# Wavelets and paracommutators

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## 1. Introduction

Let $P$ denote the collection of all dyadic cubes in $\mathbf{R}^d$, $E=\{0,1\}^d\setminus\{0,0\}$ and $\Lambda=P\times E$. The construction of the wavelet bases $\{\psi_\lambda\}_{\lambda\in\Lambda}$ on $L^2(\mathbf{R}^d)$ is a celebrated result in mathematical analysis. Y. Meyer [7] and G. David [3] have used the bases $\{\psi_\lambda\}$ to study the boundedness of the Calderón–Zygmund operators, and have successfully simplified the proofs of the "$T1$ Theorem" and the "$Tb$ Theorem". It is well known that operators of the form

$$Tf(x)=\int_{R^d} K(x,y)f(y)\,dy$$

can be compact; for example $T\in S_2$, i.e. $T$ is Hilbert–Schmidt, if and only if $K\in L^2(\mathbf{R}^d\times\mathbf{R}^d)$ . In this paper we will study the compactness and the Schatten–von Neumann properties by wavelet bases.

Precisely speaking, we consider the bilinear form $T(f,g)=\langle T(f),g\rangle$ on $\mathcal{D}\times\mathcal{D}$ with the distributional kernel $K(x,y)$. Let $\alpha_{\lambda,\lambda'}=T(\psi_\lambda,\psi_{\lambda'})$, for $\lambda,\lambda'\in\Lambda$. Then the bilinear form $T$ is determined by the infinite matrix $\{\alpha_{\lambda,\lambda'}\}$. In other words, $\sum_\lambda\sum_{\lambda'}\alpha_{\lambda,\lambda'}\psi_\lambda\otimes\psi_{\lambda'}$ gives a decomposition for $T$, called the standard decomposition. We will also consider non-standard decompositions for $T$. And we will find out the conditions for $T\in S_p$ (the Schatten–von Neumann class). Then we will consider an important example: paracommutators , which are defined and studied systematically by Janson and Peetre [5]. Supplementary results are given by Peng [9], [10], and Peng–Qian [11].

In Section 2 we will give some notations and definitions, and in Section 3 we will give some lemmas for the $S_p$ estimates. In Section 4 we will revisit the paracommutator, and will give simplified proofs for most results on paracommutators by wavelet basis expansions.

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## 2. Preliminaries

### 2.1. Multidimensional wavelets (see David [3])

*Definition* 2.1. A multiscale analysis (MSA) of $L^2(\mathbf{R}^d)$ is an increasing sequence $V_j$, $j\in\mathbf{Z}$, of closed subspaces of $L^2(\mathbf{R}^d)$, with the properties:

(1) $\bigcap_{j\in\mathbf{Z}} V_j=\{0\}$ and $\bigcup_{j\in\mathbf{Z}} V_j$ is dense;

(2) $f(x)\in V_j \Leftrightarrow f(2x)\in V_{j+1}$;

(3) $f(x)\in V_0 \Leftrightarrow f(x-k)\in V_0$, for each $k\in\mathbf{Z}^d$;

(4) There is a function $g\in V_0$ such that the sequence of functions $g(x-\underline{k})$, $\underline{k}\in\mathbf{Z}^d$, is a Riesz basis of $V_0$.

*Definition* 2.2. A MSA is "$\gamma$-regular" (for some $\gamma\in N$) if one can choose $g$ in Definition 2.1 such that

$$|\partial^\alpha g(x)|\le C_{M,\alpha}(1+|x|)^{-M} \quad \text{for all } M,\ |\alpha|\le\gamma.$$

**Theorem 2.1.** *Given a MSA of $L^2(\mathbf{R}^d)$, of regularity $\gamma$, one can find a function $\varphi=\psi^0(x)$ and $2^d-1$ functions $\psi^\varepsilon(x)$, $\varepsilon\in E$, such that $|\partial^\alpha\psi^\varepsilon(x)|\le C_{M,\alpha}(1+|x|)^{-M}$ for $|\alpha|\le\gamma$, all $M\ge 0$, and such that if $\psi_\lambda(x)=\psi_Q^\varepsilon(x)=2^{jd/2}\psi^\varepsilon(2^jx-\underline{k})$, for $\lambda=(Q,\varepsilon)\in\Lambda$, $Q\in P$ and $Q=\prod_{i=1}^d[k_i2^{-j},(k_i+1)2^{-j}]$ then $\{\varphi_Q\}_{\mathrm{length}(Q)=2^{-j}}$ is an orthogonal basis of $V_j$ and $\{\psi_\lambda\}_{\lambda\in\Lambda}$ is an orthogonal basis of $L^2(\mathbf{R}^d)$.*

The functions $\{\psi^\varepsilon\}$ are called wavelets and $\{\psi_\lambda^\varepsilon\}_{\lambda\in\Lambda}$ is called a wavelet basis. An example of an orthogonal wavelet basis of regularity $\gamma=0$ is the Haar basis. An example of an orthogonal wavelet basis of regularity $\gamma=\infty$ is the Meyer basis $\{\psi_\lambda\}_{\lambda\in\Lambda}$, see David [3]. It is not only an orthogonal basis for $L^2(\mathbf{R}^d)$, but also an unconditional basis for many functionspaces, so it is called a universal unconditional basis.

### 2.2. Frames

Let $H$ be a Hilbert space, $\{e_\lambda\}_{\lambda\in\Lambda}$ is a frame on $H$ if $T(\{\alpha_\lambda\})=\sum\alpha_\lambda e_\lambda$ is a bounded operator from $l^2(\Lambda)$ to $H$ and $T$ is onto. If $\{e_\lambda\}_{\lambda\in\Lambda}$ is a frame on $H$, then for all $x\in H$, $x=\sum\langle L^{-1}(x),e_\lambda\rangle e_\lambda$, where $L(x)=\sum\langle x,e_\lambda\rangle e_\lambda$ and $\|x\|^2\approx\sum|\langle x,e_\lambda\rangle|^2$.

Let us consider an important example—a so called smooth frame.

Let $\widehat{\varphi}\in C_0^\infty(\mathbf{R}^d)$ such that $\operatorname{supp}\widehat{\varphi}\subset\{\xi:|\xi|\le\frac12\}$, $|\widehat{\varphi}(\xi)|\ge C>0$ on $\{\xi:|\xi|\le\frac14\}$, and $\widehat{\psi}(\xi)=\widehat{\varphi}(\xi)-\widehat{\varphi}(2\xi)$ such that $\sum_{j=-\infty}^{\infty}|\widehat{\psi}(\xi/2^j)|=1$ for $\xi\ne0$. Now we denote $\Lambda=P$ and $\lambda=Q\in P$. Then $\psi_\lambda=\psi_Q=\psi_{j,\underline{k}}(x)=2^{jd/2}\psi(2^jx-\underline{k})$ form a frame (called a smooth frame) on $L^2(\mathbf{R}^d)$ by the Plancherel and Pólya Theorem. And for any

$f \in L^2(\mathbf{R}^d)$, we have $f = \sum_{\lambda\in\Lambda} \langle f, \psi_\lambda\rangle \psi_\lambda$, see Frazier and Jawerth [4]. We will show that this frame is universal for many function spaces.

### 2.3. Function spaces and sequence spaces

By the orthogonal wavelet basis in 2.1 or the smooth frame in 2.2, we can give a realization of the isomorphism between $L^2(\mathbf{R}^d)$ and $l^2(\Lambda)$.

For the wavelet basis $\{\psi_\lambda\}_{\lambda\in\Lambda}$, where $\Lambda = P\times E$, we denote $|\lambda| = \text{length}(Q)$ for $\lambda = (Q, \varepsilon) \in P\times E$, and we have two kinds of projections $P_j$ and $Q_j$,

$$P_j\colon L^2(\mathbf{R}^d) \to V_j, \quad P_j(f) = \sum_{|\lambda|>2^{-j}} \langle f, \psi_\lambda\rangle\psi_\lambda = \sum_{|\lambda|=2^{-j}} \langle f, \varphi_\lambda\rangle\varphi_\lambda,$$

$$Q_j\colon L^2(\mathbf{R}^d) \to W_j = V_{j+1}\ominus V_j, \quad Q_j(f) = \sum_{|\lambda|=2^{-j}} \langle f, \psi_\lambda\rangle\psi_\lambda,$$

and

$$P_j \uparrow 1\,(j\to+\infty), \quad P_j\downarrow 0\,(j\to-\infty),$$

$$Q_j = P_{j+1} - P_j, \quad \sum_{-\infty}^{+\infty} Q_j = 1.$$

For the smooth frame given in 2.2, where we denote $\widehat{\varphi}_j(\xi) = \widehat{\varphi}(2^{-j}\xi)$ and $\widehat{\psi}_j(\xi) = \widehat{\psi}(2^{-j}\xi)$, we have also the two kinds of operators $P_j$ and $Q_j$:

$$\widehat{P_j f}(\xi) = \widehat{\varphi}_j(\xi)\hat{f}(\xi),$$

$$\widehat{Q_j f}(\xi) = \widehat{\psi}_j(\xi)\hat{f}(\xi),$$

moreover,

$$P_j \uparrow 1\,(j\to+\infty), \quad P_j\downarrow 0\,(j\to-\infty),$$

$$Q_j = P_{j+1} - P_j, \quad \sum_{-\infty}^{+\infty} Q_j = 1.$$

Now we introduce some sequence spaces:

$$l^p(\mathbf{Z}^d) = \left\{ (\alpha_{\underline{k}})_{\underline{k}\in\mathbf{Z}^d} : \sum_{\underline{k}\in\mathbf{Z}^d} |\alpha_{\underline{k}}|^p < +\infty \right\} \quad \text{for } 0<p<\infty$$

(modification if $p=\infty$).

$$l^p(\underline{k}_1)l^q(\underline{k}_2)=\left\{(\alpha_{\underline{k}_1,\underline{k}_2})_{\underline{k}_1,\underline{k}_2\in\mathbf{Z}^d}:\sum_{\underline{k}_1\in\mathbf{Z}^d}\left[\sum_{\underline{k}_2\in\mathbf{Z}^d}|\alpha_{\underline{k}_1,\underline{k}_2}|^q\right]^{p/q}<+\infty\right\}$$

for $0<p$, $q<\infty$ (modification if $p=\infty$ or $q=\infty$).

Let $\Lambda=P\times E$ or $\Lambda=P$, we introduce:

$$l^p(\Lambda)=\left\{(\alpha_\lambda)_{\lambda\in\Lambda}:\sum|\alpha_\lambda|^p<\infty\right\}$$

for $0<p<\infty$ (modification if $p=\infty$).

$$l_p^{sq}(\Lambda)=\left\{(\alpha_\lambda)_{\lambda\in\Lambda}:\sum_{j=-\infty}^{\infty}2^{jsq}\left(\sum_{|\lambda|=2^j}|\alpha_\lambda|^p\right)^{q/p}<+\infty\right\}$$

for $s\in R$, $0<p$, $q<\infty$ (modification if $p=\infty$ or $q=\infty$).

$$l_p^q(\Lambda)=l_p^{0,q}(\Lambda),\quad l^p(\Lambda)=l_p^p(\Lambda).$$

$$\mathrm{BMO}(\Lambda)=\mathrm{QCM}_d=\left\{\{\alpha_\lambda\}_{\lambda\in\Lambda}:\sup_R\frac{1}{|R|}\sum_{Q\subset R:\lambda=(Q,\varepsilon)\text{ or }Q}|\alpha_\lambda|^2|Q|<+\infty\right\}$$

(for $\mathrm{QCM}_d$, see Rochberg and Semmes [12]).

$$l^p(\lambda)l^q(\lambda')=\left\{(\alpha_{\lambda,\lambda'})_{\lambda,\lambda'\in\Lambda}:\sum_\lambda\left[\sum_{\lambda'}|\alpha_{\lambda,\lambda'}|^q\right]^{p/q}<\infty\right\}$$

for $0<p$, $q<\infty$ (modification if $p=\infty$ or $q=\infty$).

Now let us show that both the Meyer wavelet basis $\{\psi_\lambda\}$ and the smooth frame given in 2.2 are universal unconditional bases for $\mathrm{BMO}(\mathbf{R}^d)$ and Besov spaces $B_p^{s,q}(\mathbf{R}^d)$.

First we give two lemmas, which will be used again in Section 4 below, the proof of them can be found in Triebel [15].

**Lemma 2.1.** *Let $\Omega$ be a compact subset of $\mathbf{R}^d$. Then for $0<r<p<\infty$, there exist $C_1$ and $C_2$ such that*

$$\sup_{z\in\mathbf{R}^d}\frac{|\nabla\varphi(x-z)|}{1+|z|^{d/r}}\le C_1\sup_{z\in\mathbf{R}^d}\frac{|\varphi(x-z)|}{1+|z|^{d/r}}\le C_2M_r(\varphi)(x)$$

*holds for all $\varphi\in L_\Omega^p=\{\varphi\in L^p:\operatorname{supp}\widehat{\varphi}\subset\Omega\}$, where $M_r(\varphi)(x)=\left[M|\varphi|^r(x)\right]^{1/r}$, and $M$ is the Hardy–Littlewood maximal operator.*

**Lemma 2.2.** (The Plancherel and Pólya Theorem) *Let* $0<p<\infty$, $a>0$. *For any* $a'>a$, *there exist* $C_1$ *and* $C_2$ *such that*

$$C_1\Big(\sum_{\underline{k}\in\mathbf{Z}^d}|\varphi(\underline{k}/a')|^p\Big)^{1/p}\le\|\varphi\|_p\le C_2\Big(\sum_{\underline{k}\in\mathbf{Z}^d}|\varphi(\underline{k}/a')|^p\Big)^{1/p}$$

*holds for all* $\varphi\in\{\varphi\in S':\operatorname{supp}\widehat{\varphi}\subset B(0,a)\}$

Let $\{\psi_\lambda\}_{\lambda\in\Lambda}$ denote the Meyer wavelet basis ($\Lambda=P\times E$) or the smooth frame given in 2.2 ($\Lambda=P$), we have

$$f\in\mathrm{BMO}(\mathbf{R}^d)\quad\text{if and only if}\quad\sup_{R\in P}\frac{1}{|R|}\sum_{\{Q\subset R:\lambda=(Q,\varepsilon)\text{ or }Q\}}|\langle f,\psi_\lambda\rangle|\,|Q|<+\infty,$$

i.e.

$$\mathrm{BMO}(\mathbf{R}^d)\longleftrightarrow\mathrm{BMO}(\Lambda),\quad\text{isomorphism.}$$

$$f\in B_p^{s,q}(\mathbf{R}^d)\quad\text{if and only if}\quad(\langle f,\psi_\lambda\rangle)_{\lambda\in\Lambda}\in l_p^{s,q}(\Lambda)$$

i.e.

$$B_p^{s,q}(\mathbf{R}^d)\longleftrightarrow l_p^{s,q}(\Lambda)\longleftrightarrow l_p^q(\Lambda),\quad\text{isomorphisms.}$$

### 2.4. The Schatten–von Neumann class $S_p$

Let $T$ be a bounded operator from one Hilbert space $H_1$ to another Hilbert space $H_2$. If $\mathcal{K}_n$ denotes the set of the operators of rank $\le n$, then the singular number $s_n=s_n(T)$ is defined by

$$s_n=\inf\{\|T-T_n\|:T_n\in\mathcal{K}_n\}$$

If $T$ is a compact operator, then $(T^*T)^{1/2}$ has its eigenvalues

$$s_0\ge s_1\ge\ldots\ge s_n\ge\ldots,\ s_n\to0.$$

The Schatten–von Neumann class $S_p$ is the set

$$S_p=\Big\{T:\Big(\sum s_n^p\Big)^{1/p}<\infty\Big\}.$$

We denote the set of all bounded operators by $S_\infty$, and denote the set of all compact operators by $\mathcal{K}$.

For further information on $S_p$, see e.g. McCarthy [6].

**2.5. NWO (nearly weakly orthonormal) sequences** (Rochberg and Semmes [12])

A nearly weakly orthonormal sequence (NWO) is a function sequence $\{e_\lambda\}_{\lambda\in\Lambda}$ in $L^2(\mathbf{R}^d)$ such that if

$$\sup_{|\xi_Q-x|\le|\lambda|}\left\{|Q|^{-1/2}|\langle f,e_\lambda\rangle|\right\}=f^*(x),$$

then $\|f^*\|_2\le C\|f\|_2$, where $|\lambda|=|Q|^{1/d}$ for $\lambda=(Q,\varepsilon)$ or $Q$.

It is easy to see that any $\{\varphi_\lambda\}$ and $\{\psi_\lambda\}$ in Theorem 2.1 are NWO, and so are $\{\varphi_\lambda\}$ and $\{\psi_\lambda\}$ in the smooth frame in 2.2.

**Proposition 2.1.** *If* $A=\sum_{\lambda\in\Lambda}s_\lambda\langle\,\cdot\,,e_\lambda\rangle f_\lambda$, *with* $\{e_\lambda\}$, $\{f_\lambda\}$ *are* NWO, then

$$\|A\|\le C\|\{s_\lambda\}\|_{\mathrm{BMO}(\Lambda)}$$

$$\|A\|_{S_p}\le C_p\left(\sum_{\lambda\in\Lambda}|s_\lambda|^p\right)^{1/p}\quad\text{for } 0<p<\infty,$$

and

$$\left(\sum_{\lambda}|<Te_\lambda,f_\lambda>|^p\right)^{1/p}\le C_p\|T\|_{S_p}\quad\text{for } 1<p<\infty.$$

## 3. Estimates of operators on $L^2(\mathbf{R}^d)$

First we consider the operators on $l^2(\mathbf{Z}^d)$.

**Lemma 3.1.** *If* $2\le p\le\infty$, $1/p+1/p'=1$, *then*

(1) $$(\alpha_{\underline{k}_1,\underline{k}_2})\in l^p(\underline{k}_1)l^{p'}(\underline{k}_2)\cap l^p(\underline{k}_2)l^{p'}(\underline{k}_1)$$

*implies*

$$(\alpha_{\underline{k}_1,\underline{k}_2})\in S_p;$$

(2) $$(\alpha_{\underline{k}_1,\underline{k}_1+\underline{k}_2})\in l^{p'}(\underline{k}_2)l^p(\underline{k}_1)$$

*implies*

$$(\alpha_{\underline{k}_1,\underline{k}_2})\in S_p;$$

(3) *if* $0<p\leq 2$, *then*

$$(\alpha_{\underline{k}_1,\underline{k}_2})\in l^p(\underline{k}_1)l^2(\underline{k}_2) \quad \text{or} \quad l^p(\underline{k}_2)l^2(\underline{k}_1)$$

*implies*

$$(\alpha_{\underline{k}_1,\underline{k}_2})\in S_p.$$

*Proof.* (1) is proved by Russo [13].

An easy argument shows that $(\alpha_{\underline{k}_1,\underline{k}_1+\underline{k}_2})\in l^1(\underline{k}_2)l^\infty(\underline{k}_1)$ or $(\alpha_{\underline{k}_1+\underline{k}_2,\underline{k}_2})\in l^1(\underline{k}_1)l^\infty(\underline{k}_2)$ implies that $(\alpha_{\underline{k}_1,\underline{k}_2})\in S_\infty$.

If $p=2$, the Hilbert–Schmidt norm shows that

$$\|(\alpha_{\underline{k}_1,\underline{k}_2})\|_{S_2}^2=\sum_{\underline{k}_1}\sum_{\underline{k}_2}|\alpha_{\underline{k}_1,\underline{k}_2}|^2.$$

The interpolation theorems (cf. Bergh and Löfström [1]) show (2).

(3) is the consequence of the fact: $\|T\|_{S_p}^p=\inf\sum_\alpha\|T\varphi_\alpha\|^p$ for $0<p\leq 2$ (cf. McCarthy [6]).

Now we consider operators on $L^2(\mathbf{R}^d)$.

We start with an operator $T$ from $\mathcal{D}(\mathbf{R}^d)$ to $\mathcal{D}'(\mathbf{R}^d)$, where $\mathcal{D}(\mathbf{R}^d)$ is a test function space in $\mathbf{R}^d$ containing some family of wavelets or frame which will be specified in a concrete case.

Let $\{\psi_\lambda\}$ be a wavelet basis or the smooth frame in 2.2. Then we can give two decompositions for $T$ (see [2]).

(1) Standard decomposition:

$$T=\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}T_{ij} \quad \text{(molecular decomposition)}$$

where $T_{ij}=Q_iTQ_j$, and

$$T_{ij}=\sum_{|\lambda|=2^{-i}}\sum_{|\lambda'|=2^{-j}}\alpha_{\lambda\lambda'}\psi_\lambda\otimes\psi_{\lambda'} \quad \text{(atomic decomposition)}$$

where $\alpha_{\lambda\lambda'}=\langle T\psi_\lambda,\psi_{\lambda'}\rangle$.

(2) Non-standard decomposition

$$T=T_1+T_2+T_3$$

where

$$T_1=\sum_{|\lambda|>4|\lambda'|}\beta_{\lambda\lambda'}\varphi_\lambda\otimes\psi_{\lambda'},$$

$$T_2 = \sum_{1/4\le|\lambda|/|\lambda'|\le 4} \alpha_{\lambda\lambda'}\psi_\lambda\otimes\psi_{\lambda'},$$

$$T_3 = \sum_{|\lambda'|\ge 4|\lambda|} \gamma_{\lambda\lambda'}\psi_\lambda\otimes\varphi_{\lambda'}$$

with $\beta_{\lambda\lambda'}=\langle T\varphi_\lambda,\psi_{\lambda'}\rangle$, $\alpha_{\lambda\lambda'}=\langle T\psi_\lambda,\psi_{\lambda'}\rangle$, and $\gamma_{\lambda\lambda'}=\langle T\psi_\lambda,\varphi_{\lambda'}\rangle$.

For example, in the proofs of the "$T1$ Theorem" (cf. Meyer [7]) and the "$Tb$ Theorem" (cf. David [3]), the Calderón–Zygmund Operator $T$ has been split into three parts $T=T_1+T_2+T_3$, where $T_1$ and $T_3$ are paraproducts, for $T_2$, let $\alpha_{\lambda\lambda'}=\langle T_2\psi_\lambda,\psi_\lambda\rangle$. Then $\alpha_{\lambda\lambda'}$ satisfies

$$|\alpha_{\lambda\lambda'}|\le C(|Q|\wedge|R|)^{\alpha/d}|Q|^{d/2}|R|^{d/2}(|Q|^{1/d}+|R|^{1/d}+\operatorname{dist}(Q,R))^{-d-\alpha}, \tag{3.1}$$

and by the Schur lemma, $T_2$ is bounded on $L^2(\mathbf{R}^d)$.

If $T=T_b$ is an operator with a symbol $b$, let $\widehat{\varphi}\in C_0^\infty$, $\operatorname{supp}\widehat{\varphi}\subset\{\xi:1-\varepsilon\le|\xi|\le 2+\varepsilon\}$, such that $\sum_{l=-\infty}^{\infty}\widehat{\varphi}(\xi/2^l)=1$ for $\xi\ne 0$. Then $b=\sum_{l=-\infty}^{\infty}b_l$ with $b_l=b*\varphi_l$, thus we have the standard decomposition for $T_b$:

$$T_b=\sum_l\sum_j\sum_i T_{ij}(b_l) \quad \text{(sub-molecular decomposition)},$$

where $T_{ij}(b_l)=Q_iT_{b_l}Q_j$, and

$$T_{ij}(b_l)=\sum_{|\lambda|=2^{-j}}\sum_{|\lambda'|=2^{-j}}\alpha^l_{\lambda\lambda'}\psi_\lambda\otimes\psi_{\lambda'} \quad \text{(sub-atomic decomposition)}.$$

Similarly we have also the non-standard decomposition for $T_b$. By using the above decompositions, one can estimate the molecules or sub-molecules via atoms or sub-atoms by Lemma 3.1, then estimate $T$ or $T_b$ via molecules or sub-molecules by Lemma 3.2 below.

**Lemma 3.2.** *If* $2\le p\le\infty$, $1/p+1/p'=1$, *then*

$$(\|T_{i+j,j}\|_{S_p})\in l^{p'}(i)l^p(j)$$

*or*

$$(\|T_{i,i+j}\|_{S_p})\in l^{p'}(j)l^p(i)$$

*implies that* $T\in S_p$.

*If* $0<p\le 2$, *then*

$$(\|T_{i,j}\|_{S_p})\in l^p(i)l^p(j)$$

*implies that*

$$T\in S_p.$$

These two steps of decompositions have another advantage, i.e. they can be used to get the converse estimates (see Timotin [14]).

**Lemma 3.3.** *For $1 \le p \le \infty$, if there exist $i_0 \in \mathbf{Z}$ and $\{S_{i,i+i_0}\}$ such that $\|T_{i,i+i_0}\|_{S_p} \ge C\|S_{i,i+i_0}\|_{S_p}$, then*

$$\|T\|_{S_p}^p \ge \sum_i \|T_{i,i+i_0}\|_{S_p}^p \ge C \sum_i \sum_{\underline{k} \in \mathbf{Z}^d} |\alpha_{i,i+i_0}(\underline{k}, \underline{k}+\underline{k}_0)|^p$$

*where $\alpha_{i,j}(\underline{k}_1, \underline{k}_2) = S_{i,j}(\psi_{i,\underline{k}_1}, \psi_{j,\underline{k}_2})$.*

To estimate $T_1$ and $T_3$ in the non-standard decomposition, we need a result on paraproducts. Paraproducts were introduced by Bony. Nowadays there exist hundreds of equivalent definitions. Here we use the following definition:

$$\pi_b(f) = \sum_{|\lambda| > 4|\lambda'|} \langle b, \psi_{\lambda'} \rangle \langle f, \varphi_\lambda \rangle \psi_{\lambda'}$$

where $\varphi_\lambda, \psi_\lambda$ are the functions in the Meyer wavelet basis or the smooth frame.

Notice that both $\varphi_\lambda$ and $\psi_\lambda$ are NWO, by Proposition 2.1, we have

$$\|\pi_b\|_{S_p} \approx \|b\|_{B_p^{d/p,p}(\mathbf{R}^d)} \quad \text{for } 1 \le p < \infty$$

and

$$\|\pi_b\| \approx \|b\|_{\mathrm{BMO}}.$$

*Remark.* In fact the $S_p$-estimates of paraproducts hold also for $0 < p < 1$, see Peng [9].

## 4. Paracommutators

The paracommutator is an operator of the form:

$$(T_b^{st}(A)f)^\wedge(\xi) = (2\pi)^{-d} \int_{\mathbf{R}^d} \hat{b}(\xi-\eta) A(\xi,\eta) |\xi|^s |\eta|^t \hat{f}(\eta)\, d\eta. \tag{4.1}$$

It is defined in Janson–Peetre [5].

We adopt the notation of [5] for the norm $\|A(\xi,\eta)\|_{M(U\times V)}$ of the Schur multiplier $A(\xi,\eta) \in M(U \times V)$ and the $S_p$-norm $\|K(\xi,\eta)\|_{S_p(U\times V)}$.

As in [5], we let $\Delta_j = \{\xi\colon 2^j \le |\xi| \le 2^{j+1}\}$ and $\overline{\Delta}_j = \Delta_{j-1} \cup \Delta_j \cup \Delta_{j+1}$.

For convenience, we list here some assumptions on $A(\xi,\eta)$, which come from Janson–Peetre [5], Peng [9] and [10].

A0: There exists an $r > 1$ such that $A(r\xi, r\eta) = A(\xi,\eta)$.

A1: $\|A\|_{M(\Delta_j \times \Delta_k)} \le C$, for all $j, k \in \mathbf{Z}$.

A2: There exist $A_1, A_2 \in M(\mathbf{R}^d \times \mathbf{R}^d)$ and $\delta>0$ such that

$$A(\xi,\eta)=A_1(\xi,\eta) \quad \text{for } |\eta|<\delta|\xi|$$
$$A(\xi,\eta)=A_2(\xi,\eta) \quad \text{for } |\xi|<\delta|\eta|.$$

A3($\alpha$): There exist $\alpha>0$ and $\delta>0$ such that if $B=B(\xi_0,r)$ with $r<\delta|\xi_0|$, then $\|A\|_{M(B\times B)}\le C(r/|\xi_0|)^\alpha$.

A4: There exist no $\xi\neq 0$ such that $A(\xi+\eta,\eta)=0$ for a.e. $\eta$.

A$4\frac{1}{2}$: For every $\xi_0\neq 0$ there exist $\eta_0\in\mathbf{R}^d$ and $\delta>0$ such that, with $B_0=B(\xi_0+\eta_0,\delta|\xi_0|)$ and $D_0=B(\eta_0,\delta|\xi_0|)$, $A(\xi,\eta)^{-1}\in M(B_0\times D_0)$.

A5: For every $\xi_0\neq 0$ there exist $\delta>0$ and $\eta_0\in\mathbf{R}^d$ such that, with $U=\{\xi\colon |\xi/|\xi|-\xi_0/|\xi||<\delta \text{ and } |\xi|>|\xi_0|\}$ and $V=B(\eta_0,\delta|\eta_0|)$, $A(\xi,\eta)^{-1}\in M(U\times V)$.

A10: For any $0\neq\theta\in\mathbf{R}^d$, there exist a positive $\delta<\frac{1}{2}$ and a subset $V_\theta$ of $\mathbf{R}^d$ such that if $N_r$ denote the number of integer points contained in $V_\theta\cap B_r$, where $B_r=B(0,r)$, then

$$\overline{\lim_{r\to\infty}}\frac{N_r}{r^d}>0$$

and for every $\underline{n}\in V_0$,

$$\|A(\xi+\underline{n}+\theta,\eta+\underline{n})^{-1}\|_{M(B\times B)}\le C|n|^\alpha, \quad \text{where } B=B(0,\delta).$$

We adopt the notation of Peng [9] for $\Lambda_p$, $0<p<1$, instead of using $M$ in A1, A3($\alpha$) and A$4\frac{1}{2}$, we can list the conditions of $\mathrm{A}_p1$, $\mathrm{A}_p3(\alpha)$ and $\mathrm{A}_p4\frac{1}{2}$ by using $\Lambda_p$.

The paracommutators contain many examples, cf. Janson–Peetre [5] and Peng–Qian [11].

Now let us estimate the $S_p$-norm of a paracommutator by its molecular-atomic structure. Here we use smooth frame expansion.

First we estimate the direct results. We split the operator (4.1) into three parts via the molecules $T_{ij}=Q_iT_b^{st}Q_j$:

$$T_b^{st}(A)=T_1+T_2+T_3,$$

where

$$T_1=\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{i-3}T_{i,j}=\sum_{j=-\infty}^{-3}\sum_{i=-\infty}^{\infty}T_{i,i+j}$$

$$T_2=\sum_{i=-\infty}^{\infty}\sum_{j=i-2}^{i+2}T_{i,j}$$

$$T_3=\sum_{i=-\infty}^{\infty}\sum_{j=i+3}^{\infty}T_{i,j}=\sum_{i=-\infty}^{-3}\sum_{j=-\infty}^{\infty}T_{i+j,j}.$$

To estimate them, we need the following three lemmas.

**Lemma 4.1.** *Suppose that $A$ satisfies* A1, A3($\alpha$), $\widehat{\varphi}\in C_0^\infty$, $\operatorname{supp}\widehat{\varphi}\subset\{\xi:1-\varepsilon\le|\xi|\le 2+\varepsilon\}$ *and* $\widehat{\varphi}_l(\xi)=\widehat{\varphi}(\xi/2^{l-2})$, *then for* $|i-j|\le 3$,

$$\|A(\xi,\eta)\widehat{\varphi}_l(\xi-\eta)\|_{M(\Delta_i\times\Delta_j)}\le C\left(\frac{2^l}{2^i}\right)^\alpha. \tag{4.2}$$

**Lemma 4.2.** *Suppose that $A$ satisfies* A1, *$\psi$ is the function of the smooth frame in* 2.2, $\widehat{\varphi}\in C_0^\infty$, $\widehat{\varphi}(\xi)\equiv 1$ *on* $\overline{\Delta}_0$, *and* $\widehat{\varphi}_i(\xi)=\widehat{\varphi}(\xi/2^{i-2})$. *Then for* $i>j+3$, $x\in Q_{2^{-i}}^{-\underline{k}_1}$, $x'\in Q_{2^{-i}}^{2^{i-j}\underline{k}_2-\underline{k}_1}$ *and $r>0$ small enough,*

$$|\langle T_b^{st}(\psi_{i,\underline{k}_1}),\psi_{j,\underline{k}_2}\rangle|\le C2^{-id/2+is+jt}M_r(b_i)(x)M_r(\psi_j)(x'), \tag{4.3}$$

*where* $b_i=\varphi_i*b, Q_{2^{-i}}^{\underline{k}}$ *is the cube with side* $2^{-i}$ *and center* $2^{-i}\underline{k}$.

*Remark.* For $j>i+3$, we have the same result.

**Lemma 4.3.** *Suppose that $A$ satisfies* A1, A3($\alpha$) *with* $\alpha>0$, *$\psi$ is the function of the smooth frame in* 2.2, $\widehat{\varphi}\in C_0^\infty$, $\operatorname{supp}\widehat{\varphi}(\xi)\subset\overline{\Delta}_0$, *and* $\widehat{\varphi}_i(\xi)=\widehat{\varphi}(\xi/2^{i-2})$, *then for* $|i-j|\le 3$, $x\in Q_{2^{-i}}^{-\underline{k}_1}$, $x'\in Q_{2^{-i}}^{2^{i-j}\underline{k}_2-\underline{k}_1}$ *and $r>0$ small enough,*

$$|\langle T_{b_l}^{st}(\psi_{i,\underline{k}_1}),\psi_{j,\underline{k}_2}\rangle|\le C\left(\frac{2^l}{2^i}\right)^\alpha 2^{-id/2+is+jt}M_r(b_l)(x)M_r(\psi_j)(x') \tag{4.4}$$

*where* $b_l=\varphi_l*b$.

*Proof of Lemma* 4.1. By the Fourier transform:

$$\widehat{\varphi}_l(\xi-\eta)=\int\varphi_l(x)e^{-ix\xi}e^{ix\eta}\,dx,$$

so $\|\widehat{\varphi}_l(\xi-\eta)\|_{M(\mathbf{R}^d\times\mathbf{R}^d)}\le C\|\varphi\|_1=C<\infty$.

Let $Q_l^n$ denote the cube with side $2^l$ and center $2^l n$ for $n\in\mathbf{Z}^d$, then

$$\begin{aligned}\|A(\xi,\eta)\widehat{\varphi}_l(\xi-\eta)\|_{M(\Delta_i\times\Delta_j)}&=\left\|A(\xi,\eta)\widehat{\varphi}_l(\xi-\eta)\sum_{|n-m|\le c}\chi_{\Delta_i\cap Q_l^n}(\xi)\chi_{\Delta_j\cap Q_l^m}(\eta)\right\|_M\\&\le\sum_{|m|\le c}\sup_n\|A(\xi,\eta)\widehat{\varphi}_l(\xi-\eta)\|_{M(\Delta_i\cap Q_l^n\times Q_j\cap Q_l^{n-m})}\\&\le C\left(\frac{2^l}{2^i}\right)^\alpha.\end{aligned}$$

*Proof of Lemma* 4.2. By Lemma 2.4,

$$
\begin{aligned}
&\left|\langle T_b^{st}(\psi_{i,\underline{k}_1}),\psi_{j,\underline{k}_2}\rangle\right| = \left|\langle T_{b_i}^{st}(\psi_{i,\underline{k}_1}),\psi_{j,\underline{k}_2}\rangle\right| \\
&\quad = \left|\iint \hat{b}_i(\xi-\eta)A(\xi,\eta)|\xi|^s|\eta|^t\widehat{\psi}_{i,\underline{k}_1}(\xi)\widehat{\psi}_{j,\underline{k}_2}(\eta)\,d\xi\,d\eta\right| \\
&\quad \le C2^{is}2^{jt}2^{-id/2}\left|\int \widehat{\psi}_i(\xi)\hat{b}_{i,-\underline{k}_1}*\widehat{\psi}_{j,2^{i-j}\underline{k}_2-\underline{k}_1}(\xi)\,d\xi\right| \\
&\quad = C2^{is}2^{jt}2^{-id/2}\left|\int b_j\left(y-\frac{\underline{k}_1}{2^i}\right)\psi_j\left(y+\frac{2^{i-j}\underline{k}_2-\underline{k}_1}{2^i}\right)\psi_i(y)\,dy\right| \\
&\quad \le C2^{is}2^{jt}2^{-id/2}\int \frac{\left|b_i\left(y-\frac{\underline{k}_1}{2^i}-x+x\right)\right|}{1+\left|2^i\left(y-\frac{\underline{k}_1}{2^i}-x\right)\right|^{d/r}}\frac{\left|\psi_j\left(y+\frac{2^{i-j}\underline{k}_2-\underline{k}_1}{2^i}-x'+x'\right)\right|}{1+\left|2^i\left(y+\frac{2^{i-j}\underline{k}_2-\underline{k}_1}{2^i}-x'\right)\right|^{d/r}} \\
&\qquad \times\left[1+(2^i|y|+2)^{d/r}\right]\cdot\left[1+(2^i|y|+2)^{d/r}\right]|\psi_j(y)|\,dy \\
&\quad \le C2^{is}2^{jt}2^{-id/2}M_r(b_i)(x)M_r(\psi_j)(x')
\end{aligned}
$$

where $\hat{b}_{i,-\underline{k}_1}(\xi)=\hat{b}(\xi)\widehat{\varphi}(\xi/2^i)e^{-i2\pi\xi\cdot\underline{k}_1/2^{i+2}}$.

The proof of Lemma 4.3 is similar, we omit it here. Now we deal with $T_1$.

**Lemma 4.4.** *Suppose that $A$ satisfies* A1, A$_p$1 *if* $0<p<1$, $t>\max(-d/2,-d/p)$ *and* $0<p\le\infty$, *then*

$$\|T_1\|_{S_p}\le C\|b\|_{B_p^{s+t+d/p,p}(\mathbf{R}^d)}. \tag{4.5}$$

*Proof.* This is a consequence of Lemma 3.1 , Lemma 3.2 and Lemma 4.2. We give the proof only for $2\le p\le\infty$. We choose $r<p$. Integrating over $x\in Q_{2^{-i}}^{-\underline{k}_1}$ and $x'\in Q_{2^{-i}}^{2^{i-j}\underline{k}_2-\underline{k}_1}$ in (4.3), we get

$$
\begin{aligned}
&|\langle T_b^{st}(\psi_{i,\underline{k}_1}),\psi_{j,\underline{k}_2}\rangle| \\
&\quad \le C2^{js+jt}2^{id/2}\left(\int_{Q_{2^{-i}}^{-\underline{k}_1}} M_r(b_i)(x)^p\,dx\right)^{1/p}\left(\int_{Q_{2^{-i}}^{2^{i-j}\underline{k}_2-\underline{k}_1}} M_r(\psi_j)(x')^{p'}\,dx\right)^{1/p'}.
\end{aligned}
$$

By the atomic decomposition of $T_{ij}$ and Lemmas 3.1 and 3.2 we have

$$
\begin{aligned}
\|T_{i,j}\|_{S_p} &\le \left[\sum_{\underline{k}_2}\left(\sum_{\underline{k}_1}|\langle T_b^{st}(\psi_{i,\underline{k}_1}),\psi_{j,\underline{k}_2+2^{j-i}\underline{k}_1}\rangle|^p\right)^{p'/p}\right]^{1/p'} \\
&\le C2^{js+jt}2^{id/2}\|b_j\|_p\|\psi_j\|_{p'} \\
&\le C2^{js+jt}2^{jd/p}\|b_i\|_p.
\end{aligned}
$$

By the molecular decomposition of $T_1$ and Lemma 3.2 we have

$$\begin{aligned}\|T_1\|_{S_p} &\le \left\{ \sum_{j=-\infty}^{-3} \left( \sum_{i=-\infty}^{\infty} \|T_{i,i+j}\|_{S_p}^p \right)^{p'/p} \right\}^{1/p'} \\ &\le \left\{ \sum_{j=-\infty}^{-3} \left( \sum_{i=-\infty}^{\infty} 2^{i(s+t+d/p)p} \|b_i\|_p^p 2^{j(t+d/p)p} \right)^{p'/p} \right\}^{1/p'} \\ &\le C\|b\|_{B_p^{s+t+q/p,p}(\mathbf{R}^d)}.\end{aligned}$$

For $T_3$ we have the same estimates.

Now we turn to estimate $\|T_2\|_2$.

**Lemma 4.5.** *Suppose that $A$ satisfies* A1, A3($\alpha$) (A$_p$1,A$_p$3($\alpha$) *if* $0<p<1$), $s+t+d/p<\alpha$ *and* $0<p\le\infty$. *Then*

$$\|T_2\| \le C\|b\|_{B_p^{s+t+(d/p),p}(\mathbf{R}^d)}. \tag{4.6}$$

*Remark.* It should be noticed that there is no assumption $\alpha>0$ in Lemma 4.5.

The proof is similar to the one of Lemma 4.4.

For $p=\infty$, the sub-atomic and sub-molecular decomposition of $T_2(b)$, Lemmas 3.1, 3.2 and 4.3 give us:

$$\|T_2\|_\infty \le C\|b\|_{B_\infty^{s+t,\infty}(\mathbf{R}^d)}.$$

For $0<p\le 2$, again the sub-atomic and sub-molecular decomposition of $T_2(b)$, Lemmas 3.1, 3.2 and 4.3 give us:

$$\|T_2\|_p \le C\|b\|_{B_p^{s+t+(d/p),p}(\mathbf{R}^d)}.$$

The interpolation theorem gives us (4.6).

Lemmas 4.4 and 4.5 yield the following

**Theorem 4.1.** *Suppose that $A$ satisfies* A1 *and* A3($\alpha$) *with some* $\alpha>0$ (*or* A$_p$1 *and* A$_p$3($\alpha$) *if* $0<p<1$) *and that* $s,t>\max(-d/2,-d/p)$, $s+t+d/p<\alpha$ *and* $0<p\le\infty$. *Then*

$$\|T_p^{st}(A)\|_{S_p} \le C\|b\|_{B_p^{s+t+(d/p),p}(\mathbf{R}^d)}. \tag{4.7}$$

If $p=\infty$ and $s=t=0$, we have

**Theorem 4.2.** *Suppose that $A$ satisfies* A1, A2, A3($\alpha$). *Then*

$$\|T_b(A)\|\le C\|b\|_{\mathrm{BMO}}. \tag{4.8}$$

The proof of Theorem 4.2 can be found in Janson–Peetre [5]. The main idea is the same as in the proofs of the "$T1$ Theorem" or the "$Tb$ Theorem", i.e. by the non-standard decomposition $T_b(A)=T_b(A_1)+T_b(A_2)+T_b(A_3)=T_1+T_2+T_3$, $T_1$ and $T_2$ can be estimated by the results of paraproducts and $T_3$ can be estimated by Theorem 4.1.

Secondly we estimate the converse results.

**Theorem 4.3.** *Suppose that $A$ satisfies* A0, A4$\frac{1}{2}$ *and* $1\le p\le\infty$. *Then*

$$\|b\|_{B_p^{s+t+(d/p),p}(\mathbf{R}^d)}\le C\|T_b^{st}(A)\|_{S_p}. \tag{4.9}$$

*Proof.* By A4$\frac{1}{2}$, there exist finite sets of points $\{\xi_0^{(j)}\}_{j=1}^J$ in $\Delta_0$ and $\{\eta_0^{(j)}\}_{j=1}^J$ in $\mathbf{R}^d$ with corresponding open ball $B(\xi_0^{(j)},\delta^{(j)})$ and $B(\eta_0^{(j)},\delta^{(j)})$ such that $\eta^{(j)}\neq 0$, $\eta_0^{(j)}\neq-\xi_0^{(j)}$, $\bigcup_{j=1}^J B(\xi_0^{(j)},\delta^{(j)})\supset\Delta_0$, $\delta^{(j)}<\frac{1}{4}\min(|\xi_0^{(j)}+\eta_0^{(j)}|,|\eta_0^{(j)}|,1)$, and, with $B_j=B(\xi_0^{(j)}+\eta_0^{(j)},\delta^{(j)})$ and $D_j=B(\eta_0^{(j)},\delta^{(j)})$, $A^{-1}\in M(B_j\times D_j)$. Let $\overline{B}_j=2B_j$ and $\overline{D}_j=2D_j$. We may assume that $\overline{B}_j\subset\overline{\Delta}_0$, $\overline{D}_j\subset\overline{\Delta}_{i_0}$ for some $i_0$, where $\overline{\Delta}_i=\Delta_{i-1}\cup\Delta_i\cup\Delta_{i+1}$. Now we choose the positive functions $h_j'(\xi)$ and $h_j(\eta)$ such that $h_j',h_j\in C_0^\infty(\mathbf{R}^d)$, $\operatorname{supp}h_j'=\overline{B}_j$, $h_j'(\xi)\ge C>0$ on $B_j$, $\operatorname{supp}h_j=\overline{D}_j$ and $h_j(\eta)\ge C>0$ on $D_j$.

Let

$$\widehat{\psi}(\xi)=\sum_{j=1}^J\int|\xi+\eta|^s|\eta|^t h_j'(\xi+\eta)h_j(\eta)\,d\eta.$$

Then $\widehat{\psi}\in C_0^\infty(\mathbf{R}^d)$, $\operatorname{supp}\widehat{\psi}\subset\{\frac{1}{2}\le|\xi|\le 2+\frac{1}{2}\}$ and $\widehat{\psi}(\xi)\ge C>0$ on $\Delta_0$, thus $\psi$ can be used to define the norm of $B_p^{s,p}$.

Thus

$$\|T_b^{st}(A)\|_{S_p}^p\ge\tfrac{1}{3}\sum_i\sum_{l=-1}^{1}\|T_{i+l,i+i_0}\|_{S_p}^p \tag{4.10}$$

and

$$\begin{aligned}\|T_{i+l,i+i_0}\|_{S_p}^p&=\|\hat{b}(\xi-\eta)A(\xi,\eta)|\xi|^s|\eta|^t\|_{S_p(\Delta_{i+l}\times\Delta_{i+i_0})}^p\\&\ge C\left\|\hat{b}(\xi-\eta)\sum_{j=1}^J|\xi|^s|\eta|^t h_j'\left(\frac{\xi}{2^{i+l}}\right)h_j\left(\frac{\eta}{2^{i+i_0}}\right)\right\|_{S_p}^p\\&=C\|S_i\|_{S_p}^p.\end{aligned} \tag{4.11}$$

We claim that

$$\|S_i\|_{S_p} \geq C2^{i(s+t+d/p)}\|b*\psi_i\|_p. \tag{4.12}$$

Therefore

$$\|T_b^{st}(A)\|_{S_p} \geq C\|b\|_{B_p^{s+t(d/p),p}(\mathbf{R}^d)}.$$

By the homogeneity, it suffices to prove (4.12) for $i=0$. Note that $S_0$ is an operator on $L^2((3T)^d)$. We choose an orthonomal basis of $L^2((3T)^d)$ $\{e_{\underline{k}}\}_{\underline{k}\in\mathbf{Z}^d}$, $e_{\underline{k}}(\eta)=1/((6\pi)^{d/2})e^{i\underline{k}\cdot\eta/3}$. Then $S_0$ is determined by $(\alpha_{\underline{k}_1,\underline{k}_2})$, where $\alpha_{\underline{k}_1,\underline{k}_2}=\langle S_0(e_{\underline{k}_1}),e_{\underline{k}_2}\rangle$. Note that

$$|\alpha_{\underline{k}_1,\underline{k}_2}|=\left|\iint \hat{b}(\xi-\eta)\sum_{j=1}^{J}|\xi|^s|\eta|^t h_j'(\xi)h_j(\eta)e_{\underline{k}}(-\xi)e_{\underline{k}}(\eta)\,d\xi d\eta\right|=C\left|b*\psi\left(-\frac{\underline{k}}{3}\right)\right|,$$

then we have

$$\|S_0\|_{S_p}^p \geq \sum_{\underline{k}}|\alpha_{\underline{k},\underline{k}}|^p = C\sum_{\underline{k}}\left|b*\psi\left(-\frac{\underline{k}}{3}\right)\right|^p \geq C\|b*\psi\|_{S_p}^p \quad \text{by Lemma 2.5 and 3.3,}$$

i.e. (4.12) holds.

*Remark.* For statement and proof of Theorem 4.3 for $0<p<1$, see Peng [9]. If $p=\infty$, we have also the BMO result.

**Theorem 4.4.** *Suppose that $A$ satifies* A0, A1, A3($\alpha$) *and* A5. *Then*

$$\|b\|_{\mathrm{BMO}} \leq C\|T_b(A)\|. \tag{4.13}$$

Its proof can be found in [5]. It can be proved also by the non-standard decomposition.

For completeness we list also the following theorems which describe the Janson–Wolff phenomenon and compactness of paracommutators. They can also be proved by taking the special frame of $L^2(\mathbf{R}^d)$.

**Theorem 4.5.** *Suppose that $A$ satisfies* A10($\alpha$), $1\leq p\leq d/(\alpha-s-t)$ *and* $T_b^{st}(A)\in S_p$. *Then $b$ must be a polynomial.*

For the proof, see Theorem 4 of Peng [9], which contains the case $0<p<1$.

**Theorem 4.6.** *Suppose that $A$ satisfies* A0, A1, A3($\alpha$) *and* A4$\frac{1}{2}$, $s,t>0$, *and* $s+t<\alpha$; *then* $T_b^{st}(A)$ *is compact if and only if* $b\in b_\infty^{s+t}$. *And suppose that $A$ satisfies* A0, A1, A2, A3($\alpha$) *and* A5; *then* $T_b(A)$ *is compact if and only if* $b\in$CMO.

For the proof of Theorem 4.6, see Peng [10].

Finally we give another example of paracommutator. We consider the classical Toeplitz operator $T_b=P(\bar{b}f)$, where the symbol $b$ is an analytic function, $f\in H^2(\mathbf{R})$ (Hardy space) and $P$ denotes the projection from $L^2(\mathbf{R})$ to $H^2(\mathbf{R})$. Let $I^s$ denote the fractional integral operator defined by $\widehat{I^s f}(\xi)=|\xi|^{-s}\hat{f}(\xi)$. Let us study the operator $J_b^{st}=I^{-s}T_bI^t$. It is easy to check that $J_b^{st}$ is a paracommutator with $A(\xi,\eta)=\chi(\xi>0,\eta>0)$, which satisfies A0, A1, A3(0), A4$\frac{1}{2}$ and A10(0). So we have

**Theorem 4.7.** (i) *If* $s\in\mathbf{R}, t>\max(-\frac{1}{2},-1/p)$, $0<p\leq\infty$ *and* $s+t+1/p<0$ *then* $J_b^{st}\in S_p$ *if and only if* $b\in B_p^{s+t+(1/p),p}$.

(ii) *If* $0<p<\infty$ *and* $s+t+1/p\geq 0$ *then* $J_b^{st}\in S_p$ *only if* $b\equiv 0$.

(iii) *If* $s\in\mathbf{R}, t>0$ *and* $s+t<0$, *then* $J_b^{st}$ *is compact if and only if* $b\in b_\infty^{s+t}$.

(iv) *If* $t>0$, *then* $J_b^{-tt}$ *is bounded if and only if* $b\in L^\infty$, *and* $J_b^{-tt}$ *is compact only if* $b\equiv 0$.

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