Boundedness properties of the operators of best approximation by analytic and meromorphic functions

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1. Introduction

In this paper we continue the study of properties of the operator of best approximation by analytic functions, which plays an important role in many questions arising in Hankel and Toeplitz operators [1], [2], [20], interpolation problems [1], [2], [14], prediction theory [20], electrical engineering and control theory [11], [8], [7], [4].

This operator can be defined as follows. If φ is a bounded function on the unit circle $\mathbf{T}$, then the distance $\operatorname{dist}_{L^\infty(\mathbf{T})}(\varphi, H^\infty)$ is attained, i.e. there exists a function in f in H^∞ such that

$$\|\varphi - f\|_{L^\infty(\mathbf{T})} = \operatorname{dist}_{L^\infty(\mathbf{T})}(\varphi, H^\infty).$$

Such a function f (which is not unique in general) is called a *best approximation of φ by analytic functions.*

It turns out that it is natural to consider the notion of best approximation for functions φ not necessarily bounded on $\mathbf{T}$. Namely, let $\varphi \in \mathrm{BMO}$ (the space BMO of functions of bounded mean oscillation can be defined in several ways, for example

$$\mathrm{BMO} = \{\varphi = \psi_1 + \tilde{\psi}_2 \colon \psi_1, \psi_2 \in L^\infty(\mathbf{T})\},$$

where $\tilde{\psi}$ is the harmonic conjugate of ψ, see [9]). Then it is well known that there exists a function f analytic in the unit disc $\mathbf{D}$ such that $\varphi - f \in L^\infty(\mathbf{T})$ (in the last inclusion f is identified with its boundary values on $\mathbf{T}$). Among such functions f there exists a function f_0 that minimizes the norm $\|\varphi - f_0\|_{L^\infty(\mathbf{T})}$. Such a function f_0 (not unique in general) is called a best approximation of φ by analytic functions in the sup-norm.

If φ belongs to the space VMO of functions of vanishing mean oscillation

$$(\mathrm{VMO} = \{\varphi = \psi_1 + \tilde{\psi}_2 \colon \psi_1, \psi_2 \in C(\mathbf{T})\})$$

then the best approximation is unique (see [1], [20]). The *operator* $\mathscr{A}$ *of best approximation by analytic functions* is defined on the class VMO by

$$\mathscr{A}\varphi = f_0,$$

where f_0 is the best approximation of ψ by analytic functions. The operator $\mathscr{A}$ is nonlinear but it is homogeneous, i.e.

$$\mathscr{A}(\lambda f) = \lambda \mathscr{A} f, \quad \lambda \in \mathbf{C}.$$

Note that in the study of the best approximation operator $\mathscr{A}$ an important role is played by the so-called *Hankel operators.* Given a function φ on $\mathbf{T}$ of class BMO, the Hankel operator H_φ is defined on the Hardy class H^2 by

$$H_\varphi f = \mathbf{P}_- \varphi f, \quad f \in H^2,$$

where $\mathbf{P}_-$ is the orthogonal projection from L^2 onto $H^2_- \stackrel{\text{def}}{=} L^2 \ominus H^2$. It is well known (Nehari's theorem, see [14], [20]) that for $\varphi \in \mathrm{VMO}$

$$\|H_\varphi\| = \|\varphi - \mathscr{A}\varphi\|_{L^\infty(\mathbf{T})},$$

the right-hand side being equivalent to $\|\mathbf{P}_- \varphi\|_{\mathrm{BMO}}$.

Besides the operator $\mathscr{A}$ we shall also consider the operators $\mathscr{A}_m$ of best approximation by meromorphic functions of degree at most m which can be defined in the following way. Let $\tilde{\mathscr{R}}_m$ be the set of functions ψ in BMO such that $\mathbf{P}_- \psi$ is a rational function of degree at most m. Given $\varphi \in \mathrm{VMO}$, the *best approximation of* φ *by meromorphic functions of degree at most* m is, by definition, a function $\mathscr{A}_m \varphi \in \tilde{\mathscr{R}}_m$ such that

$$\|\varphi - \mathscr{A}_m \varphi\|_{L^\infty(\mathbf{T})} = \inf\{\|\varphi - \psi\|_{L^\infty(\mathbf{T})} \colon \psi \in \mathscr{R}_m\}.$$

It follows from a theorem of Adamyan, Arov, and Krein [3] that such a best approximation exists and is unique. Clearly, $\mathscr{A} = \mathscr{A}_0$. A deep result of Adamyan, Arov, and Krein claims that

$$\|\varphi - \mathscr{A}_m \varphi\|_{L^\infty(\mathbf{T})} = s_m(H_\varphi),$$

where $\{s_m(H_\varphi)\}_{m \geq 0}$ is the sequence of singular values of H_φ.

In [20] a systematic study of hereditary properties of the operator $\mathscr{A}$ was undertaken. Namely, the problem considered there was to find for which spaces X of functions on $\mathbf{T}$ it is true that

(1) $$\varphi \in X \Rightarrow \mathscr{A}\varphi \in X.$$

Earlier in [23] and [5] the above implication was proved for some function spaces, in particular, in [5] (1) was proved for the Hölder classes, see [20] for detailed references. With the help of techniques of Hankel operators three big classes of Banach spaces were found in [20] for which the implication (1) holds, see § 2 for more detail. Note that after [20] there appeared two more papers [24], [25], dedicated to the hereditary problems for the best approximation operator $\mathscr{A}$, in which other classes of functions spaces satisfying (1) were found.

The continuity problem for the operator $\mathscr{A}$ posed in [12] is very important in applications. This problem was solved in [18] for the second class of spaces described in [20] (see § 2 for more detail). Namely, it was proved in [18] that given a function space X from the second class described in [20], a function φ is a continuity point for $\mathscr{A}$ if and only if the singular value $s_0(H_\varphi)$ of the Hankel operator H_φ has multiplicity one.

In [10] the continuity problem for the operators $\mathscr{A}_m$ was considered in the norm of the space of functions $\mathscr{F}l^1$ with absolutely convergent Fourier series. It was proved there that $\mathscr{A}_m$ is continuous at φ in the norm of $\mathscr{F}l^1$ if and only if the singular value $s_m(H_\varphi)$ of the Hankel operator H_φ has multiplicity one. In [19] the same problem was considered for the second class of spaces described in [20] (see § 2).

It was also shown in [18], [19] that if we consider the continuity problem in the L^∞-norm, the situation is quite different, namely a function φ can be a discontinuity point for $\mathscr{A}$ even if the singular value $s_0(H_\varphi)$ has multiplicity one. It was conjectured there that any function φ in $C(\mathbf{T})$ is a discontinuity point for $\mathscr{A}$ in the L^∞-norm unless $\varphi \in H^\infty$. This conjecture has been proved recently by Merino [13] and Papadimitrakis [15] by quite different methods. Note that the method of Merino allows one to prove the same result for the operators $\mathscr{A}_m$, $m \in \mathbf{Z}_+$. Namely for any $\varphi \in C(\mathbf{T}) \setminus \tilde{\mathscr{R}}_m$ the operator $\mathscr{A}_m$ is discontinuous at φ.

For the first class of spaces X considered in [20] (this class contains VMO and the Besov spaces $B_p^{1/p}$, $0 < p < \infty$, see § 2 for more detail) it was shown that $\mathscr{A}$ has a property even stronger than (1), namely in this case $\mathbf{0}$ is a continuity point for $\mathscr{A}$ in the norm of X, or in other words A is *bounded* on X, i.e.

(2) $$\|\mathscr{A}\varphi\|_X \leqq \text{const} \cdot \|\varphi\|_X, \quad \varphi \in X.$$

Note that the boundedness property of $\mathscr{A}$ is also important in applications.

However for other spaces X satisfying (1) (e.g. for the second and the third class of spaces considered in [20]) it was unclear whether (2) holds. In [18], [19] a problem was posed which spaces X with property (1) satisfy (2). In particular it was asked there whether (2) is true for the Hölder—Zygmund classes Λ_α or for the Besov classes B_p^s, $0<p<\infty$, $s>1/p$. The main result of this paper claims that the answer is negative. The proof will be given in § 3. The construction is close in spirit to that of [13]. The same result is valid for the operators $\mathscr{A}_m$.

In § 2 we give needed information on properties of the best approximation operator $\mathscr{A}$ and the Besov classes.

In § 4 we formulate open questions.

2. Preliminaries

1. Best approximation operators. As mentioned in the introduction in [20] three big classes of functions X were found which are invariant under the best approximation operator $\mathscr{A}$. The first class contains the so-called $\mathscr{R}$-spaces, i.e. the function spaces which can be described in terms of best rational approximation in BMO (see [20] for detail). The spaces VMO and the Besov spaces $B_p^{1/p}$ are examples of $\mathscr{R}$-spaces. This follows from the compactness criterion for the Hankel operators (H_φ is compact if and only if $\mathbf{P}_-\varphi\in\mathrm{VMO}$ (Hartman's theorem, see [14])) and from the description of the Hankel operators of Schatten—von Neumann class $\mathfrak{S}_p$ (H_φ belongs to $\mathfrak{S}_p$ if and only if $\mathbf{P}_-\varphi\in B^{1/p}$ (see [16] for $p\geqq 1$ and [17], [22] for $0<p<1$)).

The second class of functions considered in [20] can be described as function spaces X satisfying the following properties:

(A_1) *If $f\in X$ then $\bar{f}\in X$ and $\mathbf{P}_-f\in X$;*

(A_2) *X is a Banach algebra with respect to the pointwise multiplication;*

(A_3) *The set of trigonometric polynomials is dense in X;*

(A_4) *Each multiplicative linear functional on X coincides with point evaluation at some point ζ in $\mathbf{T}$: $f\mapsto f(\zeta)$.*

This class of spaces contains the Besov spaces B_p^s, $1\leqq p<\infty$, $s>1/p$, B_1^1 (see the definition below), the space $\mathscr{F}l^1$ of functions with absolutely convergent Fourier series, the separable Hölder—Zygmund classes λ_α, and many others (see [20]).

The third class of spaces contains some non-separable spaces (in particular, the Hölder—Zygmund classes Λ_α) and some non-normed spaces (such as Carleman classes).

Let φ be a function in VMO such that $\mathbf{P}_-\varphi\neq\mathbf{0}$, then the function $\varphi-\mathscr{A}\varphi$ has constant modulus with negative winding number with respect to 0 (as shown in [20], $\varphi-\mathscr{A}\varphi$ belongs to $QC\overset{\text{def}}{=}L^\infty\cap\mathrm{VMO}$, and it is easy to extend the notion of winding number for such functions). Conversely if g is a function in VMO analytic in the unit disc such that $|\varphi-g|=\text{const}$ a.e. and the winding number $\operatorname{wind}(\varphi-g)$ is negative, then $\mathscr{A}\varphi=g$ (for continuous g this is Poreda's theorem [21]).

There is a similar fact for the operators $\mathcal{A}_m$, $m\in\mathbf{Z}_+$. Let $\varphi\in\mathrm{VMO}\setminus\tilde{\mathcal{R}}_m$ such that the singular value $s_m(H_\varphi)$ has multiplicity μ and

$$s_k(H_\varphi) = s_{k+1}(H_\varphi) = \ldots = s_{k+\mu-1}(H_\varphi), \quad k \leqq m \leqq k+\mu-1.$$

then $\varphi-\mathcal{A}_m\varphi$ has constant modulus and winding number $-(2k+\mu)$. Conversely, if g is a function in $\tilde{\mathcal{R}}_m$ such that $\varphi-g$ has constant modulus and $-\mathrm{wind}\,(\varphi-g)>2m$, then $\mathcal{A}_m\varphi=g$ (see [3], [10]).

It was proved in [19] that if X is a function space satisfying (A_1)—(A_4) and the multiplicity μ of the singular value $s_m(H_\varphi)$ is equal to one, then φ is a continuity point for $\mathcal{A}_m$ is the norm of X. If μ is greater than one and $m\neq k+(\mu-1)/2$, then $\mathcal{A}_m$ is discontinuous at φ (see [19]). In the case $\mu>1$ and $m=k+(\mu-1)/2$ the situation is less clear. In [10] it is proved that φ is a discontinuity point for $X=\mathcal{F}l^1$ and in [19] the same is proved for $X=\Lambda_\alpha$, $\alpha>0$, $\alpha\notin\mathbf{Z}$. The same is also true for the Besov space B_1^1 (the last fact was not mentioned in [19] but the methods of [19] also work for B_1^1).

2. Besov classes. The Besov class B_p^s, $0<p<\infty$, $s>0$, $s>1/p-1$, can be defined by

$$B_p^s = \left\{f\in L^p\colon \int_{-1}^{1}\int_{\mathbf{T}} \frac{|(\Delta_t^n f)(\zeta)|^p}{|t|^{1+sp}}\,|d\zeta|\,dt < \infty\right\},$$

where n is an integer, $n>s$, $(\Delta_t f)(\zeta)=f(e^{it}\zeta)-f(\zeta)$, $\Delta_t^n=\Delta_t\Delta_t^{n-1}$. There are many other equivalent definitions of Besov classes. We shall need the following one. Let f be a function on $\mathbf{T}$ analytic in $\mathbf{D}$ (i.e. $\mathbf{P}_-f=\mathbf{0}$). Then $f\in B_p^s$ if and only if

$$\iint_{\mathbf{D}} (1-|z|)^{(n-s)p-1}|f^{(n)}(z)|^p\,dx\,dy < \infty,$$

where $n>s$.

The Besov class B_∞^s is simply the Hölder—Zygmund class Λ_s which can be defined by

$$\Lambda_s = \{f\colon |(\Delta_t^n f)(\zeta)| \leqq \mathrm{const}\cdot|t|^s\}, \quad n > s.$$

A function f analytic in $\mathbf{D}$ belongs to Λ_s if and only if

$$|f^{(n)}(\zeta)| \leqq \mathrm{const}\cdot(1-|z|)^{s-n}, \quad n > s.$$

3. Boundedness properties in Besov norms

In this section we shall prove that the best approximation operator $\mathcal{A}$ is unbounded on the Hölder classes Λ_s, $s>0$, and Besov classes B_p^s, $0<p<\infty$, $s>1/p$. Recall that the operator $\mathcal{A}$ is bounded on $B_p^{1/p}$ (see [20]). The same is true for the operators $\mathcal{A}_m$. Note that for $s<1/p$ the Besov class B_p^s is not contained in BMO.

Theorem 1. *Let* $X=\Lambda_s$, $s>0$, *or* $X=B_p^s$, $0<p<\infty$, $s>1/p$. *Then there exists a sequence of functions* $\{\varphi_n\}$ *such that*

$$\|\varphi_n\|_X \leqq \text{const}$$

but

$$\lim_{n\to\infty} \|\mathcal{A}\varphi_n\|_X = \infty.$$

The construction given below is close in spirit to (but is different from) that in [13].

Consider the conformal mapping ω_α, $0<\alpha<1$, from the unit disc $\mathbf{D}$ onto the disc $\{\zeta\colon |1-\zeta|<1\}$ defined by

$$\omega_\alpha(z) \overset{\text{def}}{=} 1+\frac{z-\alpha}{1-\alpha z}.$$

Define the functions ϱ_α and η_α on $\mathbf{T}$ by

$$\varrho_\alpha(z) = \frac{(\omega_\alpha(z)-\bar{z}^2)}{|\omega_\alpha(z)-\bar{z}^2|}\left(|\omega_\alpha(z)-\bar{z}^2|-1\right);$$

$$\eta_\alpha(z) = \bar{z}^2+\varrho_\alpha(z).$$

Note that for α close to one $\omega_\alpha(z)-\bar{z}^2$ is separated away from 0 since $\omega_\alpha(z)$ is close to 0 when z lies outside a small neighbourhood of 1.

Lemma 1. *For* α *close to* 1 *the following equality holds:*

$$\mathcal{A}\eta_\alpha = \omega_\alpha.$$

Proof. Let us show that $\eta_\alpha-\omega_\alpha$ has constant modulus and negative winding number, which will imply the desired conclusion (see § 2). We have

$$\eta_\alpha(z)-\omega_\alpha(z) = \bar{z}^2+\frac{(\omega_\alpha(z)-\bar{z}^2)}{|\omega_\alpha(z)-\bar{z}^2|}\left(|\omega_\alpha(z)-\bar{z}^2|-1\right)-\omega_\alpha(z)$$

$$= -\frac{(\omega_\alpha(z)-\bar{z}^2)}{|\omega_\alpha(z)-\bar{z}^2|}.$$

So $\eta_\alpha-\omega_\alpha$ is unimodular. Let us show that its winding number is equal to -1. Clearly,

$$\operatorname{wind}(\eta_\alpha-\omega_\alpha) = \operatorname{wind}\left(\omega_\alpha(z)-\bar{z}^2\right) = \operatorname{wind}\left(\bar{z}^2(z^2\omega_\alpha(z)-1)\right)$$
$$= -2+\operatorname{wind}\left(z^2\omega_\alpha(z)-1\right) = -2+\operatorname{wind}_1(z^2\omega_\alpha(z)),$$

where wind_1 means the winding number with respect to 1.

Let us now show that $\operatorname{wind}_1\left(z^2\omega_\alpha(z)\right)=1$. Suppose that α is close to one. Let ϑ be a positive number such that $\cos\vartheta=\alpha$ and $\tau=e^{i\vartheta}$. Then it is easy to check that $\omega_\alpha(\tau)=1-\bar{\tau}$. So when ζ is moving along $\mathbf{T}$ from 1 to τ, $\omega_\alpha(\zeta)$ is moving along the circle $\{\zeta\colon |1-\zeta|=1\}$ from 2 to $1-\bar\tau$, while ζ^2 lies in a small neighbourhood of 1. Therefore $\zeta^2\omega_\alpha(\zeta)$ is moving along a curve close to the arc $[2, 1-\bar\tau]$ of the circle $\{\zeta\colon |1-\zeta|=1\}$. Next, when ζ is moving on $\mathbf{T}$ from τ to $\bar\tau$, $\omega_\alpha(\zeta)$ is moving within a small neighbourhood of 0 from $1-\bar\tau$ to $1-\tau$ while ζ^2 has modulus 1. Therefore $\zeta^2\omega_\alpha(\zeta)$ is moving within a small neighbourhood of 0. Finally, when ζ is moving along $\mathbf{T}$ from $\bar\tau$ to 1, $\omega_\alpha(\zeta)$ is moving along the circle $\{\zeta\colon |1-\zeta|=1\}$ from $1-\tau$ to 2 while ζ^2 lies in a small neighbourhood of 1. Therefore $\zeta^2\omega_\alpha(\zeta)$ is moving along a curve close to the arc $[1-\tau, 2]$ of the circle $\{\zeta\colon |1-\zeta|=1\}$.

The above reasoning shows that $\operatorname{wind}_1(z^2\omega_\alpha(z))=1$. □

Now to prove Theorem 1, it is sufficient to show that

$$\lim_{\alpha\to 1}\frac{\|\omega_\alpha\|_X}{\|\varrho_\alpha\|_X}=\infty.$$

Lemma 2. *Let* $X=B_p^s$, $0<p\leqq\infty$, $s>1/p$. *Then*

$$\|\omega_\alpha\|_X\asymp(1-\alpha)^{1/p-s}.$$

Recall that B_∞^s means the Hölder—Zygmund class Λ_s

Proof. To prove the result we shall use the following norm for functions in B_p^s analytic in $\mathbf{D}$.

$$\|f\|_{B_p^s}=\|f\|_{L^\infty}+\Big(\iint_{\mathbf{D}}(1-|z|)^{(n-s)p-1}|f^{(n)}(z)|^p\,dx\,dy\Big)^{1/p}<\infty \tag{3}$$

for $p<\infty$, and

$$\|f\|_{\Lambda_s}=\|f\|_{L^\infty}+\sup_{\zeta\in\mathbf{D}}\frac{|f^{(n)}(\zeta)|}{(1-|\zeta|)^{s-n}}$$

for $p=\infty$, where $n>s$. The case $p=\infty$ is simpler. Let us obtain the lower estimate for $\|\omega_\alpha\|_{B_p^s}$ in the case $p<\infty$:

$$\|\omega_\alpha\|_{B_p^s}\geqq\text{const}\cdot(1-\alpha)^{1/p-s}.$$

We have

$$\|\omega_\alpha\|_{B_p^s}^p\geqq\iint_\Omega(1-|z|)^{(n-s)p-1}|f^{(n)}(z)|^p\,dx\,dy,$$

where $\Omega=\{\zeta=r^{i\vartheta}\in\mathbf{D}\colon 1-\alpha<r<1,\ 0\leqq\vartheta\leqq 1-\alpha\}$. Then

$$\begin{aligned}\|\omega_\alpha\|_{B_p^s}^p&\geqq\text{const}\cdot\iint_\Omega(1-|z|)^{(n-s)p-1}\left(\frac{1-\alpha}{|1-\alpha z|^{n+1}}\right)^p dx\,dy\\&\asymp\text{const}\cdot(1-\alpha)^{-np}\cdot\iint_\Omega(1-|z|)^{(n-s)p-1}\,dx\,dy\\&\asymp\text{const}\cdot(1-\alpha)^{-np}(1-\alpha)\cdot\int_0^{1-\alpha}r^{(n-s)p-1}\,dr=\text{const}\cdot(1-\alpha)^{1-sp},\end{aligned}$$

which yields the desired lower estimate.

The upper estimate can be easily obtained if we split the integral on the right-hand side in (3) as the sum of integrals over the sets $\{\zeta=r^{i\vartheta}\in\mathbf{D}: 1-(k+1)(1-\alpha)<r<1-k(1-\alpha), m(1-\alpha)<\vartheta<(m+1)(1-\alpha)\}$ and observe that the integral over Ω yields the main contribution. □

Lemma 3. *Let* $n\in\mathbf{Z}_+$. *Then*

$$\|\omega^{(n)}\|_{L^\infty}\asymp(1-\alpha)^{-n}.$$

This can be proved by direct computations.

Theorem 1 will be proved if we prove the following fact.

Lemma 4. *There exists a number* $\gamma>1/p-s$ *such that*

$$\|\varrho_\alpha\|_X\leqq\text{const}\cdot(1-\alpha)^\gamma.$$

Proof. Let us first estimate $\|\varrho_\alpha\|_{L^\infty}$. We claim that

$$\|\varrho_\alpha\|_{L^\infty}\leqq\text{const}\cdot(1-\alpha)^{1/2}. \tag{4}$$

Indeed let $\tau=e^{i\vartheta}$ be as in the proof of Lemma 1, $\operatorname{Re}\tau=\alpha$. Then for ζ in the arc $[-\tau,\tau]$ of the unit circle

$$\big||\omega_\alpha(\zeta)-\zeta^2|-1\big|\leqq\sup\{|\lambda^2|: \lambda\in[-\tau,\tau]\}\leqq\text{const}\cdot(1-\alpha)^{1/2}.$$

But if $\zeta\notin[-\tau,\tau]$, then $\omega_\alpha(\zeta)$ lies on the arc $[1-\tau,1+\tau]$ of the circle $\{\zeta: |1-\zeta|=1\}$. So

$$\big||\omega_\alpha(\zeta)-\zeta^2|-1\big|\leqq\sup\{|\lambda|: \lambda\in[1-\tau,1+\tau]\}\leqq\text{const}\cdot(1-\alpha)^{1/2}.$$

Let us represent ϱ_α as $\varrho_\alpha=f_\alpha(g_\alpha-1)$, where

$$f_\alpha(z)=\frac{(\omega_\alpha(z)-\bar z^2)}{|\omega_\alpha(z)-\bar z^2|},\quad g_\alpha(z)=|\omega_\alpha(z)-\bar z^2|.$$

Since $\omega_\alpha(z)-\bar z^2$ is separated away from zero, we have $\|f_\alpha\|_X\leqq\text{const}\cdot\|\omega_\alpha\|_X$. Such estimates are well-known for experts, they can be proved by direct computations and they can also be easily obtained with the help of the technique of $\bar\partial$-extensions developed by Dyn'kin [6].

We shall use in the proof the following formula which can easily be established by induction.

$$(\Delta_t^n\varphi\psi)(\zeta)=\sum_{k=0}^n\binom{n}{k}(\Delta_t^{n-k}\varphi)(e^{ikt}\zeta)(\Delta_t^k\psi)(\zeta). \tag{5}$$

To obtain the desired estimate, we consider several cases.

1. The case $p=\infty$. We shall work with the following semi-norm on X:

$$\|f\|_{s,n}=\sup_{t\neq0}\frac{|(\Delta_t^nf)(\zeta)|}{|t|^s}.$$

Let us first estimate $\|g_\alpha\|_{s,n}$. Let δ be a positive number which will be specified later. Suppose that $|t|\geqq(1-\alpha)^\delta$. We have

$$\frac{|(\Delta_t^n g_\alpha)(\zeta)|}{|t|^s} \leqq \text{const}\cdot\|\varrho_\alpha\|_{L^\infty}\cdot(1-\alpha)^{-\delta s} \leqq \text{const}\cdot(1-\alpha)^{1/2-\delta s}.$$

Suppose now that $|t|<(1-\alpha)^\delta$. Then

$$\frac{|(\Delta_t^n g_\alpha)(\zeta)|}{|t|^s} \leqq \text{const}\cdot\|g_\alpha^{(n)}\|_{L^\infty}|t|^{n-s} \leqq \text{const}\cdot\|\omega_\alpha^{(n)}\|_{L^\infty}|t|^{n-s}$$
$$\leqq \text{const}\cdot(1-\alpha)^{-n}(1-\alpha)^{(n-s)\delta} = \text{const}\cdot(1-\alpha)^{\delta n-\delta s-n}.$$

Choose now δ so that $1/2-\delta s=\delta n-\delta s-n$. So $\delta=1+1/2n$ and we have

(6) $$\|g_\alpha\|_{s,n} \leqq \text{const}\cdot(1-\alpha)^{-s+1/2-s/2n}.$$

Note that $1/2-s/2n>0$.

Let us now estimate $\|\varrho_\alpha\|_{s,n}$. To this end we apply formula (5) with $\varphi=f_\alpha$, $\psi=g_\alpha-1$. Let $0<k<n$, then $s=s_1+s_2$, where $0<s_1<n-k$, $0<s_2<k$. We have

(7) $$\frac{|(\Delta_t^{n-k}\varphi)(e^{itk}\zeta)(\Delta_t^k\psi)(\zeta)|}{|t|^s} \leqq \text{const}\cdot\|\varphi\|_{n-k,s_1}\|\varphi\|_{k,s_2}$$
$$\leqq \text{const}\cdot(1-\alpha)^{-s_1}(1-\alpha)^{-s_2+1/2-s_2/2k} = \text{const}\cdot(1-\alpha)^{-s+1/2-s_2/2k}$$

by (6) and Lemma 2. Next, if $k=0$, then

(8) $$\frac{|(\Delta_t^n\varphi)(\zeta)\psi(\zeta)|}{|t|^s} \leqq \text{const}\cdot\|\omega_\alpha\|_{\Lambda_s}\cdot\|\varrho_\alpha\|_{L^\infty} \leqq \text{const}\cdot(1-\alpha)^{-s+1/2}$$

by (4) and Lemma 2. Finally, if $k=n$, then

$$\frac{|\varphi(e^{int}\zeta)(\Delta_t^n\psi)(\zeta)|}{|t|^s} \leqq \text{const}\cdot(1-\alpha)^{-s+1/2-s/2n}$$

by (6), which together with (7) and (8) proves the lemma for $p=\infty$.

2. The case $2<p<\infty$. We shall work with the following semi-norm on X:

$$\|f\|_{p,s,n} = \left(\int_{-1}^1\int_{\mathbf{T}}\frac{|(\Delta_t^n f)(\zeta)|^p}{|t|^{1+sp}}|d\zeta|\,dt\right)^{1/p}.$$

Let us first estimate $\|g_\alpha\|_{p,s,n}$. Let δ be a positive number whose choice will be specified later. Let $\Omega_1=\{t\in[-1,1]\colon |t|\geqq(1-\alpha)^\delta\}$, $\Omega_2=\{t\in[-1,1]\colon |t|<(1-\alpha)^\delta\}$. We have

$$\int_{\Omega_1}\int_{\mathbf{T}}\frac{|(\Delta_t^n g_\alpha)(\zeta)|^p}{|t|^{1+sp}}|d\zeta|\,dt \leqq \text{const}\int_{\Omega_1}\int_{\mathbf{T}}\frac{(1-\alpha)^{p/2}}{|t|^{1+sp}}|d\zeta|\,dt$$
$$\leqq \text{const}\cdot(1-\alpha)^{p/2}\int_{(1-\alpha)^\delta}^1 t^{-1-sp}\,dt \leqq \text{const}\cdot(1-\alpha)^{p(1/2-\delta s)}$$

by (4). Next,

$$\int_{\Omega_1}\int_{\mathbf{T}}\frac{|(\Delta_t^n g_\alpha)(\zeta)|^p}{|t|^{1+sp}}\,|d\zeta|\,dt \leqq \text{const}\cdot\|\omega_\alpha^{(n)}\|_{L^\infty}^p \int_0^{(1-\alpha)^\delta} t^{np-sp-1}\,dt$$

$$\leqq \text{const}\cdot(1-\alpha)^{-np}(1-\alpha)^{\delta(np-sp)} \leqq \text{const}\cdot(1-\alpha)^{p(\delta n-\delta s-n)}.$$

Let us choose now δ so that $p(1/2-\delta s)=p(\delta n-\delta s-n)$, i.e. $\delta=1+\frac{1}{2n}$. Then

(9) $$\|g_\alpha\|_{p,s,n} \leqq \text{const}\cdot(1-\alpha)^{1/2-s-s/2n}.$$

Now we can pick $n>s/(1-2/p)$. Then $1/2-s-s/2n>1/p-s$.

We shall again use formula (5) with $\varphi=f_\alpha$, $\psi=g_\alpha-1$. Let $0<k<n$. Let us represent s as $s=s_1+s_2$, where $0<s_1<(n-k)(1-2/p)$, $0<s_2<k(1-2/p)$. This is possible because of the above choice of n. We have

(10) $$\int_{-1}^{1}\int_{\mathbf{T}}\frac{|(\Delta_t^{n-k}\varphi)(e^{ikt}\zeta)(\Delta_t^k\psi)(\zeta)|^p}{|t|^{1+sp}}\,|d\zeta|\,dt$$

$$\leqq\left(\int_{-1}^{1}\int_{\mathbf{T}}\frac{|(\Delta_t^{n-k}\varphi)(e^{ikt}\zeta)|^{2p}}{|t|^{1+2s_1p}}\,|d\zeta|\,dt\right)^{1/2}\left(\int_{-1}^{1}\int_{\mathbf{T}}\frac{|(\Delta_t^k\psi)(\zeta)|^{2p}}{|t|^{1+2s_2p}}\,|d\zeta|\,dt\right)^{1/2}$$

$$\leqq \|\varphi\|_{2p,s_1,n-k}^p\cdot\|\psi\|_{2p,s_2,k}^p \leqq \text{const}\cdot(1-\alpha)^{1/2-ps_1}(1-\alpha)^{\mu p}$$

by (9) and Lemma 2, where $\mu>1/2p-s_2$. So the integral (10) is less than or equal to $\text{const}\cdot(1-\alpha)^{\gamma p}$, where $\gamma=1/2p-s_2+\mu>1/p-s$.

Let now $k=0$. Then

$$\left(\int_{-1}^{1}\int_{\mathbf{T}}\frac{|(\Delta_t^n\varphi)(\zeta)\psi(\zeta)|^p}{|t|^{1+sp}}\,|d\zeta|\,dt\right)^{1/p}$$

$$\leqq \|\varphi\|_{p,s,n}\|\psi\|_{L^\infty} \leqq \text{const}\cdot(1-\alpha)^{1/p-s}(1-\alpha)^{1/2}$$

by (4) and Lemma 2. Finally, let $k=n$. We have

$$\left(\int_{-1}^{1}\int_{\mathbf{T}}\frac{|\varphi(e^{int}\zeta)(\Delta_t^n\psi)(\zeta)|^p}{|t|^{1+sp}}\,|d\zeta|\,dt\right)^{1/2} \leqq \text{const}\cdot(1-\alpha)^{1/2-s-s/2n}$$

by (9) which completes the proof of the lemma for $p>2$.

3. The case $p\leqq 2$. First of all let us note that

(11) $$\|g_\alpha\|_{p,s,n} \leqq \text{const}\cdot(1-\alpha)^{1/p-s}$$

for any $s,p,n>0$. This follows easily from Lemma 2. Let q be a positive number such that $qp>4$, $q'=q/(q-1)$. We have by the Hölder inequality

$$\|g_\alpha\|_{p,s,n}^p=\int_{-1}^{1}\int_{\mathbf{T}}\frac{|(\Delta_t^n g_\alpha)(\zeta)|^{p/2}}{|t|^{sp/2}}\cdot\frac{|(\Delta_t^n g_\alpha)(\zeta)|^{p/2}}{|t|^{sp/2}}\,|d\zeta|\,\frac{dt}{|t|}$$

$$\leqq\left(\int_{-1}^{1}\int_{\mathbf{T}}\frac{|(\Delta_t^n g_\alpha)(\zeta)|^{pq/2}}{|t|^{spq/2}}\,|d\zeta|\,\frac{dt}{|t|}\right)^{1/q}\left(\int_{-1}^{1}\int_{\mathbf{T}}\frac{|(\Delta_t^n g_\alpha)(\zeta)|^{pq'/2}}{|t|^{spq'/2}}\,|d\zeta|\,\frac{dt}{|t|}\right)^{1/q'}=$$

$$=\|g_\alpha\|_{pq/2,s,n}^{p/2}\cdot\|g_\alpha\|_{pq'/2,s,n}^{p/2} \leqq \text{const}\cdot(1-\alpha)^{\gamma p/2}\cdot(1-\alpha)^{(2/pq'-s)p/2}$$

by (11), where $\gamma > 2/pq - s$ (such a γ exists since $pq/2 > 2$ and the lemma has already been proved for $p > 2$). It follows that

$$\|g_\alpha\|_{p,s,n} \leqq \text{const} \cdot (1-\alpha)^{(\gamma + 2/pq' - s)/2}.$$

Clearly, $(\gamma + 2/pq' - s)/2 > (2/pq - s + 2/pq' - s)/2 = 1/p - s$ which completes the proof of the lemma. □

A similar result can be proved for the operators $\mathscr{A}_m$.

Theorem 2. *Let* $X = \Lambda_s$, $s > 0$, *or* $X = B_p^s$, $0 < p < \infty$, $s > 1/p$, $m \in \mathbf{Z}_+$. *Then there exists a sequence of functions* $\{\varphi_n\}$ *such that*

$$\|\varphi_n\|_X \leqq \text{const}$$

but

$$\lim_{n \to \infty} \|\mathscr{A}_m \varphi_n\|_X = \infty.$$

The proof of Theorem 2 is similar to that of Theorem 1. The only difference is that in the definition of ϱ_α and η_α we have to replace $\bar{z}^2$ by $\bar{z}^{m+1}$. Then it can be shown that

$$\text{wind}\,(\eta_\alpha - \omega_\alpha) = -m,$$

and so $\mathscr{A}_m \eta_\alpha = \omega_\alpha$. All above estimates work in this situation too.

4. Open problems

As we have mentioned in the introduction the $\mathscr{R}$-spaces satisfy the boundedness property (2). It is also clear that if X is an $\mathscr{R}$-space, then $X \cap L^\infty$ satisfies the boundedness property. The following question seems very interesting.

Question 1. *Are there other Banach spaces X imbedded in* VMO *that satisfy the boundedness property* (2)?

In particular it is interesting to learn whether certain classical spaces satisfying (1) have the boundedness property.

Question 2. *Does the space* $X = \mathscr{F}l^1$ *of functions with absolutely convergent Fourier series satisfy the boundedness property?*

I believe the answer to Question 2 should be negative. The following result gives an estimate for the norm of $\mathscr{A}\varphi$ in $\mathscr{F}l^1$ for functions in $\tilde{\mathscr{R}}_m$.

Proposition. *Let* $\varphi \in \tilde{\mathscr{R}}_m$. *Then*

$$\|\mathscr{A}\varphi\|_{\mathscr{F}l^1} \leqq \text{const} \cdot m\, \|\varphi\|_{\mathscr{F}l^1}.$$

Proof. Since $B_1^1 \subset \mathscr{F}l^1$ and the operator $\mathscr{A}$ is bounded on B_1^1, it follows that

$$\|\mathscr{A}\varphi\|_{\mathscr{F}l^1} \leqq \text{const} \cdot \|\mathscr{A}\varphi\|_{B_1^1} \leqq \text{const} \cdot \|\varphi\|_{B_1^1}.$$

Next, it follows from the nuclearity criterion for the Hankel operators (see § 2 and [16]) that

$$\|\mathbf{P}_-\varphi\|_{B_1^1} \leqq \text{const} \cdot \|H_\varphi\|_{\mathfrak{S}_1} \leqq \text{const} \cdot \text{rank}\, H_\varphi \cdot \|H_\varphi\| \leqq \text{const} \cdot m \|\varphi\|_{L^\infty},$$

since $\varphi \in \tilde{\mathscr{R}}_m$. The same inequality is true for the function $\mathbf{P}_- \bar{\varphi}$. Thus

$$\|\varphi\|_{B_1^1} \leqq \text{const} \cdot m \|\varphi\|_{L^\infty} \leqq \text{const} \cdot m \|\varphi\|_{\mathscr{F}l^1}. \quad \square$$

The Besov spaces $B_p^{1/p}$ play an important role in these questions. The space B_1^1 is both an $\mathscr{R}$-space and satisfies the properties (A_1)—(A_4) (see § 2). So this space satisfies the boundedness property, and the continuity points of the operators $\mathscr{A}_m$ admit a characterization in terms of the multiplicities of the corresponding singular values of Hankel operators (see § 2). So the space B_1^1 is the most convenient space to work with the operators $\mathscr{A}_m$. The spaces $B_p^{1/p}$ with $p>1$ do not satisfy the properties (A_1)—(A_4) since they contain unbounded functions. The question of whether the continuity points of the operator $\mathscr{A}$ (and the operators $\mathscr{A}_m$) admit a similar description remains unsolved. It follows from the results of [19] that if $s_0(H_\varphi)$ has multiplicity greater than one, then φ is not a continuity point.

Question 3. *Let φ be a function in $B_p^{1/p}$ such that the singular value $s_0(H_\varphi)$ has multiplicity one. Is it true that the operator $\mathscr{A}$ is continuous at φ?*

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Received Oct. 15, 1990

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