^n}\, dx \leqq \frac{C}{w(Q_0)} \int_{|x| \geqq 2} \frac{w(x)}{|x|^{n+1}}\, dx \\
&+ \sum_{j \geqq j_0} \int_{2^j \leqq |x| \leqq 2^{j+1}} \frac{w(x)}{|x|^n} \psi(2^{-j}x) \left| \int_{\mathbf{R}^n} e^{iP(x-y)} \varphi(y) a(y)\, dy \right| dx \\
&\leqq C + \sum_{j \geqq j_0} \|T_j(a)\|_{L^p(\mathbf{R}^n, dx)} \left(\int_{2^j \leqq |x| \leqq 2^{j+1}} \frac{w^{1+\varepsilon}(x)}{|x|^{n(1+\varepsilon)}}\, dx \right)^{1/(1+\varepsilon)},
\end{aligned}
$$

where T_j are given as in Lemma 4.3 and $p = 1 + \frac{1}{\varepsilon} \geqq 2$. Invoking Lemma 4.4, we obtain

$$
\begin{aligned}
\int_{2^j \leqq |x| \leqq 2^{j+1}} \frac{w^{1+\varepsilon}(x)}{|x|^{n(1+\varepsilon)}}\, dx &\leqq C 2^{-jn(1+\varepsilon)+jn} \frac{1}{2^{jn}} \int_{2^j \leqq |x| \leqq 2^{j+1}} w^{1+\varepsilon}(x)\, dx \\
&\leqq C 2^{-jn\varepsilon} \operatorname*{ess\,inf}_{|x| < 2^{j+1}} w^{1+\varepsilon}(x) \leqq C 2^{-jn\varepsilon} w(Q_0)^{1+\varepsilon}.
\end{aligned}
$$

Note that

$$\|T_j\|_{L^\infty \to L^\infty} \leqq C.$$

Using Lemma 4.3 (taking $\alpha = \alpha_0$) and interpolation we get

$$\|T_j(a)\|_{L^p(\mathbf{R}^n, dx)} \leqq C 2^{jn/p} (|a_{\alpha_0}|\, 2^{j(m-1)})^{-(1/2p(m-1))} w(Q_0)^{-1},$$

and

$$
\begin{aligned}
(4.10) \quad I_2 &\leqq C + C \sum_{j \geqq j_0} 2^{jn/p} (|a_{\alpha_0}|\, 2^{j(m-1)})^{-(1/2p(m-1))} w(Q_0)^{-1} 2^{-jn\varepsilon/(1+\varepsilon)} \cdot w_0(Q_0) \\
&\leqq C + C |a_{\alpha_0}|^{-(1/2p(m-1))} \sum_{j \geqq j_0} 2^{-j/2p} \leqq C + |a_{\alpha_0}|^{-(1/2p(m-1))} b^{-1/2p} \leqq C.
\end{aligned}
$$

Combining (4.7), (4.8) and (4.10), we see that Proposition 3.1 is proved.

5. Proof of the main theorem

We shall need the following theorem ([14], [15]):

Theorem 5.1. *Let* $w \in A_1$, R_j *be the Riesz transforms, i.e.*

$$(R_j f)\hat{\ }(\xi) = \frac{i\xi_j}{|\xi|} \hat{f}(\xi), \quad j = 1, \ldots, n.$$

Then for $f \in H_w^1$, *we have*

$$\|R_j f\|_{H_w^1} \leqq C\|f\|_{H_w^1}, \quad \text{and} \tag{5.1}$$

$$\|f\|_{H_w^1} \sim \|f\|_{L_w^1} + \sum_{j=1}^n \|R_j f\|_{L_w^1}. \tag{5.2}$$

Now we are ready to prove Theorem 1. Let $f \in H_w^1$, $\{a_j\}$ be a collection of H_w^1 atoms and $\{\lambda_j\}$ a sequence of numbers such that

$$f(x) = \sum_j \lambda_j a_j(x),$$

and

$$\sum_{j=1}^{\infty} |\lambda_j| \leqq C\|f\|_{H_w^1}.$$

By Proposition 3.1, we have

$$\begin{aligned}\|Tf\|_{L_w^1} &\leqq \sum_j |\lambda_j| \, \|T(a_j)\|_{L_w^1} \\ &\leqq C \sum_j |\lambda_j| \leqq C\|f\|_{H_w^1}.\end{aligned} \tag{5.3}$$

Since T commutes with R_j, we also have

$$\|R_j Tf\|_{L_w^1} = \|T(R_j f)\|_{L_w^1} \leqq C\|R_j f\|_{H_w^1}, \quad j = 1, \ldots, n, \tag{5.4}$$

where we applied (5.3) on $R_j(f)$. By (5.1) we obtain

$$\|Tf\|_{L_w^1} + \sum_{j=1}^n \|R_j(Tf)\|_{L_w^1} \leqq C\|f\|_{H_w^1}. \tag{5.5}$$

Now invoking (5.2), we get

$$\|Tf\|_{H_w^1} \leqq C\|f\|_{H_w^1}.$$

This completes the proof of Theorem 1. One may check to see that all the constants that appeared above depend only on the degree of the polynomial and the A_1 constant of the weight.

References

1. CHANILLO, S., Weighted norm inequalities for strongly singular convolution operators, *Trans. Amer. Math. Soc.* **281** (1984), 77—107.
2. CHANILLO, S. and CHRIST, M., Weak (1, 1) bounds for oscillatory singular integrals, *Duke Math. J.* **55** (1987), 141—155.
3. CHANILLO, S., KURTZ, D. and SAMPSON, G., Weighted L^p estimates for oscillating kernels, *Ark. Mat.* **21** (1983), 233—257.
4. CHANILLO, S. KURTZ, D. and SAMPSON, G., Weighted weak (1, 1) and weighted L^p estimates for oscillating kernels, *Trans. Amer. Math. Soc.* **295** (1986), 127—145.
5. COIFMAN, R. and FEFFERMAN, C., Weighted norm inequalities for maximal functions and singular integrals, *Studia Math.* **51** (1974), 241—250.

6. HU, Y., Weighted L^p estimates for oscillatory integrals, *preprint.*
7. HU, Y., Oscillatory singular integrals on weighted Hardy spaces, *preprint.*
8. MUCKENHOUPT, B., Weighted norm inequalities for the Hardy maximal function, *Trans. Amer. Math. Soc.* **165** (1972), 207—226.
9. PAN, Y., Hardy spaces and oscillatory singular integrals, *Rev. Mat. Iberoamericana* **7** (1991), 55—64.
10. PHONG, D. H. and STEIN, E. M., Hilbert integrals, singular integrals and Radon transforms I, *Acta Math.* **157** (1986), 99—157.
11. RICCI, F. and STEIN, E. M., Harmonic analysis on nilpotent groups and singular integrals, I, *J. Funct. Anal.* **73** (1987), 179—194.
12. STEIN, E. M., Oscillatory integrals in Fourier analysis, in *Beijing Lectures in Harmonic Analysis,* Princeton Univ. Press, Princeton, 1986.
13. STEIN, E. M. and WEISS, G., *Introduction to Fourier analysis on Euclidean spaces,* Princeton Univ. Press, Princeton, 1971.
14. STRÖMBERG, J. O. and TORCHINSKY, A., *Weighted Hardy spaces, Lecture Notes in Math.* **1381**, Springer-Verlag, Berlin—Heidelberg, 1989.
15. WHEEDEN, R., A boundary value characterization of weighted H^1, *L'Enseignement Math.* **24** (1976), 121—134.

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