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Area integral estimates for solutions and normalized adjoint solutions to nondivergence form elliptic equations

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Abstract. We show that the L^p norms, $0<p<\infty$, of the nontangential maximal function and area integral of solutions and normalized adjoint solutions to second order nondivergence form elliptic equations, are comparable when integrated on the boundary of a Lipschitz domain with respect to measures, which are respectively A_∞ with respect to the corresponding harmonic measure or normalized harmonic measure.

0. Introduction and definitions

In this work we shall prove inequalities comparing the area integral and nontangential maximal functions for solutions and normalized adjoint solutions to second order nondivergence form linear elliptic operators. Such types of inequalities have been treated recently by many authors, [BG], [GW], [D], [DJK], [BM], [Mc], [MU], [Z]. The most general setting up to now is that of weak solutions to second order divergence form operators [DJK] and subharmonic functions in Lipschitz domains [Z]. As in [DJK], the constants in our estimates will depend only on ellipticity and the Lipschitz character of the domain, and the estimates will be obtained via the classical approach of good λ inequalities. The main difficulty in our case is the lack of interior estimates for the gradients of solutions and normalized adjoint solutions, and to make up for it, the area integrals are defined with respect to certain weights for which a Cacciopoli type inequality holds.

Before we state the main theorem we shall need some definitions and remarks.

Recall that a bounded connected domain D in $\mathbf{R}^n$ is called a Lipschitz domain if its boundary ∂D can be covered by finitely many open right circular cylinders whose bases have positive distance from ∂D and corresponding to each cylinder I there is

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a coordinate system (x,s) with $x\in\mathbf{R}^{n-1}$, $s\in\mathbf{R}$ with s axis parallel to the axis of I, and a function $\varphi\colon\mathbf{R}^{n-1}\to\mathbf{R}$ satisfying a Lipschitz condition ($|\varphi(x)-\varphi(z)|\le m|x-z|$) such that $I\cap D=\{(x,s):s>\varphi(x)\}\cap I$, and $I\cap\partial D=\{(x,s):s=\varphi(x)\}\cap I$.

Denote by $Lu=\sum_{i,j=1}^n a_{ij}(X)D_{ij}u$ a uniformly elliptic operator with coefficients satisfying $a_{ij}(X)=a_{ji}(X)$ for all X in $\mathbf{R}^n$, $i,j=1,\dots,n$, and for some $\lambda>0$

$$\lambda|\xi|^2\le\langle A(X)\xi,\xi\rangle\le\lambda^{-1}|\xi|^2\quad\text{for all } X,\xi\in\mathbf{R}^n,$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbf{R}^n$, and $A(X)$ is the coefficient matrix $(a_{ij}(X))$.

Throughout this work we will assume that the coefficient matrix $A(X)$ is smooth in $\mathbf{R}^n$, though our estimates will not depend on this qualitative property. We will also assume that the Lipschitz domain D is contained in the unit ball and "centered" at the origin.

We say that a function u is L-harmonic in D if u is smooth and $Lu=0$ in D. For each continuous function f on ∂D, there is a unique L-harmonic function u in D, continuous in $\overline{D}$ with $u=f$ on ∂D. The L-harmonic measure at $X\in D$ is the representing measure of the functional $f\to u(X)=\int_{\partial D}f\,d\omega^X$. In particular ω will denote the L-harmonic measure at the origin. For each f smooth in $\overline{D}$ there is a unique solution to the nonhomogeneous problem $Lu=-f$ in D, $u=0$ on ∂D, which can be represented as

$$u(X)=\int_D g(X,Y)f(Y)\,dY\quad\text{for all } X \text{ in } D,$$

where $g(\cdot,\cdot)$ denotes the Green's function for L in D. Recall that from the Pucci–Aleksandrov–Bakelman inequality [P] we have

$$\text{(0.1)}\qquad\left(\int_D g(X,Y)^{n/(n-1)}\,dY\right)^{(n-1)/n}\le C(\lambda,n)\operatorname{diam}(D)\quad\text{for all } X \text{ in } D,$$

where $\operatorname{diam}(D)$ denotes the diameter of D. Also, when v is smooth in $\overline{D}$, v can be represented as

$$\text{(0.2)}\qquad v(X)=\int_{\partial D}v\,d\omega^X-\int_D g(X,Y)Lv(Y)\,dY\quad\text{for all } X \text{ in } D.$$

We will denote by $B_r(X)$ the ball centered at X with radius r and for $Q\in\partial D$, $\Delta_r(Q)=B_r(Q)\cap\partial D$, $T_r(Q)=B_r(Q)\cap D$, δ the distance function to the boundary of D, and for $Q\in\partial D$ and $\alpha>0$ we set $\Gamma_\alpha(Q)=\{X\in D:|X-Q|\le(1+\alpha)\delta(X)\}$. We will set $B(X)=B_{\delta(X)}(X)$ for $X\in D$. The letter G will denote the Green's function for L in $B_{20}(0)$ and with pole at a fixed point $X_0 \in \partial B_9(0)$, i.e.; $G(Y)=G(X_0,Y)$, and G the measure associated to G, i.e.; $G(F)$ denotes the integral of G over F. The normalized Green's function for L in D is defined as $\tilde{g}(X,Y)=g(X,Y)/G(Y)$. For a function v defined on D, the nontangential maximal function and area function of aperture α are defined respectively as

$$N_\alpha(v)(Q)=\sup\{|v(X)| : X\in\Gamma_\alpha(Q)\},$$

$$S_\alpha(v)(Q)^2=\int_{\Gamma_\alpha(Q)}\frac{\delta(X)^2}{G(B(X))}|\nabla v(X)|^2 G(X)\,dX$$

for $Q\in\partial D$.

A measure μ on ∂D is A_∞ with respect to a measure σ on ∂D if there are constants M and θ such that for all $Q\in\partial D$, $r>0$, and $E\subset\Delta_r(Q)$ the following holds

$$\frac{\mu(E)}{\mu(\Delta_r(Q))}\le M\left(\frac{\sigma(E)}{\sigma(\Delta_r(Q))}\right)^\theta.$$

Our main theorem is the following:

Theorem 1. *Let u be L-harmonic in D, $\alpha,\beta>0$, and μ a measure on ∂D which is A_∞ with respect to ω. Then, for each $0<p<\infty$ there is a constant C depending only on λ, n,p, the A_∞ constants of μ, and the Lipschitz character of D such that the following holds*

$$\|S_\beta(u)\|_{L^p(d\mu)}\le C\|N_\alpha(u)\|_{L^p(d\mu)}.$$

Moreover, if $u(0)=0$,

$$\|N_\alpha(u)\|_{L^p(d\mu)}\le C\|S_\beta(u)\|_{L^p(d\mu)}.$$

We say that a function $\widetilde{w}$ is a normalized adjoint solution for L in D if $\widetilde{w}$ is smooth and $\widetilde{w}G$ is an adjoint solution for L in D, i.e.; $D_{ij}(a_{ij}\widetilde{w}G)=0$ in D. In ([B], Theorem 4.4), it is shown that normalized adjoint solutions satisfy the Harnack inequality, that is, if $\widetilde{w}$ is a nonnegative normalized adjoint solution in B_{2r}, there is a constant C depending on λ and n such that $\sup\{\widetilde{w}(X):X\in B_r\}\le C\inf\{\widetilde{w}(X): X\in B_r\}$. It was also shown in ([FGMS], Theorem I.1.6) that when $\widetilde{w}$ is a normalized adjoint solution in $T_{2r}(Q)$, $Q\in\partial D$, which vanishes continuously on $\Delta_{2r}(Q)$, then for some constants $\alpha\in(0,1)$ and C depending on λ, n, and the Lipschitz character of D,

$$\widetilde{w}(X)\le C\left(\frac{|X-Q|}{r}\right)^\alpha \sup\{\widetilde{w}: X\in T_{2r}(Q)\}\quad\text{for all } X \text{ in } T_{2r}(Q).$$

The Harnack inequality implies that normalized adjoint solutions can not attain neither a maximum or a minimum in the interior of D. These and standard approximation arguments, together with the above control on the modulus of continuity at the boundary of D of normalized adjoint solutions which vanish on an open set of the boundary, imply that given f a continuous function on the boundary of D, there exists a unique normalized adjoint solution $\widetilde{w}\in C(\bar{D})$ such that $\widetilde{w}=f$ on ∂D. The normalized harmonic measure $\widetilde{\omega}^X$ at $X\in D$ is the representing measure of the functional

$$f \to \widetilde{w}(X) = \int_{\partial D} f\, d\widetilde{\omega}^X.$$

In particular, $\widetilde{\omega}$ will denote the normalized harmonic measure at the origin. In this case we also obtain the following similar result.

Theorem 2. *Let $\widetilde{w}$ be a normalized adjoint solution for L in D, $\alpha,\beta>0$, and μ a measure on ∂D which is A_∞ with respect to $\widetilde{\omega}$. Then, for each $0<p<\infty$ there is a constant C depending only on λ, n, p, the A_∞ constants of μ, and the Lipschitz character of D such that the following holds*

$$\|S_\beta(\widetilde{w})\|_{L^p(d\mu)} \le C\|N_\alpha(\widetilde{w})\|_{L^p(d\mu)}.$$

Moreover, if $\widetilde{w}(0)=0$,

$$\|N_\alpha(\widetilde{w})\|_{L^p(d\mu)} \le C\|S_\beta(\widetilde{w})\|_{L^p(d\mu)}.$$

In the first section we will prove Theorem 1 and in the second we will indicate how to obtain Theorem 2 from arguments similar to those in the first theorem.

We will say that two objects A and B (numbers or functions) are equivalent and write $A\approx B$ if there exists a positive constant C depending at most on ellipticity, dimension, α,β, constants in the A_∞ condition, and the Lipschitz character of D such that $C^{-1}A\le B\le CA$. Analogously, the notation $A\lesssim B$ will mean that for some C as above $A\le CB$.

1. Proof of Theorem 1

We will first need to recall some results about L-harmonic measure.

Lemma 1. (Doubling property of L-harmonic measure). *There exists $r_0>0$ depending on the Lipschitz character of D such that for all $0<r<r_0$ and $X\notin T_{4r}(Q)$ the following holds, $\omega^X(\Delta_{2r}(Q))\approx\omega^X(\Delta_r(Q))$. In particular, $\omega(\Delta_{2r}(Q))\approx \omega(\Delta_r(Q))$ for all $r>0$.*

The reader can find the proof of this result in ([FGMS], Theorem I.2.3). As the authors of this work point out, a consequence of the above estimate is that all the results and lemmas in [CFMS] for divergence form operators can then be shown, using the same arguments as in [CFMS]. (See also the arguments in [B1], where these results are proved with constants which depend also on the modulus of continuity of the coefficients, but which after the works of [FGMS] and [FS] can be carried out without this dependence), to hold for operators in nondivergence form with measurable coefficients. Hence, as in Theorem 1.4, Lemmas 2.1, 2.4, and 2.5 in [DJK], (where they follow from [CFMS]) we have:

(1.1) Let $Q\in\partial D$, $r>0$, and $X\in D$ with $|X-Q|=r\approx\delta(X)$. Then, $\omega^X(\Delta_r(Q))\approx 1$.

(1.2) There exists r_0 depending on the Lipschitz character of D such that if $Q\in\partial D$, $0<5r<r_0$, $s<r/2$, $F\subset\Delta_s(Q)$ Borel set, and $X\in D$ with $|X-Q|\approx r\approx\delta(X)$, then the following holds:

$$\frac{\omega(F)}{\omega(\Delta_s(Q))}\approx\frac{\omega^X(F)}{\omega^X(\Delta_s(Q))}.$$

We will also need the following facts:

(1.3) (Carleson estimate for normalized adjoint solutions.) Let $\widetilde{w}$ be a normalized adjoint solution on $T_{2r}(Q)$ which vanishes continuously on $\Delta_{2r}(Q)$ for some $Q\in\partial D$. Then, there is a constant r_0 depending on the Lipschitz character of D, such that if $r\leq r_0$, $\sup\{\widetilde{w}(X):X\in T_r(Q)\}\leq\widetilde{w}(Y)$, where Y is any point lying on $\partial B_r(Q)\cap\Gamma_1(Q)$. See [FGMS], Theorem I.1.6.

(1.4) (Doubling property of adjoint solutions.) Let $v\in L^1_{\mathrm{loc}}(B_{3r})$ be a nonnegative superadjoint solution for L on B_{3r} ($D_{ij}(a_{ij}v)\leq 0$ on B_{3r} in the distribution sense), then

$$\int_{B_{2r}}v(Y)\,dY\approx\int_{B_r}v(Y)\,dY.$$

See [FS], Lemma 2.0.

Lemma 2. *Let $g(X,Y)$ denote the Green's function for L in D. Then there is a constant r_0 depending on the Lipschitz character of D, such that for all $Q\in\partial D$, $r\leq r_0$, $Y\in\partial B_r(Q)\cap\Gamma_1(Q)$, and $X\notin T_{4r}(Q)$, the following holds*

$$\tilde{g}(X,Y)\frac{G(B(Y))}{\delta(Y)^2}\approx\omega^X\big(\Delta_r(Q)\big).$$

Proof. To show the first inequality, consider a test function $\varphi=1$ on $B_{r/2}(Q)$ and supported in $B_r(Q)$. From Lemma 1 and (0.2) we have

$$\omega\big(\Delta_r(Q)\big)\lesssim\int_{\partial D}\varphi\,d\omega^X=\int_D g(X,Y)L\varphi(Y)\,dY\lesssim\frac{1}{r^2}\int_{T_r(Q)}\tilde{g}(X,Y)G(Y)\,dY.$$

Since $\tilde{g}(X,Y)$ is a nonnegative normalized adjoint solution which vanishes on ∂D, and G is an adjoint solution for L, the first half of the lemma follows from (1.3), the Harnack inequality for normalized adjoint solutions, and (1.4).

To show the second inequality set $u(X)=(1/r^2)\int_{T_r(Q)} g(X,Y)\,dY$. This function satisfies $Lu=-(1/r^2)\chi_{T_r(Q)}$ on D, $u=0$ on ∂D, where $\chi_{T_r(Q)}$ is the characteristic function of $T_r(Q)$. Because D is a Lipschitz domain, there exist two truncated cones with a common axis $V_1(0)$ and $V_2(0)$ in $\mathbf{R}^n$, with vertex at 0 and whose opening and height only depend on the Lipschitz character of D, such that $\overline{V}_1(0)\setminus\{0\}\subset V_2(0)$, and satisfying that for all $Q\in\partial D$ there are two cones $V_1(Q)$ and $V_2(Q)$ which are congruent with $V_1(0)$ and $V_2(0)$ respectively, such that $\overline{V}_2(Q)\cap\overline{D}=\{Q\}$.

Let $h=2r_0$ denote the height of $V_2(0)$ and $\varkappa(Q)$ a unit vector in the direction of the interior axis of $V_1(Q)$. For fixed $r\leq h/2$ and $Q\in\partial D$ let $\zeta(Q)=Q-r\varkappa(Q)$. It is easy to see that there is a number $\sigma\in(0,1)$ which depends only on D so that $B_{\sigma r}(\zeta(Q))\subset\overline{V}_1(Q)$. We introduce the following auxiliary functions, $v(X)=e^{-\alpha}-e^{-\alpha(|X-\zeta(Q)|/\sigma r)^2}$ and $h(X)=v(X)^{1/2}$, where $\alpha>0$ is to be chosen. An easy calculation shows that we can choose α depending on λ, n and σ so that $Lv(X)\leq 0$ on $\mathbf{R}^n\setminus B_{\sigma r}(\zeta(Q))$, $h>0$ on $\overline{D}$, $Lh\leq 0$ in D, and $Lh\leq-\beta/r^2$ on $T_r(Q)$, where β depends on λ, n and D. Hence, $L[h-\beta u](X)\leq 0$ on D and $h-\beta u\geq 0$ on ∂D. Thus, the maximum principle implies that $h-\beta u\geq 0$ on D. In particular $u(X)\lesssim 1$ for all $X\in\partial B_r(Q)\cap D$. From (1.1), we get $1\lesssim\omega^X(\Delta_{2r}(Q))$ for all $X\in\partial B_r(Q)\cap D$. The last two estimates and the maximum principle applied to $C\omega^X(\Delta_{2r}(Q))-u(X)$ on $D\setminus B_r(Q)$ imply that $u(X)\lesssim\omega^X(\Delta_{2r}(Q))$ for all $X\notin T_r(Q)$, and from Lemma 1, $u(X)\lesssim\omega^X(\Delta_r(Q))$ for all $X\notin T_{4r}(Q)$. On the other hand, (1.4) and the Harnack inequality for normalized adjoint solutions show that for $Y\in\partial B_r(Q)\cap\Gamma_1(Q)$ and $X\notin T_{4r}(Q)$, $\tilde{g}(X,Y)(G(B(Y)))/(\delta(Y)^2)\lesssim u(X)$, which proves the lemma.

Lemma 3. *Let u satisfy $Lu=0$ in D, ϕ be a test function supported in D, and $v\in L^1_{\mathrm{loc}}(D)$ be a nonnegative superadjoint solution for L in D. Then for any constant β the following holds*

$$\int|\nabla u(X)|^2\phi^2(X)v(X)\,dX\lesssim\int|u(X)-\beta|^2\left[|\nabla\phi(X)|^2+|D^2\phi(X)|\right]v(X)\,dX.$$

Proof. The above estimate follows from the identities $L(u-\beta)^2=2\langle A\nabla u,\nabla u\rangle$, $L[(u-\beta)^2\phi^2]=(u-\beta)^2L\phi^2+\phi^2L(u-\beta)^2+8\phi(u-\beta)\langle A\nabla u,\nabla\phi\rangle$, and integration by parts.

Lemma 4. *Let u satisfy $Lu=0$ in D and α and β be positive numbers with $\alpha<\beta$. Then, there are constants C and c depending on λ,α,β,n, the A_∞ constants of μ, and the Lipschitz character of D such that for all $\gamma>C$ and $t>0$ the following* *holds*

$$\mu\big(\{Q\in\partial D: S_\alpha(u)(Q)>\gamma t, N_\beta(u)(Q)\le t\}\big)\le Ce^{-\gamma^2 c}\mu\big(\{Q\in\partial D: S_{2\alpha}(u)(Q)>t\}\big).$$

Once this lemma has been established, the first half of Theorem 1 follows from standard arguments, and the fact that L^p norms of area functions of different apertures with respect to a given doubling measure are equivalent. In order to prove this lemma, we adopt the strategies in [Z], and [BM].

We may assume without loss of generality that $t=1$. Define

$$E_u=\{Q\in\partial D: N_\beta(u)(Q)\le 1\}\quad\text{and}\quad \mathbf{R}_\gamma(H)=\bigcup\{\Gamma_\gamma(Q): Q\in H\}$$

for $\gamma>0$ and $H\subset\partial D$. We now introduce a measure Λ_u on D, which is defined by

$$\Lambda_u(F)=\int_{\mathbf{R}_\alpha(E_u)\cap F} g(0,X)|\nabla u(X)|^2 G(X)\,dX,\quad \text{for } F\subset D \text{ Borel set.}$$

We recall that a positive measure Λ on D is a Carleson measure with respect to a given measure σ on ∂D if

$$\|\Lambda\|=\sup\left\{\frac{\Lambda(T_r(Q))}{\sigma(\Delta_r(Q))}: r>0, Q\in\partial D\right\}<\infty.$$

Lemma 5. *Under the above assumptions, the measure Λ_u is a Carleson measure with respect to ω, whose Carleson norm $\|\Lambda_u\|$ is bounded by a constant which is independent of u, but depends on α,β,D,λ, and n.*

To see that the above statement holds first let r be such that $0<5r<r_0$ and $Q_0\in\partial D$ be fixed, where r_0 is as in Lemma 4. Let δ^* be the regularized distance function such that $\delta(X)\approx\delta^*(X)$, $|\nabla\delta^*(X)|\lesssim 1$, and $|D^2\delta^*(X)|\lesssim\delta(X)^{-1}$ for all X in D (see Theorem 6.2 of [St]). Now, for $\varepsilon>0$, let $D_\varepsilon=\{X\in D:\delta(X)>\varepsilon\}$ and $\Omega_\varepsilon=\{X\in D:\delta^*(X)>\varepsilon\}$. Let $\varphi\in C_0^\infty(\mathbf{R}^n)$ be such that $\varphi(X)=\varphi(|X|)$, $\operatorname{spt}(\varphi)\subset B_1(0)$, and its integral over $\mathbf{R}^n$ is equal to 1. For $\varepsilon>0$ and small we define a function ϕ_ε as

$$\phi_\varepsilon(X)=\int_W \big(\tau\delta^*(X)\big)^{-n}\varphi\left(\frac{X-Y}{\tau\delta^*(X)}\right)dY,\quad X\in\mathbf{R}^n,$$

where W is the subset of D given by $W=\mathbf{R}_{(\alpha+\beta)/2}(E_u\cap\Delta_{(2+\alpha)r}(Q_0))\cap D\setminus D_{2r}\cap\Omega_{\varepsilon/2}$. For τ sufficiently small, depending only on the Lipschitz character of D,α, and β, the functions ϕ_ε turns out to be a smoothing of the characteristic function of the set $\mathbf{R}_\alpha(E_u\cap\Delta_{(2+\alpha)r}(Q_0))\cap D\setminus D_r\cap\Omega_\varepsilon$, which contains $R_\alpha(E_u)\cap T_r(Q_0)\cap\Omega_\varepsilon$. Indeed, $\phi_\varepsilon\in C_0^\infty(D)$ satisfies the properties that

$$\phi_\varepsilon(X)=1\quad\text{for } X\in\mathbf{R}_\alpha(E_u\cap\Delta_{(2+\alpha)r}(Q_0))\cap D\setminus D_r\cap\Omega_\varepsilon,$$

$$\operatorname{spt}(\phi_\varepsilon)\subset\mathbf{R}_\beta(E_u\cap\Delta_{(2+\alpha)r}(Q_0))\cap D\setminus D_{3r}\cap\Omega_{\varepsilon/4}\subset D.$$

Moreover, $|\nabla\phi_\varepsilon(X)|\lesssim\delta(X)^{-1}$, $|D^2\phi_\varepsilon(X)|\lesssim\delta(X)^{-2}$ for all X in D.

Since $\Lambda_u(T_r(Q_0))=\lim_{\varepsilon\to 0}\Lambda_u(T_r(Q_0)\cap\Omega_\varepsilon)$ it suffices to show that $\Lambda_u(T_r(Q_0)\cap\Omega_\varepsilon)\lesssim\omega(\Delta_r(Q_0))$ independently of $\varepsilon>0$. From the above remarks, Lemma 3, and observing that $|u|\le 1$ on $\mathbf{R}_\beta(E_u)$, we conclude that for ε small

$$\Lambda_u\big(T_r(Q_0)\cap\Omega_\varepsilon\big)\lesssim\int_{W_0}\delta(X)^{-2}g(0,X)\,dX,$$

where W_0 denotes the union of the supports of $\nabla\phi_\varepsilon$ and $D^2\phi_\varepsilon$. To estimate the right hand side of the last inequality, let $\varkappa>0$ be a fixed number such that $\varkappa\le\min\{\frac{1}{2},2\alpha/(\alpha+3),2(\beta-\alpha)/3(2\beta+3)\}$. Let $\{I_j\}$ be a Whitney decomposition of D so that $\{I_j\}$ is a family of dyadic cubes in $\mathbf{R}^n$ with disjoint interiors and $n^{1/2}/\varkappa l(I_j)\le d(I_j,\partial D)\le 4n^{1/2}/\varkappa l(I_j)$, where $l(I_j)$ is the side length of the cube I_j. Let X_j be the center of I_j, $r_j=\frac{1}{2}n^{1/2}l(I_j)$, $B_j=B_{r_j}(X_j)$, and $B_j^*=B_{2r_j}(X_j)$. Let $Q_j\in\partial D$ be such that $|X_i-Q_j|=\delta(X_j)$, and $\Delta_j=\Delta_{r_j}(Q_j)$.

Let $J=\{j\in N:I_j\cap W_0\ne\phi\}$. Then for each $j\in J$, we have $\delta(X_j)\approx r_j$, and by Harnack's inequality for normalized adjoint solutions $\tilde g(0,X)\approx\tilde g(0,X_j)$ for $X\in B_j$. On the other hand, from (1.4) we have $G(B(X))\approx G(B(X_j))$ for $X\in B_j$. These and Lemma 2 imply

$$\int_{I_j}\delta(X)^{-2}g(0,X)\,dX\lesssim\omega(\Delta_j).$$

We now use the following fact: there are constants c and C, which are independent of u,r, and ε, such that

$$\sum_{j\in J}\chi_{\Delta_j}\le C\chi_{\Delta_{cr}(Q_0)}. \tag{1.5}$$

Assuming that this holds, we obtain

$$\Lambda_u\big(T_r(Q_0)\cap\Omega_\varepsilon\big)\lesssim\sum_{j\in J}\omega(\Delta_j)\lesssim\omega\big(\Delta_{cr}(Q_0)\big)\lesssim\omega\big(\Delta_r(Q_0)\big),$$

as required. We refer the reader to [Z] for the proof of (1.5).

When $5r\ge r_0$, we have from the doubling property of L-harmonic measure that $1\lesssim\omega(\Delta_r(Q_0))$. Thus, to show that $\Lambda_u(T_r(Q_0))\lesssim\omega(\Delta_r(Q_0))$ it suffices to check that $\Lambda_u(D)\lesssim 1$. From the previous case and by covering $D\setminus D_{r_0/10}$ with finitely many balls of radius $r_0/5$ centered at points of ∂D we have $\lambda_u(D\setminus D_{r_0/10})\lesssim 1$. On the other hand, $D_{r_0/10}$ is "large" and there are at most M cubes I_j which touch $D_{r_0/10}$. Moreover, if $J=\{j\in N:I_j\cap D_{r_0/10}\ne\phi\}$ we have $r_0\lesssim r_j$ for $j\in J$,

$$\mathbf{R}_\alpha(E_u)\cap D_{r_0/10}\subset\bigcup_{j\in J}B_j\quad\text{and}\quad\bigcup_{j\in J}B_j^*\subset\mathbf{R}_\beta(E_u)\subset\big\{X\in D:|u(X)|\le 1\big\}.$$

Now, for fixed $j\in J$, let $\phi_j\in C_0^\infty(D)$ with $\phi_j=1$ on B_j, supported in B_j^*, and $|\nabla\phi_j|+|D^2\phi_j|\lesssim 1$. Thus, from Lemma 3 we get

$$\int_{B_j} g(0,X)|\nabla u(X)|^2\,dX \lesssim \int_{B_j^*} g(0,X)\,dX.$$

The above estimate, (0.1) and the fact that J has no more than M elements concludes the proof of Lemma 5.

At this point we will need to recall the definition of the space of functions of bounded mean oscillation on ∂D. Let σ denote a doubling measure on ∂D. A function $f\in L^1(d\sigma)$ is said to lie in BMO$(d\sigma)$ provided

$$\|f\|_*=\sup_{\Delta}\inf_{a\in R}\frac{1}{\sigma(\Delta)}\int_{\Delta}|f(Q)-a|\,d\sigma<\infty.$$

The following lemmas will be useful for us.

Lemma 6. *Let μ and σ be doubling measures on ∂D with $\mu\in A_\infty(d\sigma)$, and $f\in$BMO$(d\sigma)$ with $\|f\|_*+\|f\|_{L^1(d\sigma)}\leq 1$. Then, there are constants C and c, depending on the A_∞ constants of μ and the doubling constant of σ, such that for all $t>3$*

$$\mu(\{Q\in\partial D:|f(Q)|>t\})\leq Ce^{-tc}\mu(\{Q\in\partial D:|f(Q)|>1\}).$$

For fixed $Q,P\in\partial D$ and $X\in D$, let $l_{Q,P}(X)=\min\{|X-Q|,|X-P|\}$, and $\Delta(Q,P,X)=\Delta_{l_{Q,P}(X)}(X_{Q,P})$, where $X_{Q,P}$ is one of the points P or Q such that $|X-X_{Q,P}|=l_{Q,P}(X)$.

Lemma 7. *Let σ be as above, and $K(X,Q)$ be a continuous function on $D\times\partial D$ which satisfies the following conditions: There exists a positive constant C such that*

$$\sup_{X\in D}\int_{\partial D}|K(X,Q)|\,d\sigma\leq C \tag{1.6}$$

and

$$|K(X,Q)-K(X,P)|\leq\frac{C}{\sigma(\Delta(Q,P,X))}\left(\frac{|Q-P|}{l_{Q,P}(X)}\right) \tag{1.7}$$

for all $Q,P\in\partial D$ and $X\in D$ with $l_{Q,P}(X)>2|Q-P|$. Assume that Λ is a Carleson measure with respect to σ, then the function $K\Lambda$ defined by

$$K\Lambda(Q)=\int_D K(X,Q)\,d\Lambda(X),\quad Q\in\partial D$$

is in $\mathrm{BMO}(d\sigma)$ *with* $\|K\Lambda\|_* + \|K\Lambda\|_{L^1(d\sigma)} \le M\|\Lambda\|$, *for some constant* M *which depends only on* C, D, *and the doubling constant of* σ.

The reader can find the proof of the above results in [Z], Propositions 1 and 2, where they are proved in the context of NTA domains, and where σ is the harmonic measure on D for the Laplace operator.

Let now α and β be as in Lemma 4 and fixed $\psi, \varphi \in C_0^\infty(R^n)$ with $\varphi = 1$ on $D \setminus D_{r_0/10}$, $\varphi = 0$ on $D_{r_0/5}$, $\psi = 1$ for $|X| \le 1+\alpha$, $\psi = 0$ for $|X| \ge 1+2\alpha$, $0 \le \psi$, $\varphi \le 1$, and define a function K on $D \times \partial D$ by

$$K(X,Q) = \varphi(X)\psi\left(\frac{|X-Q|}{\delta(X)}\right)\frac{\delta(X)^2}{G(B(X))}\frac{1}{\tilde g(0,X)}, \quad X \in D,\ Q \in \partial D.$$

We claim that the above function K satisfies the conditions in Lemma 7 with $\sigma = \omega$, and hence by Lemma 6, $K\Lambda_u$ lies in $\mathrm{BMO}(d\omega) \cap L^1(d\omega)$ with $\|K\Lambda_u\|_* + \|K\Lambda\|_{L^1(d\omega)} \lesssim 1$. If $\widetilde X \in \partial D$ is such that $|X - \widetilde X| = \delta(X)$, we have

$$\int_{\partial D} K(X,Q)\, d\omega \lesssim \frac{\delta(X)^2}{G(B(X))}\frac{1}{\tilde g(0,X)}\,\omega(\{Q \in \partial D : X \in \Gamma_{2\alpha}(Q)\}).$$

From Lemma 2 and the fact that the set $\{Q \in \partial D : X \in \Gamma_{2\alpha}(Q)\}$ is contained in $\Delta_{(2+2\alpha)\delta(X)}(\widetilde X)$, the right hand side of the above inequality is essentially bounded by 1. Hence, condition (1.6) holds. To check condition (1.7), let $X \in D \setminus D_{r_0/5}$ and $Q, P \in \partial D$ with $|X-Q|, |X-P| > 2|P-Q|$. Interchanging the roles of P and Q we may assume that $|X-Q| \le |X-P|$. In this case, $l_{Q,P}(X) = |X-Q|$ and $X_{Q,P} = Q$. Then we must consider three cases:

I. $|X-Q| \ge (1+2\alpha)\delta(X)$ and $|X-P| \ge (1+2\alpha)\delta(X)$. In this case we have $K(X,Q) = K(X,P) = 0$.

II. $|X-Q| < (1+2\alpha)\delta(X)$ and $|X-P| \ge (1+2\alpha)\delta(X)$. Then

$$\begin{aligned}|K(X,Q)-K(X,P)| &\le \frac{\delta(X)^2}{G(B(X))}\frac{1}{\tilde g(0,X)}\left|\psi\left(\frac{|X-Q|}{\delta(X)}\right) - \psi\left(\frac{|X-P|}{\delta(X)}\right)\right| \\ &\lesssim \frac{\delta(X)^2}{G(B(X))}\frac{1}{\tilde g(0,X)}\frac{|P-Q|}{\delta(X)} \lesssim \frac{1}{\omega(\Delta_{|X-Q|}(Q))}\frac{|Q-P|}{|X-Q|},\end{aligned}$$

where in the last inequality we used that $|X-Q| \approx \delta(X)$ and Lemma 2.

III. $|X-Q| < (1+2\alpha)\delta(X)$ and $|X-P| < (1+2\alpha)\delta(X)$. In this case the argument above goes through in the same way.

We now observe that for $\gamma \gtrsim 1$, we have

$$\begin{aligned}(1.8)\quad &\{Q \in \partial D : S_\alpha(u)(Q) > \gamma, N_\beta(u)(Q) \le 1\} \\ &\qquad \subset \{Q \in \partial D : \widetilde S_\alpha(u)(Q) > \gamma/2, N_\beta(u)(Q) \le 1\},\end{aligned}$$

where

$$\widetilde{S}_\alpha(u)(Q)^2=\int_{\Gamma_\alpha(Q)}\frac{\delta(X)^2}{G(B(X))}\left|\nabla v(X)\right|^2 G(X)\varphi(X)\,dX.$$

To see this, observe that since $G(B_1(0))\approx 1$ ([B2]), we have from the doubling property of G, that for $X\in\Gamma_\alpha(Q)\cap\operatorname{spt}(1-\varphi)$, $G(B(X))\approx 1$. Also, $\Gamma_\alpha(Q)\cap\operatorname{spt}(1-\varphi)$ can be covered by a finite number of balls $\{B_j:j=1,\dots,M\}$, with radius of size roughly equal to 1 and whose double concentric balls B_j^* are contained in $\Gamma_\beta(Q)$. Here M depends on α,β, and D. These and Lemma 3 show that (1.8) holds.

Now, we observe that for $Q\in E_u$, $\widetilde{S}_\alpha(u)(Q)^2\leq K\Lambda_u(Q)$, and $K\Lambda_u(Q)\leq S_{2\alpha}(u)(Q)^2$ for all $Q\in\partial D$. Thus, from Lemma 6 and for $\mu\in A_\infty(d\omega)$

$$\begin{aligned}\mu\big(\{Q\in\partial D:S_\alpha(u)(Q)>\gamma,N_\beta(u)(Q)\leq 1\}\big)&\leq\mu\big(\{Q\in E_u:\widetilde{S}_\alpha(u)(Q)>\gamma/2\}\big)\\ \leq\mu\big(\{Q\in\partial D:K\Lambda_u(Q)>\gamma^2/4\}\big)&\leq Ce^{-\gamma^{2c}}\mu\big(\{Q\in\partial D:S_{2\alpha}(u)(Q)>1\}\big).\end{aligned}$$

This concludes the proof of Lemma 4.

Lemma 8. *Let u satisfy $Lu=0$ in D, $u(0)=0$, $\|u\|_{L^2(d\omega)}\leq 1$, and α and β be positive numbers with $\alpha<\beta$, with β sufficiently large so that $0\in\Gamma_\beta(Q)$ for all $Q\in\partial D$. Then, there are constants C and c, depending on λ,α,β,n, the A_∞ constants of μ, and the Lipschitz character of D such for all $\gamma>C$ and $t>0$ the following holds*

$$\begin{aligned}&\mu\big(\{Q\in\partial D:N_\alpha(u)(Q)>\gamma t,S_\beta(u)(Q)\leq t,M_\mu(\chi_{G_t})\leq\tfrac12\}\big)\\&\qquad\leq Ce^{-\gamma c}\mu\big(\{Q\in\partial D:N_\alpha(u)(Q)>t\}\big),\end{aligned}$$

where $G_t=\{Q\in\partial D:S_\beta(u)(Q)>t\}$.

The proof of this lemma will follow from the techniques used in [DJK] and [MU] after we have shown the following Poincaré type inequality.

Lemma 9. *Let u satisfy $Lu=0$ on $B_{2r}(X_0)\subset D$. Then,*

$$\sup_{B_r(X_0)}\left|u(X)-u(X_0)\right|^2\lesssim\frac{r^2}{G(B_{2r}(X_0))}\int_{B_{2r}(X_0)}|\nabla u(X)|^2G(X)\,dX. \tag{1.9}$$

Proof. From (0.2) we have

$$\int_{\partial B_{2r}(X_0)}\big(u(Q)-u(X_0)\big)^2\,d\omega_{2r}^{X_0}=\int_{B_{2r}(X_0)}g_{2r}(X_0,X)2\langle A\nabla u,\nabla u\rangle\,dX, \tag{1.10}$$

where $\omega_{2r}^{X_0}$ and $g_{2r}(X_0,\cdot)$ denote respectively the L-harmonic measure and Green's function for L in $B_{2r}(X_0)$. Let k be an integer to be fixed later. The right hand side of (1.10) can be bounded by

$$\int_{B_{2r/2^k}(X_0)} \big(g_{2r}(X_0,X)-g_{2r/2^k}(X_0,X)\big)2\langle A\nabla u,\nabla u\rangle\,dX \\ +\int_{B_{2r}(X_0)\setminus B_{2r/2^k}(X_0)} g_{2r}(X_0,X)2\langle A\nabla u,\nabla u\rangle\,dX \\ +\sup_{B_{2r/2^k}(X_0)}\big|u(X)-u(X_0)\big|^2=\mathrm{I}+\mathrm{II}+\mathrm{III},$$

where we used the analogue of identity (1.10) over $B_{2r/2^k}(X_0)$ to handle the last term. We have from the regularity for solutions to $Lu=0$, that for some $\theta\in(0,1)$ depending only on λ, and n ([GT], Chapter 9),

$$\mathrm{III}\le\theta^k\big(\operatorname*{osc}_{B_r(X_0)}u\big)^2\le 2\theta^k\sup_{B_r(X_0)}\big|u(X)-u(X_0)\big|^2.$$

On the other hand, Harnack's inequality for solutions to $Lu=0$ implies that for $X\in B_r(X_0)$ the measures $\omega_{2r}^{X_0}$ and ω_{2r}^{X} are mutually absolutely continuous, and the Radon–Nikodym derivative of ω_{2r}^{X} with respect to $\omega_{2r}^{X_0}$ is essentially bounded by 1. Since

$$u(X)-u(X_0)=\int_{\partial B_{2r}(X_0)}\big(u(Q)-u(X_0)\big)\,d\omega_{2r}^{X}\quad\text{for all } X\in B_r(X_0),$$

we obtain from Schwarz's inequality

$$\sup_{B_r(X_0)}\big|u(X)-u(X_0)\big|^2\lesssim\int_{\partial B_{2r}(X_0)}\big(u(Q)-u(X_0)\big)^2\,d\omega_{2r}^{X_0}.$$

Therefore, choosing k large enough we have

$$\mathrm{III}\le\frac{1}{2}\int_{\partial B_{2r}(X_0)}\big(u(Q)-u(X_0)\big)^2\,d\omega_{2r}^{X_0}.$$

To control I and II, we observe that $\tilde g_{2r}(X_0,\cdot)-\tilde g_{2r/2^k}(X_0,\cdot)$ is a normalized adjoint solution for L on $B_{2r/2^k}(X_0)$. Thus, its maximum on $B_{2r/2^k}(X_0)$ is attained on $\partial B_{2r/2^k}(X_0)$, where $\tilde g_{2r}(X_0,\cdot)-\tilde g_{2r/2^k}(X_0,\cdot)=\tilde g_{2r}(X_0,\cdot)$. On the other hand, Harnack's inequality for normalized adjoint solutions and (1.4) show that for $X\in\partial B_{2r/2^k}(X_0)$ we have

$$\tilde g_{2r}(X_0,X)\lesssim\frac{1}{G(B_{2r}(X_0))}\int_{B_r(X_0)} g_{2r}(X_0,X)\,dX\lesssim\frac{r^2}{G(B_{2r}(X_0))},$$

where in the last inequality we used (0.1). Analogously, $\tilde{g}_{2r}(X_0,\cdot)$ attains its maximum over $B_{2r}(X_0)\backslash B_{2r/2^k}(X_0)$ somewhere on $\partial B_{2r/2^k}(X_0)$. Thus, dividing and multiplying the integrands in I and II by $G(X)$, we bound I and II by the right hand side of (1.9). Hence from (1.11) and the fact that

$$\sup_{B_r(X_0)} \left|u(X)-u(X_0)\right|^2 \lesssim \int_{\partial B_{2r}(X_0)} \left(u(Q)-u(X_0)\right)^2 d\omega_{2r}^{X_0},$$

we obtain (1.9).

Let α and β be as in Lemma 8, $Q_0\in\partial D$, $0<5r<r_0$, and $E\subset\Delta_r(Q_0)$ be a closed set. It is well known that one can construct a "sawtooth" region $\Omega=\Omega(E,Q_0,r)$ over E. The properties of Ω are that it is a Lipschitz domain satisfying

(i) For suitable α',α'',c_1,c_2, with $\alpha<\alpha'<\alpha''<\beta$

$$\bigcup\left\{\Gamma_{\alpha'}(Q)\cap B_{c_1 r}(Q):Q\in E\right\}\subset\Omega\subset\bigcup\left\{\Gamma_{\alpha''}(Q)\cap B_{c_2 r}(Q):Q\in E\right\}.$$

(ii) $\partial\Omega\cap\partial D=E$.

(iii) There exists $X_0\in\Omega$ with $d(X_0,\partial\Omega)\approx r$. In particular, there is a ball $B_{8cr}(X_0)\subset\Omega$ for some $c>0$ depending on D.

(iv) The Lipschitz constant of Ω depends only on D.

Next, we observe that in the proof of the "main lemma" in [DJK], the only necessary tools are the doubling property, and properties (1.1) and (1.2) of the harmonic measure associated to a divergence form elliptic operator with measurable coefficients. Since in our case these properties also hold, the main lemma in [DJK] also holds for L-harmonic measures. In particular, if Ω denotes the sawtooth region associated above to a closed set $E\subset\Delta_r(Q_0)$ for some $Q_0\in\partial D$, $0<5r<r_0$, and ν denotes the L-harmonic measure for Ω at X_0, we will have as in the proof of this lemma in [DJK], that if $\{I_j\}$ is a suitable Whitney decomposition of $\Delta_{2r}(Q_0)\backslash E$, that is, a disjoint family of dyadic surface caps I_j obtained from the dyadic family associated to $\Delta_{2r}(Q_0)$, whose union is $\Delta_{2r}(Q_0)\backslash E$ and whose distances to E are comparable to their respective diameters $l(I_j)$, and we define a measure $\tilde{\nu}$ on $\Delta_{2r}(Q_0)$ as

$$\tilde{\nu}(F)=\nu(E\cap F)+\sum_j \frac{\omega(F\cap I_j)}{\omega(I_j\cap\partial D)}\,\nu(Q_j\cap\partial\Omega),\quad \text{for } F\subset\Delta_{2r}(Q_0) \text{ Borel set,} \tag{1.12}$$

where Q_j is a cube in $\mathbf{R}^n$ centered at a point on "the lateral side" of $\partial\Omega\backslash E$, whose diameter and whose distance from ∂D are comparable to the diameter of I_j (Q_j is essentially located right above I_j and at distance $l(I_j)$ from I_j), that for some $\theta\in(0,1]$ depending only on λ,n, and D the following holds

$$\frac{\omega(F)}{\omega(\Delta)}\lesssim\left(\frac{\tilde{\nu}(F)}{\tilde{\nu}(\Delta)}\right)^{\theta}\quad \text{for all } F\subset\Delta_r(Q_0) \text{ Borel set.} \tag{1.13}$$

Before we start with the proof of Lemma 8 we will also need to recall that from the estimates on the kernel function associated to L on the Lipschitz domain Ω given in [FGMS] in Theorem I.2.5, and the arguments in Theorem 4.3 in [CFMS] it follows that if u is L-harmonic in Ω, the nontangential maximal function of u in Ω at each point $Q\in\partial\Omega$ is controlled from above by the Hardy–Littlewood maximal function with respect to ν of the boundary values of u on $\partial\Omega$. In particular,

$$\sup\{|u(X)|: X\in\Omega \text{ and } |X-Q|\le(1+\alpha)\tilde\delta(X)\}\lesssim M_\nu(u_{|\partial\Omega})(Q) \tag{1.14}$$

for all $Q\in\partial\Omega$, where $\tilde\delta(X)$ denotes the distance from X to $\partial\Omega$.

After all these remarks we will proceed with the proof of Lemma 8: We may assume that $t=1$. Let $\mu\in A_\infty(d\omega)$, α and β be as in Lemma 8, and set $F_u=\{Q\in\partial D: S_\beta(u)(Q)\le 1,\ M_\mu(\chi_{G_t})\le\frac12\}$, and $W=R_{(\alpha+\beta)/2}(F_u)$. We introduce the following function

$$N(u)(Q)=\sup\{|u(X)|: X\in\Gamma_\alpha(Q)\cap W\},\quad Q\in\partial D.$$

We will show that $N(u)\in\mathrm{BMO}(d\omega)$ with $\|N(u)\|_*\lesssim 1$. Since by hypothesis $\|u\|_{L^2(d\omega)}\le 1$, we get from (1.14) and the fact that the Hardy–Littlewood maximal function with respect to ω is bounded in $L^2(d\omega)$ that $\|N(u)\|_{L^1(d\omega)}\lesssim 1$. Since $N(u)\le N_\alpha(u)$ on ∂D, and $N_\alpha(u)\le N(u)$ on F_u, Lemma 8 will follow from Lemma 6.

To show the above claim, let $Q_0\in\partial D$ and $0<20r<r_0$ be fixed. We introduce the following functions

$$N^r(u)(Q)=\sup\{|u(X)|: X\in\Gamma_\alpha^r(Q)\},\quad \text{where } \Gamma_\alpha^r(Q)=\{\Gamma_\alpha(Q)\cap W, \delta(X)\ge\tau r\},$$
$$N_r(u)(Q)=\sup\{|u(X)|: X\in\Gamma_{r,\alpha}(Q)\},\quad \text{where } \Gamma_{r,\alpha}(Q)=\{\Gamma_\alpha(Q)\cap W, \delta(X)<\tau r\},$$

and

$$\widetilde N(u)(Q)=\sup\{|u(X)-u(X_0)|: X\in\Gamma_{r,\alpha}(Q)\}$$

and where τ has been chosen so that if Ω is the sawtooth region associated to $E=F_u\cap\overline\Delta_r(Q_0)$, and $Q\in E$, then $\Gamma_{r,\alpha}(Q)\subset\Omega\setminus B_{cr}(X_0)$.

Observe that if $Q\in\Delta_r(Q_0)$, and $X\in\Gamma_\alpha^r(Q)$, with $s=\delta(X)\le\frac{1}{20}r_0$, and $P\in F_u$ is such that $X\in\Gamma_{(\alpha+\beta)/2}(P)$, then there is a point $\widetilde X\in\Gamma_\alpha^r(Q_0)$ with $\delta(\widetilde X)\approx\delta(X)$, and such that $\widetilde X\in\Gamma_{(\alpha+\beta)/2}(P)$. It is well known that there exists a sequence $X=X_1,X_2,\dots,X_N=\widetilde X$ such that

$$B_{\eta s}(X_j)\subset\Gamma_{(\alpha+\beta)/2}(P),\quad \delta(X_j)\approx s,$$

and $|X_j-X_{j+1}|\le\frac12\eta s$ for all j, where N and η depend only on the Lipschitz character of D,α and β. These, (1.4) and Lemma 9 imply

$$|u(X_j)-u(X_{j+1})|^2\lesssim\frac{s^2}{G(B_{\eta s}(X_j))}\int_{B_{\eta s}(X_j)}|\nabla u(X)|^2G(X)\,dX\lesssim S_\beta(u)(P)^2\lesssim 1.$$

Thus, $|u(X)|\le|u(\widetilde{X})|+C$, where C is a constant which depends on D,α, and β. On the other hand, when $\delta(X)\ge\frac{1}{20}r_0$ we can find a sequence of points $X=X_1,X_2,\dots,X_N=0$, such that $B_\eta(X_j)\subset\Gamma_{(\alpha+\beta)/2}(P)$, and $|X_j-X_{j+1}|\le\frac{1}{2}\eta$, where N and η are as above. Again, Lemma 9, (1.4), and the fact that $G(B_1(0))\approx1$ imply $|u(X)|\le|u(0)|+C=C$. Therefore, we just showed that $|u(X)|\le N^r(u)(Q_0)+C$ for all $X\in\Gamma^r_\alpha(Q)$, and $Q\in\Delta_r(Q_0)$. A similar argument shows that

$$|u(X)|\le N^r(u)(Q)+C \quad \text{for all } X\in\Gamma^r_\alpha(Q),$$

and $Q\in\Delta_r(Q_0)$. Hence,

$$\left|N^r(u)(Q)-N^r(u)(Q_0)\right|\lesssim 1 \quad \text{for all } Q\in\Delta_r(Q_0).$$

Let now θ be the exponent in (1.13). From the above inequality we get

$$\begin{aligned}\inf_{a\in\mathbf{R}}\int_{\Delta_r(Q_0)}&\left|N(u)-a\right|^\theta d\omega\\ &\lesssim \inf_{a\in\mathbf{R}}\int_{\Delta_r(Q_0)}\left|\max\{N_r(u),N^r(u)(Q_0)\}-a\right|^\theta d\omega+\omega(\Delta_r(Q))\\ &\le \inf_{a\in\mathbf{R}}\int_{\Delta_r(Q_0)}\left|N_r(u)-a\right|^\theta d\omega+\omega(\Delta_r(Q_0))\\ &\le \int_{\Delta_r(Q_0)}\left|N_r(u)-|u(X_0)|\right|^\theta d\omega+\omega(\Delta_r(Q_0))\\ &\le \int_{\Delta_r(Q_0)}\left|\widetilde{N}(u)\right|^\theta d\omega+\omega(\Delta_r(Q_0)).\end{aligned}$$

Next, we claim that

$$\omega(\{Q\in\Delta_r(Q_0):\widetilde{N}(u)(Q)>s\})\lesssim s^{-2\theta}\omega(\Delta_r(Q_0)). \tag{1.15}$$

Assuming this claim we get that the above integral is bounded by $\omega(\Delta_r(Q_0))$. Therefore,

$$\inf_{a\in\mathbf{R}}\frac{1}{\omega(\Delta_r(Q_0))}\int_{\Delta_r(Q_0)}\left|N(u)-a\right|^\theta d\omega\lesssim 1 \quad \text{for all } 0<20r<r_0.$$

It then follows from the John–Nirenberg inequality [ST] that $N(u)\in\mathrm{BMO}(d\omega)$ with $\|N(u)\|_*\lesssim1$, which proves our first claim.

To show (1.15), let $\tilde{\nu}$ be the measure defined in $\Delta_{2r}(Q_0)$ in (1.12). Then, setting

$$H_s=\{Q\in\Delta_r(Q_0):\widetilde{N}(u)(Q)>s\},$$

we have

$$\tilde{\nu}(H_s)\leq s^{-2}\int_E \widetilde{N}(u)^2\,d\nu+\sum_j \frac{\omega(H_s\cap I_j)}{\omega(I_j\cap\partial D)}\,\nu(Q_j\cap\partial\Omega). \tag{1.16}$$

We now observe that if $H_s\cap I_j$ is nonempty, and $Q\in H_s\cap I_j$, we have $d(Q,E)\approx l(I_j)$, and if $P\in E$ is such that $|Q-P|\approx l(I_j)$, there is a point $X\in\Gamma_{r,\alpha}(Q)\cap\Gamma_{(\alpha+\beta)/2}(P)$ with $\delta(X)\approx l(I_j)$such that $|u(X)-u(X_0)|>s$. If β is then a sufficiently large multiple of α, it follows that on a surface cap around Q with diameter $\eta l(I_j)$, $\Delta_{\eta l(I_j)}(Q)$, we have $\widetilde{N}(u)(Q)>s$, where η depends on α,β and D. Thus, $\omega(H_s\cap I_j)\approx\omega(I_j\cap\partial D)$. On the other hand, since $|u(X)-u(X_0)|>s$, and $S_\beta(u)(P)\leq 1$, it follows from Lemma 9 that $|u(\cdot)-u(X_0)|>s-C$ on $Q_j\cap\partial\Omega$, for some C depending on α,β,λ, and D. Hence, we get from (1.16) that for $s>2C$

$$\tilde{\nu}(H_s)s^2\lesssim\int_E \widetilde{N}(u)^2\,d\nu+\sum_j\int_{Q_j\cap\partial\Omega}\left|u(Q)-u(X_0)\right|^2 d\nu.$$

Since the cubes Q_j have finite overlappings we have from (1.14) and (0.2)

$$\begin{aligned}\tilde{\nu}(H_s)s^2&\lesssim\int_E M_\nu\big((u-u(X_0))_{|\partial\Omega}\big)(Q)^2\,d\nu+\int_{\partial\Omega}\left|u(Q)-u(X_0)\right|^2 d\nu\\&\lesssim\int_{\partial\Omega}\left|u(Q)-u(X_0)\right|^2 d\nu=2\int_\Omega g_\Omega(X_0,Y)\langle A\nabla u,\nabla u\rangle\,dY,\end{aligned}\tag{1.17}$$

where $g_\Omega(\cdot\,,\cdot)$ denotes the Green's function for L in Ω. On the other hand,

$$1\geq\int_{\partial D\setminus G_1}S_\beta(u)(P)^2\,d\omega^{X_0}\geq\int_\Omega\frac{\delta(Y)^2}{G(B(Y))}\left|\nabla u(Y)\right|^2 G(Y)\varphi(Y)\,dY,$$

where $\varphi(Y)=\omega^{X_0}(\{P\in\partial D\setminus G_1 : Y\in\Gamma_\beta(Q)\})$. Setting $\Omega^r=\{X\in\Omega:\delta(X)\leq\tau r\}$, we have that for $Y\in\Omega^r$ there exists $\widetilde{Y}\in E\subset\partial D\setminus G_1$, such that $|Y-\widetilde{Y}|\approx\delta(Y)$ and $Y\in\Gamma_\beta(\widetilde{Y})$. The latter implies that for some η depending on α, and β, $\widetilde{\Delta}\cap\partial D\setminus G_1\subset\{P\in\partial D\setminus G_1 : Y\in\Gamma_\beta(Q)\}$, where $\widetilde{\Delta}=\Delta_{\eta\delta(Y)}(\widetilde{Y})$. Therefore,

$$\varphi(Y)\geq\omega^{X_0}(\widetilde{\Delta}\cap\partial D\setminus G_1)=\frac{\omega^{X_0}(\widetilde{\Delta}\cap\partial D\setminus G_1)}{\omega^{X_0}(\widetilde{\Delta})}\,\omega^{X_0}(\widetilde{\Delta}),$$

and from (1.2)

$$\frac{\omega^{X_0}(\widetilde{\Delta}\cap\partial D\setminus G_1)}{\omega^{X_0}(\widetilde{\Delta})}\approx\frac{\omega(\widetilde{\Delta}\cap\partial D\setminus G_1)}{\omega(\widetilde{\Delta})}. \tag{1.18}$$

By Lemma 2, $\omega^{X_0}(\widetilde{\Delta})\approx\tilde{g}(X_0,Y)G(B(Y))/\delta(Y)^2$. Recall that because $\widetilde{Y}\in E$, $\mu(\widetilde{\Delta}\cap\partial D\setminus G_1)/\mu(\widetilde{\Delta})\geq\frac{1}{2}$, and since μ is A_∞ with respect to ω we get from (1.18) $1\lesssim\omega^{X_0}(\widetilde{\Delta}\cap\partial D\setminus G_1)/\omega^{X_0}(\widetilde{\Delta})$. Hence,

$$1\gtrsim\int_{\Omega^r} g(X_0,Y)\left|\nabla u(Y)\right|^2 dY\geq\int_{\Omega^r} g_\Omega(X_0,Y)\left|\nabla u(Y)\right|^2 dY.$$

Next, from (0.2) and (0.1) we obtain

$$\int_{\Omega\setminus\Omega^r} g_\Omega(X_0,Y)\left|\nabla u(Y)\right|^2 dY\lesssim\sup_{\Omega\setminus\Omega^{r/2}}\left|u(\cdot)-u(X_0)\right|^2,$$

and an argument similar to the one we used to control the BMO norm on the upper part of $N(u)$ using Lemma 9, shows that the right hand side of the last inequality is essentially bounded by 1. In all, we get from (1.17)

$$\tilde{\nu}(H_s)\lesssim s^{-2}\quad\text{for } s\gtrsim 1,$$

and the above inequality together with (1.13) finishes the proof of our claim (1.15).

Remark. The assumption that $\|u\|_{L^2(d\omega)}\leq 1$ can actually be dropped. This follows from the fact that after proving Lemma 9, and if we had followed step by step the arguments in Lemma 2 of [DJK], we would have obtained for $t>0$ and $\gamma\gtrsim 1$

$$\mu\big(\{Q\in\partial D:N_\alpha(u)(Q)>\gamma t, S_\beta(u)(Q)\leq t, M_\mu(\chi_{G_t})\leq\tfrac{1}{2}\}\big)\\ \leq C\gamma^{-\theta}\mu\big(\{Q\in\partial D:N_\alpha(u)(Q)>t\}\big),$$

and under the assumption $\|S_\beta(u)\|_{L^p(d\mu)}\leq 1$ for some $p>0$, these would imply that for $t>0$, $\mu(\{Q\in\partial D:N_\alpha(u)(Q)>t\})\leq Ct^{-\theta'}$ for some θ' depending on α,β,D, and the A_∞ condition. Since the A_∞ condition is symmetric, [Mu], we would have $\omega(\{Q\in\partial D:N_\alpha(u)(Q)>t\})\leq Ct^{-\theta''}$ for a new θ''. Thus, $\omega(\{Q\in\partial D:N(u)(Q)>t\})\leq Ct^{-\theta''}$, and $N(u)\in L^q(d\omega)$ for some $q>0$ with $\|N(u)\|_{L^q(d\omega)}\lesssim 1$. Then the latter conclusion can replace in Lemma 6 the condition $\|\widetilde{N}(u)\|_{L^1(d\omega)}\leq 1$, and still yield the same result.

2. Proof of Theorem 2

This theorem follows in the same way from the analogues of Lemmas 4 and 8, and the fact that a comparison principle for normalized adjoint solutions vanishing at a boundary portion of a Lipschitz domain holds with constants depending only on ellipticity and the Lipschitz character of the domain (see [FGMS], Theorem I.3.7). An immediate consequence of this is that Lemma 1 holds with ω replaced by $\widetilde{\omega}$, as well as properties (1.1) and (1.2). On the other hand, we will show that the analogues of Lemmas 2, 3 and 9 also hold for normalized adjoint solutions, and therefore we would only have to follow the steps in our previous arguments to obtain Theorem 2. In particular, we have

Lemma 10. *Let $\widetilde{w}$ be a normalized adjoint solution for L in D, ϕ be a test function supported in D, and $v\in L^1_{\mathrm{loc}}(D)$ be a nonnegative superadjoint solution for L in D. Then for any constant β the following holds:*

$$\int |\nabla \widetilde{w}(X)|^2 \phi(X) G(X)\,dX \lesssim \int |\widetilde{w}(X)-\beta|^2 |D^2\phi(X)| G(X)\,dX.$$

Moreover, if $B_{2r}(X_0)\subset D$,

$$\sup_{B_r(X_0)} |\widetilde{w}(X)-\widetilde{w}(X_0)|^2 \lesssim \frac{r^2}{G(B_{2r}(X_0))}\int_{B_{2r}(X_0)} |\nabla \widetilde{w}(X)|^2 G(X)\,dX.$$

Proof. The first estimate follows from the identity $D_{ij}(a_{ij}(\widetilde{w}-\beta)^2G)=2\langle A\nabla\widetilde{w},\nabla\widetilde{w}\rangle G$, and integration by parts. To prove the second estimate we need an identity similar to (0.2). But a simple integration by parts argument shows that when D is smooth and $D_{ij}(a_{ij}\widetilde{w}G)=f$ on D we have

$$\widetilde{w}(X)G(X)=\int_{\partial D}\widetilde{w}(Q)g(Q)\frac{\partial g}{\partial\nu}(Q,X)\,d\sigma(Q)-\int_D g(Y,X)f(Y)\,dY$$

for all X in D, where $\partial u/\partial\nu(Q)=\langle A\nabla u,N\rangle$, $d\sigma$ denotes surface measure, and N the interior unit normal at Q. Since in this case $d\widetilde{\omega}^X(Q)=G(Q)(\partial/\partial\nu)\tilde{g}(Q,X)d\sigma(Q)$ (see [B1], Proposition 4.1), we obtain after dividing by G

$$\widetilde{w}(X)=\int_{\partial D}\widetilde{w}(Q)\,d\widetilde{\omega}^X(Q)-\int_D\tilde{g}(Y,X)f(Y)\,dY \quad \text{for all } X \text{ in } D. \tag{2.1}$$

Thus the Green's function of the operator $D_{ij}(a_{ij}\widetilde{w}G)=GL\widetilde{w}+2D_i\widetilde{w}D_j(a_{ij}G)$ is $\tilde{g}(Y,X)$, and the above formula gives a representation of functions in terms of $d\widetilde{\omega}^X$ and $\tilde{g}(Y,X)$ similar to (0.2). At this point, the proof of the second inequality proceeds as in Lemma 9, because normalized adjoint solutions satisfy Harnack's inequality (this implies $\operatorname{osc}_{B_{r/2}(X_0)}\widetilde{w}\le\theta\operatorname{osc}_{B_r(X_0)}\widetilde{w}$ for some $\theta\in(0,1)$ depending on ellipticity), and it is known that $\tilde{g}_{2r}(X,Y)\approx\tilde{g}_{2r}(Y,X)$ for $X,Y\in B_r(X_0)$ independently of r. (See [B2], Theorem 2.3, where this is proved for $r=1$. In our case it follows by proper scaling.)

One also needs a more general version of the result we just mentioned above:

Lemma 11. *Let Ω be a Lipschitz domain contained in $B_1(0)$ with diameter* $\operatorname{diam}(\Omega)\le 5r$, *$r>0$, and assume that $X_0\in\Omega$ with $B_{2r}(X_0)\subset\Omega$. Then, $\tilde{g}_\Omega(X,Y)\approx \tilde{g}_\Omega(Y,X)$ for $X,Y\in B_r(X_0)$, and where $\tilde{g}_\Omega(X,Y)$ is the normalized Green's function for Ω.*

See [B2], Theorems 2 and 3.

Lemma 12. *Let $g(X,Y)$ denote the Green's function for L in D. Then, there is a constant r_0 depending on the Lipschitz character of D, such that for all $Q\in\partial D$, $r\le r_0$, $Y\in\partial B_r(Q)\cap\Gamma_1(Q)$, and $X\notin T_{4r}(Q)$, the following holds*

$$\tilde{g}(Y,X)\frac{G(B(Y))}{\delta(Y)^2}\approx\tilde{\omega}^X(\Delta_r(Q)).$$

Proof. Let us fix $Q_0\in\partial D$. The inequality

$$\tilde{g}(Y,X)G(B(Y))/\delta(Y)^2\gtrsim\tilde{\omega}^X(\Delta_r(Q_0))$$

follows from (2.1), the analogous Carleson estimate for solutions to $Lu=0$ vanishing on a boundary portion of ∂D [B1], with a similar argument to the one used in the proof of the analogous inequality in Lemma 2, integration by parts, Lemma 3, and the doubling property of normalized harmonic measure.

For the opposite inequality, assume that on a neighborhood of Q_0, D coincides with $\{(x,y):y>\varphi(x)\}$ for some Lipschitz function φ, and set $\Omega=D\cup B_r(Q_0)$, $\widetilde{Q}=Q_0-re_n$, where e_n is a unit vector in the direction of the y-axis. Let $\tilde{\omega}_\Omega^Y$ denote the normalized harmonic measure associated to Ω, and $\tilde{g}_\Omega(\cdot,\cdot)$ the normalized Green's function on Ω. From the maximum principle

$$\tilde{\omega}^X(\Delta_{2r}(Q_0))\ge\tilde{\omega}_\Omega^X(\partial B_r(Q_0)\cap D^c)\quad\text{for } X\in D.$$

On the other hand, there is a constant $\tau\in(0,1)$ depending on the Lipschitz character of D such that $\Delta_{\tau r}(\widetilde{Q})\subset\partial B_r(Q_0)\cap D^c$ and

$$\tilde{\omega}_\Omega^X(\Delta_{\tau r}(\widetilde{Q}))=\int_{\Delta_{\tau r}(\widetilde{Q})}G(Q)\frac{\partial}{\partial\nu}\tilde{g}_\Omega(Q,Y)\,d\sigma(Q),$$

and for $X\notin T_{4r}(Q_0)$, $Q\in\Delta_{\tau r}(\widetilde{Q})$, $(\partial/\partial\nu)\tilde{g}_\Omega(Q,X)\approx(1/r)\tilde{g}_\Omega(Q_0+re_n,X)$ (see [B1], Lemma 4.3, Lemma 2.5; [BEF], Lemma 3). Hence

$$\tilde{\omega}^X(\Delta_{2r}(Q_0))\gtrsim\frac{1}{r}\tilde{g}_\Omega(Q_0+re_n,X)\int_{\Delta_{\tau r}(\widetilde{Q})}G(Q)\,d\sigma(Q).$$

On the other hand, if $\tilde{g}_r(\cdot,\cdot)$ denotes the Green's function for L in $B_r(Q_0)$, it follows from ([B1], Lemma 4.3, Lemma 2.5; [BEF], Lemma 3), and the Harnack inequality for normalized adjoint solutions that for $Q\in\partial B_r(Q_0)$, $(\partial/\partial\nu)\tilde{g}_r(Q,Q_0)\approx r/G(B_r(Q_0))$. Hence, from (2.1) we have

$$\begin{aligned}\int_{\Delta_{\tau r}(\tilde{Q})} G(Q)\,d\sigma(Q) &\approx \frac{G(B_r(Q_0))}{r}\int_{\Delta_{\tau r}(\tilde{Q})} \frac{\partial}{\partial\nu}\tilde{g}_r(Q,Q_0)G(Q)\,d\sigma(Q)\\ &= \frac{G(B_r(Q_0))}{r}\int_{\Delta_{\tau r}(\tilde{Q})} d\widetilde{\omega}_r^{Q_0},\end{aligned}$$

where $\widetilde{\omega}_r^{Q_0}$ denotes the normalized harmonic measure for L on $B_r(Q_0)$. Thus,

$$\widetilde{\omega}^X\big(\Delta_{2r}(Q_0)\big)\gtrsim \tilde{g}_\Omega(Q_0+re_n,X)\frac{G(B_r(Q_0))}{r^2}\,\widetilde{\omega}_r^{Q_0}\big(\Delta_{\tau r}(\widetilde{Q})\big),$$

and from the doubling property of normalized harmonic measure we get, $\widetilde{\omega}_r^{Q_0}(\Delta_{\tau r}(\widetilde{Q}))\gtrsim 1$. Finally, we only have to observe that $\tilde{g}_\Omega(Y,X)\ge\tilde{g}(Y,X)$ for $X,Y\in D$, and that for $Y\in\partial B_r(Q)\cap\Gamma_1(Q)\tilde{g}(Y,X)\approx\tilde{g}(Q+re_n,X)$.

In the case of the analogue of Lemma 4, the measure which in this case is shown to be Carleson with respect to $\widetilde{\omega}$ is

$$\Lambda_{\widetilde{w}}(F)=\int_{\mathbf{R}_\alpha(E_{\widetilde{w}})\cap F}\tilde{g}(X,0)\big|\nabla\widetilde{w}(X)\big|^2G(X)\,dX,\quad \text{for } F\subset D \text{ Borel set},$$

and where $E_{\widetilde{w}}=\{Q\in\partial D: N_\beta(\widetilde{w})(Q)\le 1\}$. The corresponding function K is given by

$$K(X,Q)=\varphi(X)\psi\bigg(\frac{|X-Q|}{\delta(X)}\bigg)\frac{\delta(X)^2}{G(B(X))}\frac{1}{\tilde{g}(X,0)},\quad X\in D,\ Q\in\partial D.$$

In this case, the first estimate turns out to be a little bit more complicated, and follows from the identity

$$\phi D_{ij}(a_{ij}\widetilde{w}^2G)=D_{ij}(a_{ij}\phi\widetilde{w}^2G)+\widetilde{w}^2GL\phi-2D_j(D_j\phi a_{ij}\widetilde{w}^2G).$$

The only new term which is not similar in the argument given in Lemma 2 is

$$\int\langle A\nabla\tilde{g}(X,0),\nabla\phi\rangle\widetilde{w}^2G(X)\,dX,$$

but here we just used the fact that $|\widetilde{w}|\le 1$ on the support of $\nabla\phi$, and apply Lemma 3 to the integrals over the corresponding dyadic cubes in the following sum

$$\sum_{j\in J}\frac{1}{l(I_j)}\int_{I_j}|\nabla\tilde{g}(X,0)|G(X)\,dX.$$

The rest of the argument from here, and in the analogue of Lemma 8, proceeds in the same way as before. The estimates that were obtained over sets which we called "large", are handled using Lemma 11.

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