Riemann's zeta-function and the divisor problem. II

Matti Jutila

1. Introduction

An interesting analogy between the divisor function $d(n)$ and the function $|\zeta(\frac{1}{2}+it)|^2$ was pointed out by F.V. Atkinson [1] fifty years ago. A celebrated result in this direction is Atkinson's formula [2] for the error term $E(T)$ in the relation

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^2\,dt = (\log(T/2\pi)+2\gamma-1)T+E(T),$$

where γ denotes Euler's constant. The most significant terms in this formula are—up to an oscillating sign—similar to those in Voronoi's formula for the error term $\Delta(x)$ in Dirichlet's divisor problem for the sum

$$\sum_{n\le x} d(n) = (\log x+2\gamma-1)x+\Delta(x).$$

More precisely, $E(T)$ is comparable with $2\pi\Delta(T/2\pi)$ in this sense. In [7] we showed that $E(T)$ should actually be compared with $2\pi\Delta^*(T/2\pi)$, where

$$\Delta^*(x) = -\Delta(x)+2\Delta(2x)-\tfrac{1}{2}\Delta(4x),$$

because the Voronoi formula for $2\pi\Delta^*(T/2\pi)$ is analogous to Atkinson's formula for $E(T)$ even as to the signs of the terms.

The function $\Delta^*(x)$ can be understood as the error in a certain divisor problem. Namely, it was observed by T. Meurman [10] that

$$\tfrac{1}{2}\sum_{n\le 4x} (-1)^n d(n) = (\log x+2\gamma-1)x+\Delta^*(x).$$

The duality between $d(n)$ and $|\zeta(\frac{1}{2}+it)|^2$ is another example of the correspondence between arithmetic and analysis, like the connection between primes and zeta-zeros. We made a local analysis of this phenomenon in [7], showing in particular that the conjecture

$$\Delta(x) \ll x^{1/4+\varepsilon} \tag{1.1}$$

would imply the estimates

$$\zeta(\tfrac{1}{2}+it) \ll t^{3/20+\varepsilon} \tag{1.2}$$

and

$$E(T) \ll T^{5/16+\varepsilon}. \tag{1.3}$$

A result of global nature, namely an estimate for the mean square of the function $E(t)-2\pi\Delta^*(t/2\pi)$, was established in [8] (or see [5], Theorem 15.6).

Our object in this paper is to give an approximate relation between smoothed variants of the functions $E(T)$ and $\Delta^*(x)$. Thus we are going to compare these functions themselves rather than the similarity of their behaviour, which was our point in [7].

Retaining the notations in [7], let

$$\Delta_0^*(x) = 2\pi\Delta^*(x^2/2\pi),$$

$$E_0(x) = E(x^2),$$

and for a parameter G with

$$T^{-3/10} \le G \le T^{-1/6}, \tag{1.4}$$

define the smoothed functions

$$\Delta_1^*(x) = G^{-1} \int_{-H}^{H} \Delta_0^*(x+u)e^{-(u/G)^2}\,du,$$

$$E_1(x) = G^{-1} \int_{-H}^{H} E_0(x+u)e^{-(u/G)^2}\,du,$$

where $H=GL$ with $L=\log T$. Note that if $x \approx T^{1/2}$, then in terms of the functions $\Delta^*(t/2\pi)$ or $E(t)$ for $t \approx T$, the smoothing is done over an interval whose length is of the order $GT^{1/2} \in [T^{1/5}, T^{1/3}]$. Now, the connection between the functions Δ_1^* and E_1 is as follows.

Theorem. *Let T be a large positive number and put $\tau = T^{1/2}$. Let G satisfy* (1.4). *Then*

$$E_1(\tau) = \int_{-u_0}^{u_1} \alpha(u)\Delta_1^*(\tau+u)\,du + O\big(G^{-11/2}T^{-5/4}L^{13/2}\big) + O\big(GT^{1/2}\big), \tag{1.5}$$

where

$$\alpha(u) = \begin{cases} aT^{1/4}|u|^{1/2}K_{1/3}(bT^{1/4}|u|^{3/2}) & \text{for } u<0, \\ cT^{1/4}u^{1/2}\left(J_{1/3}(bT^{1/4}u^{3/2}) + J_{-1/3}(bT^{1/4}u^{3/2})\right) & \text{for } u>0, \end{cases} \tag{1.6}$$

$u_0 = T^{-1/6}L$, and $u_1 = 20b^{-2}G^{-2}T^{-1/2}L^2$; here $K_{1/3}(x)$ and $J_{\pm 1/3}(x)$ are Bessel functions in the standard notation, and a, b, and c are certain numerical constants.

This is deduced by Fourier analysis from the Voronoi–Atkinson type approximate formulae for $\Delta_1^*(x)$ and $E_1(x)$ proved in [7].

To illustrate the scope of the relation (1.5), let us derive a conditional bound for $E(T)$ under the assumption that

$$\Delta(x) \ll x^{\alpha} \tag{1.7}$$

for all $x \geq 1$. Then clearly $\Delta_1^*(x) \ll x^{2\alpha}$. It is known that α must lie in the interval $\frac{1}{4} < \alpha < \frac{7}{22} + \varepsilon$; the lower bound is a classical result of Hardy, and the upper bound is due to H. Iwaniec and C. J. Mozzochi [6].

By using well-known properties of Bessel functions, it is easy to verify that

$$\int_{-u_0}^{u_1} |\alpha(u)|\,du \ll G^{-3/2}T^{-1/4}L^{3/2} \tag{1.8}$$

(see section 3 for details). Hence, by (1.5) and (1.7), we have

$$E_1(\tau) \ll G^{-3/2}T^{-1/4+\alpha}L^{3/2} + G^{-11/2}T^{-5/4}L^{13/2} + GT^{1/2}. \tag{1.9}$$

This implies an estimate for $E(T)$ because (see [7], p. 95, or [5], p. 477)

$$E_1\big((T-Y)^{1/2}\big) + O(YL) \leq \sqrt{\pi}E(T) \leq E_1\big((T+Y)^{1/2}\big) + O(YL)$$

with $Y = GT^{1/2}L^2$; in fact, the estimate (1.9) holds for $E(T)$ as well with the slight modification that the last term on the right is replaced by $GT^{1/2}L^3$. Choosing now

$$G = T^{(4\alpha-3)/10}L^{-3/5}$$

we obtain the following conditional estimate for $E(T)$.

Corollary. *Suppose that* (1.7) *holds for all* $x \geq 1$. *Then*

$$E(T) \ll T^{(1+2\alpha)/5} L^{12/5}. \tag{1.10}$$

In particular, if $\alpha = \frac{1}{4} + \varepsilon$ *is admissible for any fixed* $\varepsilon > 0$, *then*

$$E(T) \ll T^{3/10+\varepsilon}. \tag{1.11}$$

Remark 1. The constants a, b and c in (1.6) are $a = 2^{5/2} 3^{-1/2} \pi^{-1}$, $b = 2^{7/2} 3^{-1}$, and $c = 2^{5/2} 3^{-1}$.

Remark 2. The conditional estimate (1.11) improves (1.3) and implies (1.2). The best known unconditional exponent in (1.11) is $7/22 + \varepsilon$, due to D. R. Heath-Brown and M. N. Huxley [4].

Remark 3. An equation of the type (1.5) also holds if the roles of E_1 and Δ_1^* are interchanged. Then a hypothetical estimate for $E(T)$ implies a conditional estimate for $\Delta^*(x)$.

Remark 4. The conclusion (1.10) can be drawn even if (1.7) is assumed just in three short intervals. In particular, for $\alpha = \frac{1}{4} + \varepsilon$, our G is about $T^{-1/5}$, so the relevant values of u in (1.5) are $\ll T^{-1/10+\varepsilon}$, and therefore it suffices to assume (1.7) in three intervals of length $T^{2/5+\varepsilon}$ near $T/2\pi$, T/π, and $2T/\pi$.

Remark 5. In view of the analogy between $\Delta(x)$ and $E(T)$, one expects that the lower bound for the numbers α satisfying (1.7) should be the same as the lower bound for β in the estimation

$$E(T) \ll T^{\beta}. \tag{1.12}$$

Let us assume (1.7) and choose $G = T^{\alpha - 1/2}$. Then the error terms in (1.5) are $\ll T^{\alpha+\varepsilon}$, and the same estimate holds for the integral over $|u| \leq u_0$ as well. Therefore the validity of (1.12) for $\beta = \alpha + \varepsilon$ depends on the integral over $u_0 \leq u \leq u_1$ in (1.5). In this range, the function $\alpha(u)$ is oscillatory, and the oscillations cancel each other satisfactorily if the function $\Delta_1^*(\tau + u)$ is sufficiently stationary. But in the worst case the oscillations of the functions $\alpha(u)$ and $\Delta_1^*(\tau + u)$ may compensate each other to prevent cancellation. Thus the question whether (1.7) and (1.12) hold for the same numbers α and β depends ultimately on the local behaviour of the function $\Delta(x)$. This problem will be briefly discussed in the end of the paper.

2. An analytical lemma

Atkinson's formula for $E(T)$ involves functions of the type $\cos(f(T,n))$, where $f(T,n)$ can be written as a power series in $y=\sqrt{n}$, and cutting the series we end up with an approximation of the form $\cos(Ay^3+By+C)$, where $A>0$. On the other hand, similar functions with $A=0$ occur in the Voronoi formula for $\Delta^*(T/2\pi)$. Thus the term Ay^3 can be viewed as a perturbation. The following lemma expresses $\cos(Ay^3+By+C)$ in terms of functions of the form $\cos(\tilde{B}y+C)$.

Lemma. *Let $A>0$, B, and C be real numbers. Then, for all real y, we have*

$$\cos(Ay^3+By+C)=\int_{-\infty}^{\infty}\beta(u)\cos((B+u)y+C)\,du, \tag{2.1}$$

where

$$\beta(u)=\frac{1}{3\pi}A^{-1/2}|u|^{1/2}K_{1/3}\left(\frac{2|u|^{3/2}}{3\sqrt{3A}}\right) \quad \text{for } u<0 \tag{2.2}$$

and

$$\beta(u)=\frac{1}{3\sqrt{3}}A^{-1/2}u^{1/2}\left(J_{1/3}\left(\frac{2u^{3/2}}{3\sqrt{3A}}\right)+J_{-1/3}\left(\frac{2u^{3/2}}{3\sqrt{3A}}\right)\right) \quad \text{for } u>0. \tag{2.3}$$

Proof. The Fourier transform of the function e^{iAy^3} is

$$\begin{aligned}\beta(u)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i(Ay^3-uy)}\,dy\\&=\frac{1}{\pi}\int_0^{\infty}\cos(Ay^3-uy)\,dy.\end{aligned}$$

This is an *Airy integral* which can be written in terms of Bessel functions by the following formulae (see [3], p. 22, eq. (39) and (40), where the minus sign on the right of (40) should be deleted as a misprint): for $x>0$,

$$\int_0^{\infty}\cos(t^3+3tx)\,dt=(x/3)^{1/2}K_{1/3}(2x^{3/2}),$$

$$\int_0^{\infty}\cos(t^3-3tx)\,dt=(\pi/3)x^{1/2}\left(J_{1/3}(2x^{3/2})+J_{-1/3}(2x^{3/2})\right).$$

Thus $\beta(u)$ is given by (2.2)–(2.3), and by the Fourier inversion we have

$$\begin{aligned}\cos(Ay^3+By+C)&=\mathrm{Re}\{e^{i(Ay^3+By+C)}\}\\&=\mathrm{Re}\left\{e^{i(By+C)}\int_{-\infty}^{\infty}\beta(u)e^{iuy}\,du\right\}\\&=\int_{-\infty}^{\infty}\beta(u)\cos((B+u)y+C)\,du.\end{aligned}$$

3. Proof of the theorem

The proof of the theorem is based on an analysis of the following formulae (see [7], Lemma 2): for $|x-\tau|\le\frac{1}{2}\tau$, we have

(3.1)
$$\Delta_1^*(x)=(2\pi x^2)^{1/4}\sum_{n\le M}(-1)^n d(n)n^{-3/4}\exp(-2\pi nG^2)\cos(2\sqrt{2\pi n}\,x-\pi/4)+O(T^\varepsilon),$$

$$E_1(x)=(2\pi x^2)^{1/4}\sum_{n\le M}(-1)^n d(n)n^{-3/4}e(x^2,n)r(x,n)\cos(f(x^2,n))+O(T^\varepsilon), \tag{3.2}$$

where $M=G^{-2}L^2$, $e(x^2,n)=1+O(nT^{-1})$,

$$r(x,n)=\exp\{-4G^2\left(x\operatorname{arsinh}((\pi n/2)^{1/2}x^{-1})\right)^2\},$$

and

$$\begin{aligned} f(x^2,n)&=2x^2\operatorname{arsinh}\left((\pi n/2)^{1/2}x^{-1}\right)+(\pi^2n^2+2\pi nx^2)^{1/2}-\pi/4 \\ &=-\pi/4+2\sqrt{2\pi n}\,x+\frac{\sqrt{2}}{6}\pi^{3/2}n^{3/2}x^{-1}+O(n^{5/2}T^{-3/2}). \end{aligned} \tag{3.3}$$

We are going to apply (3.2) for $x=\tau$ and (3.1) for $x=\tau+u$ with $u\in[-u_0,u_1]$. Then $u\ll G^{-2}\tau^{-1}L^2$, and the factor $(2\pi x^2)^{1/4}$ on the right of (3.1) can be replaced by $(2\pi)^{1/4}\tau^{1/2}$ with an error $\ll G^{-5/2}T^{-3/4}L^{7/2}\le L^{7/2}$. Also, (3.2) remains valid if we replace $e(x^2,n)$ by 1 and $r(x,n)$ by $\exp(-2\pi nG^2)$. When (3.3) is substituted into (3.2), the error term can be omitted with an error $\ll G^{-11/2}T^{-5/4}L^{13/2}$. Therefore

$$\begin{aligned} E_1(\tau)=(2\pi)^{1/4}\tau^{1/2}&\sum_{n\le M}(-1)^n d(n)n^{-3/4}\exp(-2\pi nG^2) \\ \times\cos&\left(2\sqrt{2\pi n}\,\tau+\frac{\sqrt{2}}{6}\pi^{3/2}\tau^{-1}n^{3/2}-\pi/4\right)+O(T^\varepsilon)+O(G^{-11/2}T^{-5/4}L^{13/2}). \end{aligned} \tag{3.4}$$

By the lemma, we may write

(3.5)
$$\cos\left(2\sqrt{2\pi n}\,\tau+\frac{\sqrt{2}}{6}\pi^{3/2}\tau^{-1}n^{3/2}-\pi/4\right)=\int_{-\infty}^{\infty}\alpha(u)\cos\left(2\sqrt{2\pi n}\,(\tau+u)-\pi/4\right)du,$$

where $\alpha(u)$ is as in (1.6). Here the range of integration is split up into three parts: $(-\infty,-u_0]$, $[-u_0,u_1]$, and $[u_1,\infty)$, and the expression on the right of (3.4) is decomposed accordingly. The middle range gives

$$\int_{-u_0}^{u_1}\alpha(u)\Delta_1^*(\tau+u)\,du+O\left(T^\varepsilon\int_{-u_0}^{u_1}|\alpha(u)|\,du\right) \tag{3.6}$$

by (3.1) and the preceding discussion. The contribution of the infinite ranges is estimated as the error terms

$$\ll T^{1/4} \sum_{n\le M} d(n)n^{-3/4} \int_{-\infty}^{-u_0} |\alpha(u)|\, du \tag{3.7}$$

and

$$\ll T^{1/4} \sum_{n\le M} d(n)n^{-3/4} \left| \int_{u_1}^{\infty} \alpha(u) \cos(2\sqrt{2\pi n}\,(\tau+u)-\pi/4)\, du \right|. \tag{3.8}$$

The main term in (3.6) equals the main term in (1.5). Thus, to complete the proof of the theorem, we have to estimate the error term in (3.6) and the sums (3.7) and (3.8). For this, we need a few properties of the function $\alpha(u)$.

By the definitions of Bessel functions as power series, the functions

$$x^{1/2} K_{1/3}(x^{3/2}) \quad \text{and} \quad x^{1/2} J_{\pm 1/3}(x^{3/2})$$

are bounded for $0<x\le 1$. In fact, they are bounded for $x>1$ as well, by the asymptotic formulae (see [3], p. 24 and 85):

$$K_\nu(x) \sim (\pi/2x)^{1/2} e^{-x}$$

and

$$J_\nu(x) = (2/\pi x)^{1/2} \cos(x-\pi\nu/2-\pi/4) + O(x^{-3/2})$$

for fixed ν and $x\ge 1$. Therefore

$$\alpha(u) \ll T^{1/6} \quad \text{for all } u\ne 0, \tag{3.9}$$

$$\alpha(u) \ll T^{1/6} \exp(-bT^{1/4}|u|^{3/2}) \quad \text{for } u<0, \tag{3.10}$$

and

$$\begin{aligned}\alpha(u) = T^{1/8}u^{-1/4}\big(d\exp(ibT^{1/4}u^{3/2}) + \bar{d}\exp(-ibT^{1/4}u^{3/2})\big) \\ + O(T^{-1/8}u^{-7/4}) \quad \text{for } u\ge T^{-1/6},\end{aligned} \tag{3.11}$$

where d is a certain complex constant.

The estimate (1.8) now follows immediately form (3.9)–(3.11), and therefore the error term in (3.6) is $\ll G^{-3/2}T^{-1/4+\varepsilon}$, which can be absorbed into the error terms in (1.5). Further, the terms in (3.7) are negligibly small by (3.10). Consider finally the sum (3.8). The contribution of the error term in (3.11) to this sum is

$$\ll M^{1/4}T^{1/8}u_1^{-3/4}L \ll GT^{1/2}.$$

The main terms in (3.11) give rise to integrals

$$T^{1/8}\int_{u_1}^{\infty} u^{-1/4}\exp\big(i\big(\pm bT^{1/4}u^{3/2}\pm 2\sqrt{2\pi n}\,u\big)\big)\,du. \tag{3.12}$$

Since

$$\tfrac{3}{2}bT^{1/4}u^{1/2} > 2\sqrt{2\pi M} \quad \text{for } u \geq u_1,$$

this integral has no saddle point in the interval of integration (the parameter u_1 was specified with this condition in mind), and by the "first derivative test" (see [5], Lemma 2.1) we obtain the bound

$$\ll T^{1/8}u_1^{-1/4}(T^{1/4}u_1^{1/2})^{-1} \ll G^{3/2}T^{1/4}L^{-3/2}.$$

for (3.12). Therefore the sum (3.8) is $\ll GT^{1/2}$, and the proof of the theorem is complete.

4. Concluding remarks

The error term $O(GT^{1/2})$ in (1.5) can be improved if the main term is replaced by a weighted integral. However, this sharpening does not effect the estimation of $E(T)$ because a similar error term $O(GT^{1/2}L^3)$ appears anyway when $E(T)$ is approximated by the function E_1.

In the proof of the corollary, we estimated the integral in the theorem simply by absolute values. For a more careful estimation, let us write this integral as

$$\Delta_1^*(\tau)\int_{-u_0}^{u_1}\alpha(u)\,du + \int_{-u_0}^{u_1}(\Delta_1^*(\tau+u)-\Delta_1^*(\tau))\alpha(u)\,du. \tag{4.1}$$

It is easy to see, by (3.9)–(3.11), that

$$\int_{-u_0}^{u_1}\alpha(u)\,du \ll 1.$$

Thus the first term in (4.1) is of the same order as $\Delta_1^*(\tau)$. If the difference of Δ_1^* in the second term could be estimated more precisely than the function Δ_1^* itself, then the corollary could be sharpened.

Large values of $|\zeta(\frac{1}{2}+it)|$ can be studied in terms of the change of $E(t)$ by appealing to the inequality

$$\left|\zeta\left(\tfrac{1}{2}+iT\right)\right|^2 \ll \left(E(T+X)-E(T-X)\right)L^{-1}+XL,$$

valid for $X \geq L^2$ (this follows from Lemma 7.1 in [5]). Let G be as in the definition of E_1, and let X be of the order $GT^{1/2}L$. By averaging the above inequality as in the definition of E_1, we obtain

$$\left|\zeta\left(\tfrac{1}{2}+iT\right)\right|^2 \ll \left(E_1(\tau+GL)-E_1(\tau-GL)\right)L^{-1}+GT^{1/2}L^2. \tag{4.2}$$

Here the difference E_1 can be written (approximately) in terms of the difference of Δ_1^* by (1.5). Let us suppose that

$$\Delta(x+y)-\Delta(x) \ll y^{1/2}x^{\varepsilon} \quad \text{for } y \geq 1; \tag{4.3}$$

this very strong conjecture can be motivated by mean value considerations. Then

$$\Delta_1^*(\tau+u+GL)-\Delta_1^*(\tau+u-GL) \ll G^{1/2}T^{1/4+\varepsilon} \quad \text{for } -u_0 \leq u \leq u_1.$$

Together with (4.2), (1.5), and (1.8), this gives

$$\left|\zeta\left(\tfrac{1}{2}+it\right)\right|^2 \ll \left(G^{-1}+G^{-11/2}T^{-5/4}+GT^{1/2}\right)T^{\varepsilon}.$$

Choosing $G=T^{-1/4}$, we find that the conjecture (4.3) implies the estimate

$$\zeta\left(\tfrac{1}{2}+it\right) \ll t^{1/8+\varepsilon};$$

the same conclusion was made in [7] in a different way.

Another indication of the role played by the divisor problem in the theory of the zeta function is given by the approximate functional equation for $\zeta^2(\frac{1}{2}+it)$ whose error term in the "symmetric" case can be related to $\Delta(t/2\pi)$. This fact was discovered by Y. Motohashi [11] (we gave subsequently a different proof in [9]).

References

1. Atkinson, F. V., The mean value of the zeta-function on the critical line, *Quart. J. Math. Oxford* **10** (1939), 122–128.
2. Atkinson, F. V., The mean-value of the Riemann zeta function, *Acta Math.* **81** (1949), 353–376.

3. ERDÉLYI, A., OBERHETTINGER, F. and TRICOMI, F. G., *Higher Transcendental Functions*, vol. II, McGraw–Hill, New York, 1953.
4. HEATH-BROWN, D. R. and HUXLEY, M. N., Exponential sums with a difference, *Proc. London Math. Soc. (3)* **61** (1990), 227-250.
5. IVIĆ, A., *The Riemann Zeta-function*, John Wiley & Sons, New York, 1985.
6. IWANIEC, H. and MOZZOCHI, C. J., On the divisor and circle problems, *J. Number Theory* **29** (1988), 1–24.
7. JUTILA, M., Riemann's zeta-function and the divisor problem, *Ark. Mat.* **21** (1983), 75–96.
8. JUTILA, M., On a formula of Atkinson, in *Coll. Math. Soc. János Bolyai 34. Topics in Classical Number Theory, Budapest 1981*, pp. 807–823, North-Holland, Amsterdam, 1984.
9. JUTILA, M., On the approximate functional equation for $\zeta^2(s)$ and other Dirichlet series, *Quart. J. Math. Oxford (2)* **37** (1986), 193–209.
10. MEURMAN, T., A generalization of Atkinson's formula to L-functions, *Acta Arith.* **47** (1986), 351–370.
11. MOTOHASHI, Y., A note on the approximate functional equation for $\zeta^2(s)$. II, *Proc. Japan Acad. Sci. Ser. A Math. Sci.* **59** (1983), 469–472; III, *ibid.* **62** (1986), 410-412.

Received September 20, 1990

Matti Jutila
Department of Mathematics
University of Turku
SF-20500 Turku
Finland