Square area integral estimates for subharmonic functions in NTA domains

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Abstract. In this paper, we prove a good-λ inequality between the nontangential maximal function and the square area integral of a subharmonic function u in a bounded NTA domain D in $\mathbf{R}^n$. We achieve this by showing that a weighted Riesz measure of u is a Carleson measure, with the Carleson norm bounded by a constant independent of u. As consequences of the good-λ inequality, we obtain McConnell—Uchiyama's inequality and an analogue of Murai—Uchiyama's inequality for subharmonic functions in D.

1. Introduction

In this paper we shall prove a good-λ inequality between the nontangential maximal function and the square area integral of a subharmonic function in a bounded nontangentially accessible (NTA) domain D in $\mathbf{R}^n$, which leads to two L^p inequalities comparing these two quantities.

Throughout this paper D will denote a bounded NTA domain in $\mathbf{R}^n$, $n \geqq 2$, with structure constants M and r_0 (see [JK] for the definition). For $X \in D$, we let $\delta(X)$ denote the Euclidean distance from X to the boundary ∂D of D. For $Q \in \partial D$ and $\alpha > 0$, define the nontangential region at Q by

(1.1) $$\Gamma_\alpha(Q) = \{X \in D\colon |X-Q| < (1+\alpha)\delta(X)\}.$$

For a subset $E \subset \partial D$ and $\alpha > 0$, the so-called sawtooth region $\mathcal{R}_\alpha(E)$ over E is given by

(1.2) $$\mathcal{R}_\alpha(E) = \bigcup_{Q \in E} \Gamma_\alpha(Q).$$

The surface ball centered at $Q \in \partial D$ and $r > 0$ is defined by $\Delta(Q, r) = B(Q, r) \cap \partial D$, where $B(Q, r) = \{X \in \mathbf{R}^n\colon |X-Q| < r\}$.

Let O be a fixed point of D, we shall use the notation ω for the harmonic measure of D evaluated at O, that is, for each Borel set $E \subset \partial D$, $\omega(E)$ is the value at O of the solution of the Dirichlet problem with boundary data χ_E, where χ_E is the characteristic function of E. It is well-known that the harmonic measure ω satisfies the doubling property ([JK, Lemma 4.9]), that is, there is a constant C depending only on D and O such that

$$\omega(\Delta(Q, 2r)) \leqq C\omega(\Delta(Q, r)) \tag{1.3}$$

for all $Q\in\partial D$ and $r>0$.

We recall that a positive measure σ on ∂D is said to satisfy the A_∞ condition with respect to ω, denoted by $\sigma\in A_\infty(\omega)$, if there exist positive constants η, c_1 and c_2 such that for any surface ball Δ and any Borel set $E\subset\Delta$,

$$c_1\left(\frac{\omega(E)}{\omega(\Delta)}\right)^{1/\eta} \leqq \frac{\sigma(E)}{\sigma(\Delta)} \leqq c_2\left(\frac{\omega(E)}{\omega(\Delta)}\right)^{\eta}. \tag{1.4}$$

A constant is said to be independent in $A_\infty(\omega)$ if the constant depends only on the $A_\infty(\omega)$ constants of σ, c_1 and c_2, the dimension n, and the doubling constant of ω in (1.3) rather than on the measure σ itself. We remark that if $\sigma\in A_\infty(\omega)$, then it follows from the doubling property of ω and the first inequality of (1.4) with $E=\Delta(Q,r)$ and $\Delta=\Delta(Q,2r)$ that the measure σ satisfies the doubling property as well. The most interesting examples of $A_\infty(\omega)$ measures for us are the surface measures of the boundaries of some NTA domains, such as planar chord-arc domains, Lipschitz domains and BMO_1 domains (see [JK]).

For a harmonic function h in D, the area integral $A_\alpha h$ and the nontangential maximal function $N_\alpha h$ are given by

$$A_\alpha h(Q) = \left\{\int_{\Gamma_\alpha(Q)} \delta(X)^{2-n}|\nabla h(X)|^2\,dX\right\}^{1/2},$$

and

$$N_\alpha h(Q) = \sup\{|h(X)|:\ X\in\Gamma_\alpha(Q)\},$$

respectively. It has been shown in [DJK] (with a different but equivalent definition of $A_\alpha h$) that, for any measure $\sigma\in A_\infty(\omega)$, the $L^p(\sigma)$-norms of $A_\alpha h$ and $N_\alpha h$ are equivalent.

If u is a subharmonic function in an NTA domain D, we define the *square area integral* of u by

$$S_\alpha u(Q) = \int_{\Gamma_\alpha(Q)} \delta(X)^{2-n}\,d\mu_u(X), \quad Q\in\partial D. \tag{1.5}$$

where μ_u is the Riesz measure of u, that is,

$$d\mu_u(X) = \Delta u(X)\,dX,$$

with the Laplacian Δu of u understood in the sense of distributions.

As usual, the *nontangential maximal function* for a subharmonic function u in D is defined as

$$N_\alpha u(Q) = \operatorname{ess\,sup}\{|u(X)|:\ X\in\Gamma_\alpha(Q)\},\quad Q\in\partial D. \tag{1.6}$$

This type of (square) area integral was first introduced by McConnell [Mc] for positive subharmonic functions in the half-space of $\mathbf{R}^n$, based on the identity $|\nabla h|^2=\frac{1}{2}\Delta(|h|^2)$ for a harmonic function h. He also showed that $\|S_\alpha u\|_{L^p(\mathbf{R}^{n-1})}$ is controlled by $\|N_\alpha u\|_{L^p(\mathbf{R}^{n-1})}$ for a limited range of p. Later, Uchiyama [U] proved the same result for all $0<p<\infty$ by using a different argument. A slightly different proof is given in [K] so that it works for more general differential operators than the Laplacian (but the underlying domain is still the half-space). In this paper, we shall generalize this result to the setting of bounded NTA domains. Following Uchiyama [U], the main task is to prove the following good-λ inequality:

Theorem. *Let D be an NTA domain in $\mathbf{R}^n$, and ω be the harmonic measure of D at a fixed point $O\in D$. Then, for any $\sigma\in A_\infty(\omega)$ and $0<\alpha<\beta<\infty$, there exist constants C and c, which are independent in $A_\infty(\omega)$, such that if u is a subharmonic function in D then*

$$\begin{aligned}\sigma(\{Q\in\partial D:\ S_\alpha u(Q)>\gamma\lambda,\ N_\beta u(Q)\leqq\lambda\})\\ \leqq Ce^{-c\gamma}\sigma(\{Q\in\partial D:\ S_{2\alpha}u(Q)>\lambda\}),\end{aligned} \tag{1.7}$$

for all $\lambda>0$ and $\gamma>1$.

We remark that, when $u=|h|^2$ and h is harmonic, the same inequality was obtained earlier in [MU] for half-spaces and recently in [BM] for Lipschitz domains.

Once the theorem is established, well-known arguments lead to the following corollaries. The first one is a generalized McConnell—Uchiyama inequality ([Mc; U]).

Corollary 1. *Under the hypothesis of the Theorem, for any $0<\alpha,\beta<\infty$ and $0<p<\infty$, there exists a constant C, independent in $A_\infty(\omega)$, such that if u is a subharmonic function in D, then*

$$\|S_\alpha u\|_{L^p(\sigma)}\leqq C\|N_\beta u\|_{L^p(\sigma)}.$$

Another application of the last theorem is the following analogue of a result of Murai—Uchiyama [MU] for subharmonic functions in D (see also [K]).

Corollary 2. *Under the hypothesis of the Theorem, suppose that $0<\alpha<\beta<\infty$ and $0<p<\infty$. Then there exist constants C_1 and C_2, independent in $A_\infty(\omega)$, such that for any subharmonic function u in D,*

$$\int_{\partial D}\exp\left(C_1\frac{S_\alpha u(Q)}{N_\beta u(Q)}\right)(S_\alpha u(Q))^p\,d\sigma(Q)\leqq C_2\|S_\alpha u\|^p_{L^p(\sigma)}.$$

Before we prove the Theorem, we notice that, by replacing u by u/λ, we may assume that $\lambda=1$. In what follows, we fix α and β with $0<\alpha<\beta<\infty$, and for a subharmonic function u in D we denote

$$E_u = \{Q\in\partial D\colon\ N_\beta u(Q) \geqq 1\}, \quad \text{and} \quad \mathcal{R}_u = \mathcal{R}_\alpha(E_u). \tag{1.8}$$

Let $G(X)$ denote the Green function of D with the pole at O. (We shall use the convention that the Green function is positive and hence superharmonic in D.) We now introduce a weighted Riesz measure ν_u on D, which is defined by

$$\nu_u(W) = \int_{\mathcal{R}_u\cap W} G(X)\,d\mu_u(X) \tag{1.9}$$

for each Borel subset W of D.

The key step in the proof of the Theorem is the following:

Main Lemma. *Under the hypothesis of the Theorem, the measure ν_u defined by (1.9) is a Carleson measure, and furthermore, the Carleson norm $\|\nu_u\|_C$ is bounded by a constant which is independent of u (but dependent on α, β, O and D).*

We recall that a positive measure ν on D is a Carleson measure if

$$\|\nu\|_C = \sup_B \frac{\nu(B\cap D)}{\omega(B\cap\partial D)} < \infty, \tag{1.10}$$

where B is a ball centered at a point in ∂D, and $\|\nu\|_C$ is called the Carleson norm of ν.

In what follows, we shall use letters $N, C, c, \ldots$ to indicate constants which are not necessarily the same at each occurrence. We also use the notation $q_1 \lesssim q_2$ to mean that there is a constant $c>0$ such that $q_1 \leqq cq_2$, if the dependency of the constant c is clear in the context. By $q_1 \cong q_2$ we mean that $q_1 \lesssim q_2$ and $q_2 \lesssim q_1$.

2. Proof of the Main Lemma

We begin the proof by recalling a well-known estimate of the harmonic measure in terms of the Green function, which we refer to as the Dahlberg–Jerison–Kenig Comparison Theorem ([JK, Lemma 4.8]). This theorem states that if $Q\in\partial D, 2r<r_0$ and $|O-Q|>2r$ then there exists a constant C, depending only on D, such that

$$C^{-1}r^{2-n}\omega(\Delta(Q,r)) \leqq G(Y) \leqq Cr^{2-n}\omega(\Delta(Q,r)), \tag{2.1}$$

for all $Y\in D$ with $\delta(Y)>M^{-1}r$ and $|Y-Q|<r$. As before, r_0 and M are the structure constants in the definition of the NTA domain D, G is the Green function of D with pole at O and ω is the harmonic measure of D evaluated at the point O.

The main lemma will be an immediate consequence of the following two lemmas. The first lemma deals with balls with small radii, while the second one takes care of large balls.

Lemma 1. *Let r be such that $0<5r<\min\{r_0, \delta(O)\}$. Then there is a constant C, independent of r and u, such that if $B=B(Q_0, r)$ for some $Q_0\in\partial D$, and $\Delta=B\cap D$, then*

$$\nu_u(B) \leqq C\omega(\Delta). \tag{2.2}$$

Proof. First we recall that there exists a so-called regularized distance function $\delta^*\in\mathscr{C}^\infty(D)$ such that

$$c_1\delta(X) \leqq \delta^*(X) \leqq c_2\delta(X), \tag{2.3}$$

and

$$|\nabla\delta^*(X)| \leqq C_1, \quad |\nabla\delta^*(X)| \leqq \frac{C_2}{\delta(X)}, \tag{2.4}$$

for all $X\in D$, where the constants c_1, c_2, C_1 and C_2 depend only on the dimension n. (See Theorem 6.2 of [S].)

Now, for $\varepsilon>0$, let $D_\varepsilon=\{X\in D\colon \delta(X)>\varepsilon\}$ and $\Omega_\varepsilon=\{X\in D\colon \delta^*(X)>\varepsilon\}$. Then, by (2.3), we have $D_{\varepsilon/c_1}\subset\Omega_\varepsilon\subset D_{\varepsilon/c_2}$, so that the family $\{\Omega_\varepsilon\colon \varepsilon>0\}$ of subdomains of D increases to D as $\varepsilon\searrow 0$. Clearly,

$$\nu_u(B) \leqq \lim_{\varepsilon\searrow 0}\int_{\mathscr{R}_u\cap B\cap\Omega_\varepsilon} G(X)\,d\mu_u(X).$$

Hence, to complete the proof it is enough to show that

$$\int_{\mathscr{R}_u\cap B\cap\Omega_\varepsilon} G(X)\,d\mu_u(X) \leqq C\omega(\Delta),$$

for some constant C independent of r, u and ε.

We next observe that if $X\in\mathscr{R}_u\cap B$ then $\delta(X)\leqq|X-Q_0|<r$ and $|X-Q|<(1+\alpha)\delta(X)$ for some $Q\in E_u$, so that

$$|Q-Q_0| \leqq |X-Q|+|X-Q_0| < (2+\alpha)r.$$

Let us denote by $\Delta^*=\overline{B(Q_0, (2+\alpha)r)}\cap\partial D$ and $\tilde{D}_r=D\setminus\overline{D_r}$. Then the last fact implies that $\mathscr{R}_u\cap B\subset\mathscr{R}_\alpha(E_u\cap\Delta^*)\cap\tilde{D}_r$.

We now fix $\varepsilon>0$ and take a number $\tau>0$ (which is independent of ε) sufficiently small so that

$$\tau \leqq \min\left\{\frac{1}{2c_2}, \frac{\beta-\alpha}{2c_2}\right\}.$$

We then have

$$\begin{aligned} B(X,\tau\delta^*(X)) &\subset \mathscr{R}_{(\alpha+\beta)/2}(E_u\cap\Delta^*) &&\text{for}\quad X\in\mathscr{R}_\alpha(E_u\cap\Delta^*),\\ B(X,\tau\delta^*(X))\cap\mathscr{R}_{(\alpha+\beta)/2}(E_u\cap\Delta^*) &= \varnothing &&\text{for}\quad X\in D\setminus\mathscr{R}_\beta(E_u\cap\Delta^*),\\ B(X,\tau\delta^*(X)) &\subset \tilde{D}_{r'} &&\text{for}\quad X\in\tilde{D}_r,\\ B(X,\tau\delta^*(X))\cap\tilde{D}_{r'} &= \varnothing &&\text{for}\quad X\in D_{r''},\\ B(X,\tau\delta^*(X)) &\subset \Omega_{\varepsilon'} &&\text{for}\quad X\in\Omega_\varepsilon,\\ B(X,\tau\delta^*(X))\cap\Omega_{\delta'} &= \varnothing &&\text{for}\quad X\in D\setminus\Omega_{\varepsilon''}, \end{aligned}$$

where the constants r', r'', ε' and ε'' are given by

$$r' = (1+c_2\tau)r, \qquad r'' = \frac{1+c_2\tau}{1-c_2\tau}\,r,$$

$$\varepsilon' = c_1(c_2^{-1}-\tau)\varepsilon, \quad \varepsilon'' = \frac{c_1(c_2^{-1}-\tau)}{c_2(c_1^{-1}+\tau)}\,\varepsilon.$$

Let $\varphi\in\mathscr{C}_0^\infty(\mathbf{R}^n)$ be such that $\varphi(X)=\varphi(|X|)$, $\operatorname{spt}(\varphi)\subset\{X\in\mathbf{R}^n\colon |X|<1\}$, and its integral over $\mathbf{R}^n$ is equal to 1. Here $\mathscr{C}_0^\infty(D)$ is the space of smooth functions compactly supported in D and $\operatorname{spt}(\varphi)$ is the support of the function φ. We now define the function Φ as

$$\Phi(X) = \int_W (\tau\delta^*(X))^{-n}\varphi\left(\frac{X-Y}{\tau\delta^*(X)}\right)dY, \quad X\in D,$$

where the subset W of D is given by

$$W = \mathscr{R}_{(\alpha+\beta)/2}(E_u\cap\Delta^*)\cap\tilde{D}_{r'}\cap\Omega_{\varepsilon'}.$$

The function Φ turns out to be a smoothing of the characteristic function of the set $\mathscr{R}_\alpha(E_u\cap\Delta^*)\cap\tilde{D}_r\cap\Omega_\varepsilon$ (which contains $\mathscr{R}_u\cap B\cap\Omega_\varepsilon$). Indeed, $\Phi\in\mathscr{C}_0^\infty(\mathbf{R}^n)$ satisfies the properties that $\Phi(X)=1$ if $X\in\mathscr{R}_\alpha(E_u\cap\Delta^*)\cap\tilde{D}_r\cap\Omega_\varepsilon$, and

$$\operatorname{spt}(\Phi)\subset\mathscr{R}_\beta(E_u\cap\Delta^*)\cap\tilde{D}_{r''}\cap\Omega_{\varepsilon''}\subset D.$$

Moreover, by using (2.4), a straightforward computation shows that

$$|\nabla\Phi(X)| \lesssim \frac{1}{\delta(X)}, \quad |\Delta\Phi(X)| \lesssim \frac{1}{\delta(X)^2}, \tag{2.5}$$

for all $X\in D$.

We next note that, by the implicit function theorem, if η is a regular value of δ^* (that is, $\nabla\delta^*(X)\neq 0$ for all $X\in D$ with $\delta^*(X)=\eta$), then $\partial\Omega_\eta=\{X\in D\colon \delta^*(X)=\eta\}$ is smooth. According to Sard's theorem (see [F]), almost all values of a smooth function are regular, so we are able to choose a number η with $0<\eta\leqq\varepsilon''$ such that Ω_η is a smooth subdomain of D.

Let $\{u_i\}$ be a non-increasing sequence of $\mathscr{C}^\infty$ subharmonic functions which converges to u (see [HK]). Moreover, it was essentially shown in Chapter 3 of [HK] that the sequence (or a subsequence of it, if necessary) $\{\mu_{u_i}\}$ of Riesz measures of u_i converges vaguely to the Riesz measure μ_u of u on D, i.e.,

$$\lim_{i\to\infty}\int_D f(X)\,d\mu_{u_i}(X)=\int_D f(X)\,d\mu_u(X)$$

for all $f\in\mathscr{C}_0(D)$. Since $u\leqq 1$ on $\mathscr{R}_\beta(E_u)$, by choosing i sufficiently large, we may assume that $u_i\leqq 2$ on $\mathscr{R}_\beta(E_u)$ for all i.

Set $W_0=\operatorname{spt}(\nabla\Phi)\cup\operatorname{spt}(\Delta\Phi)$. Apply Green's theorem on Ω_η for each u_i to obtain

$$\begin{aligned}\int_{\mathscr{R}_u\cap B\cap\Omega_\varepsilon} G(X)\,d\mu_{u_i}(X) &\leqq \int_{\Omega_\eta}\Phi(X)G(X)\,\Delta u_i(X)\,dX\\ &=\int_{\Omega_\eta}\Delta\big(\Phi(X)G(X)\big)u_i(X)\,dX+\{\text{zero boundary terms}\}\\ &\leqq\left|\int_{\Omega_\eta}\Phi(X)\Delta G(X)u_i(X)\,dX\right|+\int_{W_0}|\Delta\Phi(X)|\,G(X)u_i(X)\,dX\\ &\qquad+2\int_{W_0}|\nabla\Phi(X)|\,|\nabla G(X)|\,u_i(X)\,dX\\ &\lesssim\int_{W_0}\frac{G(X)}{\delta(X)^2}\,dX+\int_{W_0}\frac{|\nabla G(X)|}{\delta(X)}\,dX.\end{aligned}$$

In the above we have used $\Phi(O)=0$ and (2.5). Therefore, since μ_{u_i} converges to μ_u vaguely on D and $G(X)$ is bounded and continuous on the compact set $\overline{\mathscr{R}_u\cap B\cap\Omega_\varepsilon}$, it can easily be seen that

$$\int_{\mathscr{R}_u\cap B\cap\Omega_\varepsilon} G(X)\,d\mu_u(X)\lesssim\int_{W_0}\frac{G(X)}{\delta(X)^2}\,dX+\int_{W_0}\frac{|\nabla G(X)|}{\delta(X)}\,dX, \tag{2.6}$$

by taking the limit as $i\to\infty$ (see also the proof of Lemma 2 below).

To estimate the right hand side of the last inequality, let $\varkappa>0$ be a fixed number such that

$$\varkappa\leqq\min\left\{\frac{1}{2},\ \frac{2\alpha}{\alpha+3}\right\}. \tag{2.7}$$

Let $D=\bigcup_{k=1}^\infty I_k$ be a Whitney decomposition of D so that $\{I_k\}$ is a family of dyadic cubes in $\mathbf{R}^n$ with disjoint interiors and

$$\frac{\sqrt{n}}{\varkappa}\,l(I_k)\leqq\operatorname{dist}(I_k,\partial D)\leqq\frac{4\sqrt{n}}{\varkappa}\,l(I_k), \tag{2.8}$$

where $l(I_k)$ is the side length of the cube I_k. Let Z_k be the center of the cube I_k, $r_k=\frac{1}{2}\sqrt{n}\,l(I_k)$, $B_k=B(Z_k, r_k)$, and $B_k^*=B(Z_k, 2r_k)$. Let $\tilde{Z}_k\in\partial D$ be such that $|Z_k-\tilde{Z}_k|=\delta(Z_k)$, and $\Delta_k=\Delta(\tilde{Z}_k, r_k)$.

Let $J=\{k\in\mathbf{N}\colon I_k\cap W_0\neq\emptyset\}$. Then for each $k\in J$, we have $\delta(Z_k)\cong r_k$, and, by Harnack's inequality, $G(X)\cong G(Z_k)$ for $X\in B_k^*$. Thus, for the first term on the right hand side of (2.6), the Dahlberg–Jerison–Kenig Comparison Theorem gives that

$$\int_{B_k}\frac{G(X)}{\delta(X)^2}\,dX \cong r_k^{n-2}G(Z_k)\cong\omega(\Delta_k).$$

For the second term, it is well-known (see [M]) that, for the harmonic function $G(X)$ on B_k^*, we have

$$\int_{B_k}|\nabla G(X)|^2\,dX\lesssim\frac{1}{r_k^2}\int_{B_k^*}G(X)^2\,dX.$$

This inequality together with Schwarz's inequality gives

$$\begin{aligned}\int_{B_k}\frac{|\nabla G(X)|}{\delta(X)}&\lesssim\frac{|B_k|^{1/2}}{r_k}\Big(\int_{B_k}|\nabla G(X)|^2\,dX\Big)^{1/2}\\&\lesssim r_k^{(n-4)/2}\Big(\int_{B_k^*}G(X)^2\,dX\Big)^{1/2}\\&\cong r_k^{n-2}G(Z_k)\cong\omega(\Delta_k).\end{aligned}$$

We now claim the following: under the notation above, there exist constants c and C, which are independent of u, r and ε, such that

$$\sum_{k\in J}\chi_{\Delta_k}\leqq C\chi_{\Delta(Q_0,cr)}. \tag{2.9}$$

Assuming that (2.9) holds, it then follows from Fubini's theorem and the doubling property of ω that

$$\begin{aligned}\int_{\mathscr{R}_u\cap B\cap\Omega_\varepsilon}G(X)\,d\mu_u(X)&\lesssim\sum_{k\in J}\Big\{\int_{B_k}\frac{G(X)}{\delta(X)^2}\,dX+\int_{B_k}\frac{|\nabla G(X)|}{\delta(X)}\,dX\Big\}\\&\lesssim\int_{\partial D}\sum_{k\in J}\chi_{\Delta_k}(Q)\,d\omega(Q)\\&\lesssim\omega(\Delta(Q_0, cr))\lesssim\omega(\Delta).\end{aligned}$$

The lemma is therefore established, once we prove (2.9).

Proof of (2.9). We first notice that $W_0 \subset W_1 \cup W_2 \cup W_3$, where

$$W_1 = (\mathcal{R}_\beta(E_u \cap \Delta^*) \setminus \mathcal{R}_\alpha(E_u \cap \Delta^*)) \cap \operatorname{spt}(\Phi),$$

$$W_2 = ((\tilde{D}_{r''} \setminus \tilde{D}_r) \cap \operatorname{spt}(\Phi)) \setminus W_1,$$

$$W_3 = ((\Omega_{\varepsilon''} \setminus \Omega_\varepsilon) \cap \operatorname{spt}(\Phi)) \setminus (W_1 \cup W_2).$$

A close look at the definitions shows that there is a constant c such that $\Delta_k \subset \Delta(Q_0, cr)$ for all $k \in J$. Now, for each $Q \in \Delta(Q_0, cr)$, let $J_Q = \{k \in J:\ Q \in \Delta_k\}$. We observe that, if J'_Q is a subfamily of J_Q with the property that there are constants c' and c'' such that $c'\varrho \leqq \delta(Z_k) \leqq c''\varrho$ for all $k \in J'_Q$ and some constant $\varrho > 0$ (which may depend on Q), then

$$|Z_j - Z_k| \leqq (2+\varkappa)c''\varrho, \quad r_k \geqq \frac{\varkappa c'}{2+\varkappa}\varrho,$$

for all $j, k \in J'_Q$. This fact implies that the cardinality of J'_Q is not greater than some integer N, which depends only on the constants c', c'' and $\varkappa$, but is independent of Q and ϱ.

We now consider two cases:

Case I. If $Q \in E_u \cap \Delta(Q_0, cr)$, then, for $k \in J_Q$ and $X \in I_k$, we have by (2.8) and (2.7) that

$$\begin{aligned} |X-Q| &\leqq |X-Z_k| + |Z_k - \tilde{Z}_k| + |\tilde{Z}_k - Q| \\ &\leqq r_k + \delta(Z_k) + r_k \leqq 3r_k + \delta(X) \\ &\leqq \left(1 + \frac{3\varkappa}{2-\varkappa}\right)\delta(X) < (1+\alpha)\delta(X). \end{aligned}$$

This implies that $I_k \subset \Gamma_\alpha(Q) \subset \mathcal{R}_\alpha(E_u \cap \Delta^*)$, namely, I_k only intersects with W_2 and W_3. Thus, if $k \in J_Q$ then either $\delta(Z_k) \cong r$ or $\delta(Z_k) \cong \varepsilon$, and hence (2.9) holds at Q, by the observation above.

Case II. For $Q \in \Delta(Q_0, cr) \setminus (E_u \cap \Delta^*)$, let $\varrho(Q) = d(Q, E_u \cap \Delta^*)$. If $k \in J_Q$ so that $I_k \cap W_1 \neq \emptyset$, then there is $X \in \Gamma_\beta(P)$ for some $P \in E_u \cap \Delta^*$, and hence

$$\begin{aligned} \varrho(Q) &\leqq |Q-P| \leqq |Q - \tilde{Z}_k| + |\tilde{Z}_k - Z_k| + |Z_k - X| + |X-P| \\ &\leqq r_k + \delta(Z_k) + r_k + (1+\beta)\delta(X) \\ &\leqq \tfrac{1}{2}((1+\beta)(2+\varkappa) + 2(1+\varkappa))\delta(Z_k). \end{aligned}$$

On the other hand, since $E_u \cap \Delta^*$ is closed, there is a $P \in E_u \cap \Delta^*$ such that $|Q-P| =$

$\varrho(Q)$. We then take $X \in I_k \setminus \Gamma_\alpha(P)$ and find that

$$\begin{aligned}\delta(Z_k) &\leqq \delta(X)+r_k \leqq (1+\alpha)^{-1}|X-P|+r_k \\ &\leqq (1+\alpha)^{-1}(|X-Z_k|+|Z_k-\tilde{Z}_k|+|\tilde{Z}_k-Q|+|Q-P|)+r_k \\ &\leqq (1+\alpha)^{-1}\big(r_k+\delta(Z_k)+r_k+\varrho(Q)\big)+r_k \\ &\leqq (1+\alpha)^{-1}\varrho(Q)+\tfrac{1}{2}\big(2(1+\alpha)^{-1}(1+\varkappa)+\varkappa\big)\delta(Z_k).\end{aligned}$$

This yields that

$$\varrho(Q) \geqq \tfrac{1}{2}\big((1+\alpha)(2-\varkappa)-2(1+\varkappa)\big)\delta(Z_k).$$

Therefore, if $k \in J_Q$, then either $\delta(Z_k) \cong \varrho(Q)$ or $\delta(Z_k) \cong r$, and so (2.9) holds at Q. This concludes the proof of (2.9) and hence the lemma is proved.

Lemma 2. *There exists a constant C, which is independent of u, such that $\nu_u(D) \leqq C$.*

Proof. We first claim that, if $r>0$ is a fixed number, then

$$\int_{\mathscr{R}_u \cap D_r} G(X)\, d\mu_u(X) \leqq C, \tag{2.10}$$

for a constant C which is independent of u, where $D_r = \{X \in D\colon \delta(X) > r\}$ as in the last lemma.

This claim can be proved in the same way as the last one, but is somewhat simpler.

Let $\{I_k\}$ be the family of dyadic Whitney cubes of D, which satisfies (2.8) with

$$\varkappa \leqq \min\left\{\frac{2}{3}, \frac{2(\beta-\alpha)}{3(2\beta+3)}\right\}.$$

We denote by B_k and B_k^* the balls corresponding to I_k as in the last lemma, and r_k the radius of B_k.

Since $\overline{D_r}$ is compact, there are at most N cubes in the family which touch D_r. Let $J = \{k \in \mathbf{N}\colon I_k \cap \mathscr{R}_u \cap D_r \neq \emptyset\}$. Then $r \lesssim r_k$, $\mathscr{R}_u \cap \overline{D_r} \subset \bigcup_{k \in J} B_k$, and

$$\bigcup_{k \in J} B_k^* \subset \mathscr{R}_\beta(E_u) \subset \{X \in D\colon u(X) \leqq 1\}.$$

Now fix $k \in J$, and let $\varphi_k \in \mathscr{C}_0^\infty(\mathbf{R}^n)$ with the properties that $\varphi_k = 1$ on B_k, $\operatorname{spt}(\varphi_k) \subset B_k^*$, and

$$|\nabla \varphi_k| \leqq \frac{C_1}{r_k}, \quad |\Delta \varphi_k| \leqq \frac{C_2}{r_k^2},$$

with some constants C_1 and C_2 depending only on the dimension n. Let $\{u_i\}$ be a non-increasing sequence of $\mathscr{C}^\infty$ subharmonic functions as in the last lemma, so that $\{u_i\}$ converges to u and μ_{u_i} converges to μ_u vaguely on D. As before, we may assume that $u_i \leqq 2$ on B_k^* for all i.

It is well-known that G is integrable over D, furthermore, the argument in [GW] shows that its gradient ∇G is also integrable over D (as D is a bounded domain). Thus, for each fixed i and k we are able to apply Green's theorem to get that

$$\int_{B_k} G(X)\,d\mu_{u_i}(X) \leqq \int_{B_k^*} \varphi_k(X)\,G(X)\,\Delta u_i(X)\,dX$$

$$= \int_{B_k^*} \Delta\big(\varphi_k(X)\,G(X)\big)\,u_i(X)\,dX + \{\text{zero boundary terms}\}$$

$$\leqq \left|\int_{B_k^*} \Delta\varphi_k(X)\,G(X)\,u_i(X)\,dX\right| + \int_{B_k^*} |\Delta\varphi_k(X)|\,G(X)\,u_i(X)\,dX$$

$$+2\int_{B_k^*} |\nabla\varphi_k(X)|\,|\nabla G(X)|\,u_i(X)\,dX$$

$$\leqq \varphi_k(O)\,u_i(O) + \frac{C_2}{r_k^2}\int_D G(X)\,dX + \frac{C_1}{r_k}\int_D |\nabla G(X)|\,dX$$

$$\leqq 2 + \frac{C_2}{r^2} + \frac{C_1}{r} = C_0,$$

Unlike the last lemma, we cannot simply take the limit as $i \to \infty$, since $G(X)$ might not be a bounded continuous function on B_k. This difficulty can be overcome as follows. For each positive integer m, we let

$$G_m(X) = \phi_m(X)\min\{m, G(X)\}, \quad X \in D.$$

where ϕ_m is a function in $\mathscr{C}_0(\mathbf{R}^n)$ such that $0 \leqq \phi_m \leqq 1$, $\phi_m = 1$ on the ball concentric with B_k but having the radius of $(1-1/m)$ times the radius of B_k, and $\phi_m = 0$ outside B_k. Then, $G_m \in \mathscr{C}_0(D)$ and

$$\int_D G_m(X)\,d\mu_{u_i}(X) \leqq \int_{B_k} G(X)\,d\mu_{u_i}(X) \leqq C_0$$

for every m. As μ_{u_i} converges to μ_u vaguely on D, we have

$$\int_D G_m(X)\,d\mu_u(X) = \lim_{i\to\infty} \int_D G_m(X)\,d\mu_{u_i}(X) \leqq C_0.$$

Since the sequence $\{G_m\}$ of positive continuous functions on D is increasing to G

on B_k, by the monotone convergence theorem we have

$$\int_{B_k} G(X)\,d\mu_u(X) = \lim_{m\to\infty} \int_D G_m(X)\,d\mu_u(X) \leqq C_0.$$

(In fact, a much stronger result is true, for details see [H], p. 114.)

Finally, summing over $k\in J$, we obtain that

$$\int_{\mathcal{R}_u \cap D_r} G(X)\,d\mu_u(X) \leqq \sum_{k\in J} \int_{B_k} G(X)\,d\mu_u(X) \leqq NC_0 = C.$$

This proves the claim.

Now, we choose the number r to be such that $0<10r<\min\{r_0, \delta(O)\}$. We can then cover the set $D\setminus D_r$ by finitely many balls $\{B_j\colon 1\leqq j\leqq N\}$ of the form $B_j=B(Q_j, 2r)$ with $Q_j\in\partial D$. Therefore, it follows from (2.2) and (2.10) that

$$\begin{aligned} v_u(D) &= \int_{\mathcal{R}_u} G(X)\,d\mu_u(X) \\ &\leqq \sum_{j=1}^N v_u(B_j) + \int_{\mathcal{R}_u\cap D_r} G(X)\,d\mu_u(X) \\ &\lesssim \sum_{j=1}^N \omega(B_j\cap\partial D)+1 \lesssim 1. \end{aligned}$$

This completes the proof of the lemma.

3. Some results about BMO functions on ∂D

In order to prove the Theorem, we need to use BMO functions on ∂D. This section is devoted to generalizing some well-known results of BMO functions on $\mathbf{R}^n$ to special types of spaces of homogeneous type $(\partial D, d, \omega)$, where D is the NTA domain we are working with, d is the Euclidean distance in $\mathbf{R}^n$, and ω is the harmonic measure of D evaluated at a fixed point $O\in D$. (The argument will work for the general case.) For the definition of spaces of homogeneous type we refer to [JK] or [ST]. In this section, by a constant C depending on D we mean that the constant C depends on the triple $(\partial D, d, \omega)$.

The following covering lemma of Vitali-type is not difficult to prove on its own, but we refer to [ST] for a proof in the general setting of spaces of homogenous type.

Lemma A. *There exists a positive constant N, depending only on D, such that if $\mathcal{F}=\{\Delta(Q,r)\}$ is a family of surface balls of ∂D (with bounded radii) then there is a pairwise disjoint, countable subfamily $\{\Delta(Q_j, r_j)\}$ of $\mathcal{F}$ such that each surface ball $\Delta(Q,r)$ in $\mathcal{F}$ is contained in one of the surface balls $\Delta(Q_j, Nr_j)$.*

By using the last lemma, a well-known argument shows that the Hardy—Littlewood maximal function with respect to ω for $f \in L^1(\omega)$,

$$M_\omega f(Q) = \sup_{Q \in \Delta} \frac{1}{\omega(\Delta)} \int_\Delta |f(P)|\, d\omega(P), \quad Q \in \partial D,$$

is weak-$L^1(\omega)$.

We next recall some more notation. For $f \in L^1(\omega)$, let

$$\|f\|_* = \sup_\Delta \inf_{a \in \mathbf{R}} \frac{1}{\omega(\Delta)} \int_\Delta |f(Q) - a|\, d\omega(Q),$$

where Δ is a surface ball of ∂D, and set

$$\|f\|_{\mathrm{BMO}} = \|f\|_{L^1(\omega)} + \|f\|_*.$$

The John—Nirenberg class of functions on ∂D is defined by

$$\mathrm{BMO}(\omega) = \{f \in L^1(\omega) \colon \|f\|_{\mathrm{BMO}} < \infty\}.$$

It is easy to see that if $f \in \mathrm{BMO}(\omega)$, then

$$\|f\|_* \leqq \sup_\Delta \frac{1}{\omega(\Delta)} \int_\Delta |f(Q) - f_\Delta|\, d\omega(Q) \leqq 2\|f\|_*, \tag{3.1}$$

where

$$f_\Delta = \frac{1}{\omega(\Delta)} \int_\Delta f(Q)\, d\omega(Q)$$

is the average of f over Δ.

The most important result we need from the general theory of spaces of homogeneous type is the so-called John—Nirenberg inequality, which we state for BMO functions on ∂D. A more general statement and a proof of it can be found in [ST].

Theorem B. *There exist positive constants C and c, which depend only on D, such that for every $f \in \mathrm{BMO}(\omega)$, every surface ball Δ, and every $\lambda > 0$,*

$$\omega(\{Q \in \Delta \colon |f(Q) - f_\Delta| > \lambda\}) \leqq C\omega(\Delta) \exp\left(\frac{-c\lambda}{\|f\|_*}\right). \tag{3.2}$$

We note that the John—Nirenberg inequality (3.2) is also valid if the harmonic measure ω replaced by a measure $\sigma \in A_\infty(\omega)$, which can easily be seen from the second inequality of (1.4).

The next proposition is a consequence of the John—Nirenberg inequality. This result was observed in the case of half-spaces by Murai and Uchiyama [MU], and our proof is on the same line as theirs.

Proposition 1. *Let $\sigma \in A_\infty(\omega)$, then there exist positive constants C and c, independent in $A_\infty(\omega)$, such that if $f \in \mathrm{BMO}(\omega)$ with $\|f\|_{\mathrm{BMO}} \leqq 1$, then*

$$\sigma(\{Q \in \partial D\colon |f(Q)| > \gamma\}) \leqq Ce^{-c\gamma}\sigma(\{Q \in \partial D\colon |f(Q)| > 1\}) \tag{3.3}$$

for all $\gamma > 1$.

Proof. For $\gamma > 1$, let $F_\gamma = \{Q \in \partial D\colon |f(Q)| > \gamma\}$. Since $\|f\|_{L^1(\omega)} \leqq 1$, $\omega(F_3) \leqq \frac{1}{3}$. The weak-$L^1(\omega)$ estimate for the Hardy—Littlewood maximal function $M_\omega f$ of f implies that the set of points of density of F_3,

$$\tilde{F}_3 = \left\{Q \in F_3\colon \lim_{r \searrow 0} \frac{\omega(\Delta(Q, r) \cap F_3)}{\omega(\Delta(Q, r))} = 1\right\},$$

has full ω-measure in F_3, i.e., $\omega(F_3 \setminus \tilde{F}_3) = 0$. Thus, for each $Q \in \tilde{F}_3$ we have

$$0 < \sup\{r \in \mathbf{R}\colon \omega(\Delta(Q, r) \cap F_3) > \tfrac{1}{2}\omega(\Delta(Q, r))\} < \infty.$$

Let N be the constant in Lemma A. By a stopping time argument, for each $Q \in F_3$, we can choose a number $r_Q > 0$ so that

$$\omega(\Delta_Q \cap F_3) > \tfrac{1}{2}\omega(\Delta_Q), \tag{3.4}$$

$$\omega(\Delta_Q^* \cap F_3) \leqq \tfrac{1}{2}\omega(\Delta_Q^*), \tag{3.5}$$

where

$$\Delta_Q = \Delta(Q, r_Q), \quad \text{and} \quad \Delta_Q^* = \Delta(Q, Nr_Q).$$

Applying Lemma A to the covering $\{\Delta_Q\}_{Q \in \tilde{F}_3}$ of $\tilde{F}_3$, there are countably many surface balls $\Delta_j = \Delta_{Q_j}$, $j = 1, 2, \ldots$, in this covering, which are pairwise disjoint, such that $\tilde{F}_3 \subset \bigcup_{j=1}^\infty \Delta_j^*$, where $\Delta_j^* = \Delta_{Q_j}^*$, or in other words,

$$F_3 \subset \bigcup_{j=1}^\infty \Delta_j^*, \quad \omega\text{-a.e.}. \tag{3.6}$$

Now, (3.1) and (3.5) give that

$$|f_{\Delta_j^*}| = \frac{1}{\omega(\Delta_j^*)}\Big|\int_{\Delta_j^*} f(Q)\,d\omega(Q)\Big|$$

$$\leqq \frac{1}{\omega(\Delta_j^*)}\Big(\Big|\int_{\Delta_j^*\cap F_3} f(Q)\,d\omega(Q)\Big| + \int_{\Delta_j^*\cap(\partial D\setminus F_3)} |f(Q)|\,d\omega(Q)\Big)$$

$$\leqq \frac{1}{\omega(\Delta_j^*)}\int_{\Delta_j^*} |f(Q)-f_{\Delta_j^*}|\,d\omega(Q)$$

$$+|f_{\Delta_j^*}|\,\frac{\omega(\Delta_j^*\cap F_3)}{\omega(\Delta_j^*)}+3\,\frac{\omega(\Delta_j^*\cap(\partial D\setminus F_3))}{\omega(\Delta_j^*)}$$

$$\leqq 2\|f\|_* + \tfrac{1}{2}|f_{\Delta_j^*}|+3,$$

so that, by using $\|f\|_*\leqq 1$, we get $|f_{\Delta_j^*}|\leqq 10$. It follows that if $\gamma>10$ then

$$\{Q\in\Delta_j^*:\ |f(Q)|>\gamma\}\subset\{Q\in\Delta_j^*:\ |f(Q)-f_{\Delta_j^*}|>\gamma-10\}.$$

Hence by (3.6) and the John—Nirenberg inequality (3.2) with the measure $\sigma\in A_\infty(\omega)$, we obtain

$$\sigma(\{Q\in\partial D:\ |f(Q)|>\gamma\}) \leqq \sum_{j=1}^{\infty}\sigma(\{Q\in\Delta_j^*:\ |f(Q)|>\gamma\})$$

$$\leqq \sum_{j=1}^{\infty}\sigma(\{Q\in\Delta_j^*:\ |f(Q)-f_{\Delta_j^*}|>\gamma-10\})$$

$$\leqq Ce^{-c(\gamma-10)}\sum_{j=1}^{\infty}\sigma(\Delta_j^*)$$

$$\leqq Ce^{-c\gamma}\sum_{j=1}^{\infty}\sigma(\Delta_j)$$

$$\leqq Ce^{-c\gamma}\sum_{j=1}^{\infty}\sigma(\Delta_j\cap F_3)$$

$$\leqq Ce^{-c\gamma}\sigma(F_3),$$

where we have used the doubling property of σ, (3.4) and the fact that $\{\Delta_j\}$ are pairwise disjoint. To finish the proof, we choose a bigger constant C, if necessary, so that $Ce^{-8c}\geqq 1$.

As an analogue to the balayage of a Carleson measure on a half-space (see [G]), the following proposition links BMO functions on ∂D and Carleson measures on D. We shall use the following notation: For fixed Q, $P\in\partial D$ and $X\in D$, let

$$l_{Q,P}(X) = \min\{|X-Q|, |X-P|\},$$

and let $\tilde{X}_{Q,P}$ be one of the points of Q and P such that $|X-\tilde{X}_{Q,P}|=l_{Q,P}(X)$.

Proposition 2. *Let $K(X, Q)$ be a continuous function on $D\times\partial D$ which satisfies the following condition: There exist positive constants C_1, C_2 and ε such that*

$$\operatorname*{ess\,sup}_{X\in D}\int_{\partial D}|K(X, Q)|\,d\omega(Q) \leqq C_1, \tag{3.7}$$

and

$$|K(X, Q)-K(X, P)| \leqq \frac{C_2}{\omega(\Delta(\tilde{X}_{Q,P}, l_{Q,P}(X)))}\left(\frac{|Q-P|}{l_{Q,P}(X)}\right)^{\varepsilon}, \tag{3.8}$$

for all Q, $P\in\partial D$ and $X\in D$ with $l_{Q,P}(X)>|Q-P|$.

Assume that ν is a Carleson measure, then the function $K\nu$ on ∂D defined by

$$K\nu(Q) = \int_D K(X, Q)\,d\nu(X), \quad Q\in\partial D \tag{3.9}$$

is in $\mathrm{BMO}(\partial D)$ *with $\|K\nu\|_{\mathrm{BMO}}\leqq C\|\nu\|_{C}$ for some constant C which depends only on C_1, C_2, ε and D.*

Proof. To see that $K\nu\in L^1(\omega)$, we apply Fubini's theorem and condition (3.7) to obtain that

$$\int_{\partial D}|K\nu(Q)|\,d\omega(Q) \leqq \int_D\int_{\partial D}|K(X, Q)|\,d\omega(Q)\,d\nu(X) \lesssim \|\nu\|_{C},$$

since $\omega(\partial D)=1$.

Now let $\Delta=\Delta(P, r)$ be a fixed surface ball in ∂D, and let $B_j=B(P, 2^j r)$ and $\Delta_j=B_j\cap\partial D$, $j=0, 1, 2, \ldots$. Then Fubini's theorem and (3.7) again give that

$$\int_\Delta\left|\int_{B_1}K(X, Q)\,d\nu(X)\right|d\omega(Q) \lesssim \nu(B_1) \lesssim \|\nu\|_{C}\,\omega(\Delta),$$

where the doubling property of ω has been used. Next, for $X\in B_{j+1}\setminus B_j$, $j\geqq 1$, and $Q\in\Delta$, the condition (3.8) implies that

$$\int_\Delta\left|\int_{B_{j+1}\setminus B_j}(K(X, Q)-K(X, P))\,d\nu(X)\right|d\omega(Q)$$

$$\lesssim \frac{\nu(B_{j+1})}{2^{\varepsilon(j-1)}\omega(\Delta_{j-1})}\,\omega(\Delta) \lesssim \frac{\|\nu\|_{C}}{2^{\varepsilon(j-1)}}\,\omega(\Delta).$$

The last inequality is because $\omega(\Delta_{j+1}) \lesssim \omega(\Delta_{j-1})$, which follows from the doubling property of ω. Summing on j, we obtain that

$$\frac{1}{\omega(\Delta)} \int_{\Delta} |Kv(Q) - a_\Delta| \, d\omega(Q) \lesssim \|v\|_C,$$

where the number a_Δ is given by

$$a_\Delta = \int_{D \setminus B_1} K(X, P) \, dv(X).$$

Hence, $\|Kv\|_* \lesssim \|v\|_C$. The conclusion of the proposition follows from the above two estimates.

A glance at the proof shows that condition (3.7) can be replaced by

$$\int_{\partial D} |Kv(Q)| \, d\omega(Q) \leqq C_1 \|v\|_C, \tag{3.10}$$

for some positive constant C_1.

4. Proof of the Theorem

We first notice that the Dahlberg–Jerison–Kenig Comparison Theorem implies

$$\delta(X)^{2-n} \omega\big(\Delta(\tilde{X}, \delta(X))\big) \lesssim G(X) \tag{4.1}$$

for all $X \in D$. We shall use this inequality and the doubling property of ω and σ over and over again, and so we will feel free to use these facts without mentioning them.

We now put

$$\theta = \min\left\{2\alpha, \frac{\alpha+\beta}{2}\right\},$$

$$\psi(X) = \max\{0, 1 - |X|\}.$$

and define the kernel function K on $D \times \partial D$ by

$$K(X, Q) = \psi\left(\frac{X-Q}{\theta\delta(X)}\right) \frac{\delta(X)^{2-n}}{G(X)}, \quad X \in D, \quad Q \in \partial D. \tag{4.2}$$

Clearly, K is a continuous function on $D \times \partial D$. Now, for the given subharmonic function u, let v_u be the measure defined by (1.9). By the Main Lemma, v_u is a Carleson measure with $\|v_u\|_C \leqq C$ for some constant C independent of u. Let Kv_u be the function on ∂D defined by (3.9), i.e.,

$$Kv_u(Q) = \int_{\mathscr{R}_u} \psi\left(\frac{X-Q}{\theta\delta(X)}\right)\delta(X)^{2-n}\,d\mu_u(X), \quad Q\in\delta D. \tag{4.3}$$

We now claim that the kernel function K satisfies the conditions (3.10) and (3.8), and hence by Proposition 2, $Kv_u \in \mathrm{BMO}(\omega)$ with $\|Kv\|_{\mathrm{BMO}} \leqq \|v_u\|_C \leqq C$. We shall keep the notation used in Proposition 2 and denote

$$\tilde{\Gamma}_\theta(X) = \{Q\in\partial D\colon X\in\Gamma_\theta(Q)\}.$$

It is easy to see that $\tilde{\Gamma}_\theta(X) \subset \Delta(\tilde{X}, (2+\theta)\delta(X))$ for $X\in D$. Then, by (4.3) and Lemma 2, we have

$$\begin{aligned}
\int_{\partial D} |Kv_u(Q)|\,d\omega(Q) &= \int_{\mathscr{R}_u} \delta(X)^{2-n} \int_{\partial D} \psi\left(\frac{X-Q}{\theta\partial(X)}\right) d\omega(Q)\,d\mu_u(X) \\
&\leqq \int_{\mathscr{R}_u} \delta(X)^{2-n} \int_{\partial D} \chi_{\tilde{\Gamma}_\theta(X)}(Q)\,d\omega(Q)\,d\mu_u(X) \\
&\lesssim \int_{\mathscr{R}_u} \delta(X)^{2-n}\,\omega\big(\Delta(\tilde{X}, \delta(X))\big)\,d\mu_u(X) \\
&\lesssim \int_{\mathscr{R}_u} G(X)\,d\mu_u(X) \leqq \|v_u\|_C.
\end{aligned}$$

Hence the condition (3.10) holds.

To check the condition (3.8), let $X\in D$ and Q, $P\in\partial D$. We need to consider three cases:

Case I. $|X-Q|>\theta\delta(X)$ and $|X-P|>\theta\delta(X)$. In this case we have $K(X,Q)=K(X,P)=0$, so there is nothing to show.

Case II(a). $|X-Q|<\theta\delta(X)$ and $|X-P|>\theta\delta(X)$. In this case $l_{Q,P}(X)=|X-Q|$ and $\tilde{X}_{Q,P}=Q$. As $K(X,P)=0$, we have

$$\begin{aligned}
|K(X,Q)-K(X,P)| = K(X,Q) &= \left(1-\frac{|X-Q|}{\theta\delta(X)}\right)\frac{\delta(X)^{2-n}}{G(X)} \\
&\lesssim \frac{1}{\omega\big(\Delta(\tilde{X}, \delta(X))\big)}\,\frac{|X-P|-|X-Q|}{|X-Q|} \\
&\lesssim \frac{1}{\omega\big(\Delta(Q, |X-Q|)\big)}\,\frac{|Q-P|}{|X-Q|},
\end{aligned}$$

where we have used the fact that

$$\Delta(Q, |X-Q|) \subset \Delta(\tilde{X}, (1+2\theta)\delta(X)),$$

which follows easily from the triangle inequality.

Case II(b). $|X-Q|>\theta\delta(X)$ and $|X-P|<\theta\delta(X)$. This is similar to the last case.

Case III. $|X-Q|<\theta\delta(X)$, $|X-P|<\theta\delta(X)$. By interchanging the roles of Q and P, we may assume that $|X-Q|\leqq|X-P|$. In this case, $l_{Q,P}(X)=|X-Q|$ and $\tilde{X}_{Q,P}=Q$, and the same reason as in Case II(a) gives that

$$\begin{aligned} |K(X,Q)-K(X,P)| &= \left|\frac{|X-P|}{\theta\delta(X)}-\frac{|X-Q|}{\theta\delta(X)}\right|\frac{\delta(X)^{2-n}}{G(X)} \\ &\lesssim \frac{1}{\omega(\Delta(\tilde{X},\delta(X)))}\frac{|X-P|-|X-Q|}{|X-Q|} \\ &\lesssim \frac{1}{\omega(\Delta(Q,|X-Q|))}\frac{|Q-P|}{|X-Q|}. \end{aligned}$$

Therefore, condition (3.8) has been verified, and thus the claim is proved.

We next observe that, if $X\in D$ then

$$\psi\left(\frac{X-Q}{\theta\delta(X)}\right) \geqq 1-\frac{\alpha}{\theta} = \min\left\{\frac{1}{2}, \frac{\beta-\alpha}{\beta+\alpha}\right\} \quad \text{for} \quad Q\in\tilde{\Gamma}_\alpha(X),$$

and

$$\psi\left(\frac{X-Q}{\theta\delta(X)}\right) \leqq 1 \quad \text{for} \quad Q\in\partial D.$$

We deduce from this and (4.3) that

$$S_\alpha u(Q) \leqq \frac{\theta}{\theta-\alpha} Kv_u(Q) \quad \text{for} \quad Q\in E_u,$$

and

$$Kv_u(Q) \leqq S_{2\alpha}u(Q) \quad \text{for} \quad Q\in\partial D.$$

Thus, by Proposition 1, for $\sigma \in A_\infty(\omega)$,

$$\begin{aligned}
&\sigma(\{Q\in\partial D\colon S_\alpha u(Q) > \gamma,\ N_\beta u(Q) \leqq 1\}) \\
&= \sigma(\{Q\in E_u\colon S_\alpha u(Q) > \gamma\}) \\
&\leqq \sigma\left(\left\{Q\in\partial D\colon Kv_u(Q) > \frac{\theta-\alpha}{\theta}\gamma\right\}\right) \\
&\leqq Ce^{-c\gamma}\sigma(\{Q\in\partial D\colon Kv_n(Q) > 1\}) \\
&\leqq Ce^{-c\gamma}\sigma(\{Q\in\partial D\colon S_{2\alpha}u(Q) > 1\}).
\end{aligned}$$

This concludes the proof of the theorem.

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