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Hyperbolicity of localizations

Tatsuo Nishitani

1. Introduction

Let $P(x,D)$ be a differential operator of order m in an open set $\Omega\subset\mathbf{R}^{n+1}$ with coordinates $x=(x_0,x')=(x_0,x_1,...,x_n)$, hence a sum of differential polynomials $P_j(x,D)$ of order j $(j\le m)$ with symbols $P_j(x,\xi)$. In [7] Ivrii–Petkov has proved a necessary condition for the Cauchy problem of $P(x,D)$ to be correctly posed which asserts that $P_{m-j}(z)$ must vanish of order $r-2j$ at z if $P_m(z)$ vanishes of order r at z with $z=(x,\xi)\in T^*\Omega\backslash 0$. This enables us to define the localization $P_{z_0}(z)$ at a multiple characteristic z_0 (of $P_m(z)$) following Helffer [4] which is a polynomial on $T_{z_0}(T^*\Omega)$.

In this note we show that $P_{z_0}(z)$ is hyperbolic, that is verifies Gårding's condition if the Cauchy problem for $P(x,D)$ is correctly posed. The proof is based on the arguments of Svensson [9].

Since $P_{z_0}(z)$ is hyperbolic one can define the localizations $P_{(z_0,z_1,...,z_s)}(z)$ successively as the localization of $P_{(z_0,z_1,...,z_{s-1})}(z)$ at z_s which are hyperbolic polynomials on $T_{z_0}(T^*\Omega)\cong...\cong T_{z_s}(T^*\Omega)$ (see Hörmander [6, II] and Atiyah–Bott–Gårding [1]). It may happen that the lineality $\Lambda_{(z_0,z_1,...,z_s)}(P_m)$ of $P_{m(z_0,z_1,...,z_s)}(z)$ is an involutive subspace with respect to the canonical symplectic structure on $T_{z_0}(T^*\Omega)$. In this case we prove that for the Cauchy problem to be correctly posed it is necessary that

$$P_{(z_0,z_1,...,z_s)}(z)=P_{m(z_0,z_1,...,z_s)}(z).$$

This argument was also used in Bernardi–Bove–Nishitani [2] with $s=1$.

2. The localization is hyperbolic

We denote by $L^{m,r}_{z_0}$ the set of pseudodifferential operators P near z_0 with symbol $P(x,\xi)$ verifying

$$P(x,\xi)\sim\sum_{j=0}^{\infty}P_{m-j}(x,\xi)$$

in every system of homogeneous symplectic coordinates around z_0, where $P_{m-j}(x,\xi)$ are positively homogeneous of degree $m-j$ in ξ and vanish of order at least $r-2j$ and $P_m(x,\xi)$ vanishes exactly to the order r at z_0. Note that we may replace in the definition "every" by "some".

Lemma 2.1 (Helffer [4]). *Let $P\in L^{m,r}_{z_0}$. Then*

$$Q(x,\xi)=\exp\left\{\frac{i}{2}\sum_{j=0}^{n}\frac{\partial^2}{\partial x_j\partial\xi_j}\right\}P(x,\xi) \tag{2.1}$$

is invariantly defined in $L^{m,r}_{z_0}/L^{m,r+1}_{z_0}$: Let χ be a homogeneous symplectic transformation around z_0 and let F be a Fourier integral operator associated with χ and $\widehat{P}=FPF^{-1}$. Then we have

$$\widehat{Q}(\chi(x,\xi))=Q(x,\xi)$$

in $L^{m,r}_{z_0}/L^{m,r+1}_{z_0}$ where $\widehat{Q}$ is associated with $\widehat{P}$ by (2.1).

Definition 2.1. We define the localization $P_{z_0}(x,\xi)$ of $P\in L^{m,r}_{z_0}$ at $z_0=(x_0,\xi_0)$ as the lowest order term of the Taylor expansion of

$$\mu^{2m}Q(x_0+\mu x,\mu^{-2}\xi_0+\mu^{-1}\xi)$$

as $\mu\to 0$ which is invariantly defined as a polynomial on $T_{z_0}(T^*\Omega)$. If y are local coordinates around the origin and $\widehat{P}(y,\eta)$ is the full symbol of P for the coordinates $(y,\eta dy)$, then we have

$$\widehat{P}_{w_0}(y'(x_0)x,{}^t y'(x_0)^{-1}\xi)=P_{z_0}(x,\xi),\ w_0=(y(x_0),{}^t y'(x_0)^{-1}\xi_0).$$

Writing $Q(x,\xi)$ as the sum of homogeneous parts $Q_{m-j}(x,\xi)$, it is clear that

$$\begin{aligned}&P_{z_0}(x,\xi)=\sum_{r-2j\geq 0}Q_{m-j,z_0}(x,\xi),\\&Q_{m-j,z_0}(z)=P_{m-j,z_0}(z)+\sum_{i<j,|\alpha|=j-i}c_\alpha P^{(\alpha)}_{m-i,z_0(\alpha)}(z)\end{aligned} \tag{2.2}$$

with some constants c_α where $Q_{m-j,z_0}(x,\xi)$ and $P_{m-j,z_0}(x,\xi)$ are defined by

$$P_{m-j,z_0}(z)=\lim_{\mu\to 0}\mu^{-(r-2j)}P_{m-j}(z_0+\mu z).$$

Let $P(x,D)=\sum_{j=0}^{m}P_j(x,D)$ be a differential operator of order m on Ω containing the origin where $P_j(x,D)$ is the homogeneous part of degree j with symbol $P_j(x,\xi)$. Assume that the plane $x_0=0$ is non-characteristic and we are concerned with the Cauchy problem with respect to $x_0=$const. Let z_0 be a multiple characteristic of P_m. By the necessary condition of Ivrii–Petkov [7] stated in the introduction we conclude that $P\in L_{z_0}^{m,r}$ with some $r\geq 2$ provided that the Cauchy problem for P is correctly posed. Then we have from Lemma 2.1 the following

Proposition 2.2. *Assume that the Cauchy problem for $P(x,D)$ is correctly posed near the origin and let $z_0\in T^*\Omega\backslash 0$ be a multiple characteristic of P_m. Then the localization $P_{z_0}(z)$ is an invariantly defined polynomial on $T_{z_0}(T^*\Omega)$.*

Let us denote by $\widetilde{P}_{z_0}(x,\xi)$ the lowest order term of the Taylor expansion of $\mu^{2m}P(x_0+\mu x,\mu^{-2}\xi_0+\mu^{-1}\xi)$ as $\mu\to 0$. Then we have

Lemma 2.3. *The following two conditions are equivalent.*
(i) *$\widetilde{P}_{z_0}(z)$ is hyperbolic with respect to $\theta=(0,e_0)$,*
(ii) *$P_{z_0}(z)$ is hyperbolic with respect to θ.*

Proof. Recall that $\widetilde{P}_{z_0}(z)=\sum_{r-2j\geq 0}P_{m-j,z_0}(z)$. Since $\widetilde{P}_{z_0}(z)$ is hyperbolic if and only if $P_{m-j,z_0}(z)$ are weaker than $P_{m,z_0}(z)=Q_{m,z_0}(z)$ (see Hörmander [6, II], Svensson [9]) the proof is immediate by (2.2).

Now our aim is to prove

Theorem 2.4. *Assume that the Cauchy problem for $P(x,D)$ is correctly posed near the origin and let $z_0\in T^*\Omega\backslash 0$ be a multiple characteristic of P_m. Then the localization $P_{z_0}(z)$ is a hyperbolic polynomial with respect to $\theta=(0,e_0)$.*

Let z_0 be a characteristic of order r_0 of $P_m(z)$ so that $P_{z_0}(z)$ is a polynomial of degree r_0. We denote by $P_{(z_0,z_1)}(z)$ the localization of $P_{z_0}(z)$ at z_1, that is the first coefficient of $\mu^{r_0}P_{z_0}(\mu^{-1}z_1+z)$ that does not vanish identically in z:

$$\mu^{r_0}P_{z_0}(\mu^{-1}z_1+z)=\mu^{r_1}(P_{(z_0,z_1)}(z)+O(\mu)),\quad \mu\to 0$$

(see Hörmander [6, II] and Atiyah–Bott–Gårding [1]). We call r_1 the order of z_1. From Lemma 3.4.2 in Atiyah–Bott–Gårding [1] it follows that $P_{(z_0,z_1)}(z)$ is again hyperbolic with respect to θ. Furthermore z_1 is a characteristic of P_{m,z_0} of order r_1 and $P_{m(z_0,z_1)}(z)$ is the principal part of $P_{(z_0,z_1)}(z)$. On the other hand Corollary 12.4.9 in Hörmander [6, II] shows that

$$d^{\nu}Q_{m-j,z_0}(z_1)=0,\quad \nu<r_1-2j$$

where $d^\nu Q(z)$ denotes the ν-th differential of Q with respect to z. Since $Q_{m-j,z_0}(z)$ are homogeneous of degree r_0-2j it is clear that

$$P_{(z_0,z_1)}(z)=\sum_{r_1-2j\geq 0} Q_{m-2j(z_0,z_1)}(z)$$

where

$$Q_{m-j(z_0,z_1)}(z)=\lim_{\mu\to 0}\mu^{-(r_1-2j)}Q_{m-j,z_0}(z_1+\mu z)$$

which is homogeneous of degree r_1-2j in z. Repeating the same arguments we get

Lemma 2.5. *Let $P_{(z_0,\dots,z_k)}(z)$ be the localization of $P_{(z_0,\dots,z_{k-1})}(z)$ at z_k where its order is r_k (≥ 2). Then we have for every j with $r_k-2j>0$*

$$d^\nu Q_{m-j(z_0,\dots,z_{k-1})}(z_k)=0,\ \nu<r_k-2j$$

and hence

$$Q_{m-j(z_0,\dots,z_k)}(z)=\lim_{\mu\to 0}\mu^{-(r_k-2j)}Q_{m-j(z_0,\dots,z_{k-1})}(z_k+\mu z)$$

exists. Moreover $P_{(z_0,\dots,z_k)}(z)$ is equal to

$$\sum_{r_k-2j\geq 0} Q_{m-j(z_0,\dots,z_k)}(z)$$

and hyperbolic with respect to θ.

Corollary 2.6. *Let z_k be a characteristic of $P_{m(z_0,\dots,z_{k-1})}(z)$ of order r_k (≥ 2). Then we have*

$$d^\nu P_{m-j(z_0,\dots,z_{k-1})}(z_k)=0,\quad \nu<r_k-2j \tag{2.3}$$

and hence

$$P_{m-j(z_0,\dots,z_k)}(z)=\lim_{\mu\to 0}\mu^{-(r_k-2j)}P_{m-j(z_0,\dots,z_{k-1})}(z_k+\mu z) \tag{2.4}$$

exists.

Proof. Assume that (2.3) and

$$Q_{m-j(z_0,\dots,z_{k-1})}(z)=P_{m-j(z_0,\dots,z_{k-1})}(z)+\sum_{i<j,|\alpha|=j-i} c_\alpha P^{(\alpha)}_{m-i(z_0,\dots,z_{k-1})(\alpha)}(z) \tag{2.5}$$

hold with $k=p$ where c_α are constants. Then it is easy to see that (2.5) with $k=p+1$ holds. Thus (2.3) with $k=p+1$ follows from Lemma 2.5. By induction on k we get the desired conclusion.

Let $\Lambda_{(z_0,\dots,z_s)}(P_m)$ be the lineality of $P_{m(z_0,\dots,z_s)}$ which is defined by

$$\Lambda_{(z_0,\dots,z_s)}(P_m)=\{z|P_{m(z_0,\dots,z_s)}(w+tz)=P_{m(z_0,\dots,z_s)}(w),\forall t\in\mathbf{R},\forall w\in T_{z_0}(T^*\Omega)\}$$

and let $\sigma=\sum_{j=0}^n d\xi_j\wedge dx_j$ be the canonical symplectic two form on $T^*\Omega$. For $S\subset T_{z_0}(T^*\Omega)$ we denote by S^σ the annihilator of S with respect to σ:

$$S^\sigma=\{z\in T_{z_0}(T^*\Omega)|\sigma(z,w)=0,\forall w\in S\}.$$

Theorem 2.7. *Assume that the Cauchy problem for $P(x, D)$ is correctly posed near the origin and*

$$\Lambda_{(z_0,\dots,z_s)}(P_m)^\sigma \subset \Lambda_{(z_0,\dots,z_s)}(P_m).$$

Then we have

$$P_{(z_0,\dots,z_s)}(z) = P_{m(z_0,\dots,z_s)}(z).$$

Example 2.1. Let

$$P(x,\xi) = (\xi_0^2 - x_1^2\xi_n^2 - \xi_1^2)(\xi_0^2 - x_1^2\xi_n^2 - 2\xi_1^2) + p_2(\xi_0, x_1\xi_n, \xi_1)\xi_n$$

where p_2 is a homogeneous polynomial of degree 2. With $z_0=(0,e_n)$ it is clear that

$$P_{4,z_0} = (\xi_0^2 - x_1^2 - \xi_1^2)(\xi_0^2 - x_1^2 - 2\xi_1^2), \quad Q_{3,z_0} = 6ix_1\xi_1 + p_2(\xi_0, x_1, \xi_1).$$

Let z_1 be $\xi_0 = x_1 = a$, $a\in\mathbf{R}$, $\xi_1=0$ so that

$$P_{4(z_0,z_1)} = 4a^2(\xi_0 - x_1)^2, \quad Q_{3(z_0,z_1)} = p_2(a,a,0).$$

Since $\Lambda_{(z_0,z_1)}(P_4)^\sigma \subset \Lambda_{(z_0,z_1)}(P_4)$ it follows from Theorem 2.7 that $p_2(a,a,0)=0$. Similarly choosing z_1 to be $\xi_0=a$, $x_1=-a$, $\xi_1=0$ we get $p_2(a,-a,0)=0$. Thus

$$p_2(\xi_0, x_1, \xi_1) = c(\xi_0^2 - x_1^2) + \xi_1 p_1(\xi_0, x_1, \xi_1)$$

where p_1 is linear. Finally one can write

$$P(x,\xi) = (\xi_0^2 - x_1^2\xi_n^2 - \xi_1^2 + c\xi_n)(\xi_0^2 - x_1^2\xi_n^2 - 2\xi_1^2) + \xi_1 L(\xi_0, x_1\xi_n, \xi_1)\xi_n$$

with a linear function L.

Example 2.2. Let

$$P(x,\xi) = (\xi_0 - x_0\xi_n)^2(\xi_0 + x_0\xi_n) + \alpha(\xi_0 - x_0\xi_n)\xi_n + \beta(\xi_0 + x_0\xi_n)\xi_n$$

where $\alpha, \beta \in \mathbf{C}$. With $z_0 = (0, e_n)$ we have

$$P_{3,z_0} = (\xi_0 - x_0)^2(\xi_0 + x_0), \quad Q_{2,z_0} = \alpha(\xi_0 - x_0) + (\beta - i)(\xi_0 + x_0).$$

Taking z_1 to be $\xi_0 = 1$, $x_0 = 1$ it follows that

$$P_{3(z_0,z_1)} = 2(\xi_0 - x_0)^2, \quad Q_{2(z_0,z_1)} = 2(\beta - i).$$

Since $\Lambda_{(z_0,z_1)}(P_3)^\sigma \subset \Lambda_{(z_0,z_1)}(P_3)$ we have $\beta = i$ by Theorem 2.7. Set

$$p_1(x,\xi) = \xi_0 - x_0\xi_n, \quad p_2(x,\xi) = (\xi_0 - x_0\xi_n)(\xi_0 + x_0\xi_n) + (\alpha + i)\xi_n$$

then $\beta = i$ implies that

$$P(x,D) = p_1^w(x,D)p_2^w(x,D)$$

where $p_j^w(x,D)$ are Weyl realizations of $p_j(x,\xi)$, see Hörmander [6, III].

3. Proof of Theorem 2.4

To prove Theorem 2.4 we construct an asymptotic solution depending on a large parameter contradicting an a priori estimate that a correctly posed Cauchy problem must satisfy. In constructing a desired phase function of the asymptotic solution we follow the arguments of Svensson [9]. We first derive an a priori estimate assuming that the Cauchy problem for $P(x, D)$ is correctly posed in both Ω^t and Ω_t for every small t where $\Omega^t = \{x \in \Omega | x_0 < t\}$ and $\Omega_t = \{x \in \Omega | x_0 > t\}$. Let $\sigma = (\sigma_0, ..., \sigma_n) \in \mathbf{Q}_+^{n+1}$ and set

$$y(\lambda) = \sum_{j=0}^{\infty} y_j \lambda^{-\varepsilon j}, \quad \eta(\lambda) = \sum_{j=0}^{\infty} \eta_j \lambda^{-\varepsilon j}, \quad y_j, \eta_j \in \mathbf{R}^{n+1}, \ \varepsilon \in \mathbf{Q}_+$$

which are assumed to be convergent in a neighborhood of $\lambda = \infty$. For a differential operator P on $C^\infty(\Omega)$ with C^∞ coefficients we set with $\varkappa \in \mathbf{Q}_+$

$$P_\lambda(y(\lambda), \eta(\lambda); x, \xi) = P(y(\lambda) + \lambda^{-\sigma} x, \lambda^{\varkappa} \eta(\lambda) + \lambda^{\sigma} \xi)$$

where $\lambda^{-\sigma} x = (\lambda^{-\sigma_0} x_0, ..., \lambda^{-\sigma_n} x_n)$ etc.

Proposition 3.1. *Assume that* $0 \in \Omega$, $y_0 = 0$ *and that the Cauchy problem for* $P(x, D)$ *is correctly posed in both* Ω^t *and* Ω_t *for every small* t. *Then for every compact set* $W \subset \mathbf{R}^{n+1}$ *and for every positive* $T > 0$ *we can find* $C > 0$, $\bar{\lambda} > 0$ *and* $p \in \mathbf{N}$ *such that*

$$|u|_{C^0(W^t)} \leq C \lambda^{(\bar{\sigma} + \varkappa)p} |P_\lambda u|_{C^p(W^t)}, \quad |u|_{C^0(W_t)} \leq C \lambda^{(\bar{\sigma} + \varkappa)p} |P_\lambda u|_{C^p(W_t)}$$

if $u \in C_0^\infty(W)$, $\lambda \geq \bar{\lambda}$, $|t| < T$ *where* $\bar{\sigma} = \max_j \sigma_j$.

Proof. Recall the following a priori estimate a proof of which is found in Hörmander [5]: for every compact set $K \subset \Omega$ there exist positive constants C, τ and $p \in \mathbf{N}$ such that

$$|u|_{C^0(K^t)} \leq C |Pu|_{C^p(K^t)}, \quad |u|_{C^0(K_t)} \leq C |Pu|_{C^p(K_t)} \tag{3.1}$$

for $u \in C_0^\infty(K)$, $|t| < \tau$. Setting $\widetilde{P}(x, D) = e^{-i\lambda^{\varkappa} <\eta(\lambda), x>} P(x, D) e^{i\lambda^{\varkappa} <\eta(\lambda), x>}$ we get from (3.1) that

$$|u|_{C^0(K^t)} \leq C_1 \lambda^{\varkappa p} |\widetilde{P} u|_{C^p(K^t)}, \quad \lambda \geq \lambda_1.$$

For a given compact set $W \subset \mathbf{R}^{n+1}$ one can find a compct set $K \subset \Omega$ so that $u(\lambda^\sigma x - y(\lambda)) \in C_0^\infty(K)$, $\forall u \in C_0^\infty(W)$ if $\lambda \geq \lambda_2$. Then the desired inequality follows from (3.1). The second estimate is proved in the same way.

Let z_0 be a characteristic of P_m of order r. We may assume that $z_0=(0,e_n)$ without restrictions. We specialize Proposition 3.1 setting

$$y(\lambda)=\lambda^{-\varkappa/2}\tilde{y}(\lambda),\quad \eta(\lambda)=e_n+\lambda^{-\varkappa/2}\tilde{\eta}(\lambda)$$

where $\tilde{y}(\lambda)=\sum_{j=j_0}^{\infty}y_j\lambda^{-j}$, $\tilde{\eta}(\lambda)=\sum_{j=j_0}^{\infty}\eta_j\lambda^{-j}$ are meromorphic in a neighborhood of $\lambda=\infty$ and $\varkappa/2+j_0>0$. With $\Gamma(\lambda)=(\tilde{y}(\lambda),\tilde{\eta}(\lambda))$ it follows that

$$\lambda^{-\varkappa(m-r/2)}P(y(\lambda)+\lambda^{-\varkappa/2+\sigma/2}x,\lambda^{\varkappa}\eta(\lambda)+\lambda^{\varkappa/2-\sigma/2}\xi)$$
$$=\tilde{P}_{z_0}(\Gamma(\lambda)+(\lambda^{\sigma/2}x,\lambda^{-\sigma/2}\xi))+O(\lambda^{-\varkappa/2+(r+1)\sigma/2}),\quad \lambda\to\infty.$$

By Lemma 2.3, $\tilde{P}_{z_0}$ is hyperbolic with respect to θ if P_{z_0} is. Now assuming that $\tilde{P}_{z_0}$ is not hyperbolic with respect to θ we look for an asymptotic solution with complex valued phase function to $Pu\sim 0$. The main step is to prove:

Proposition 3.2. *Assume that $\tilde{P}_{z_0}$ is not hyperbolic with respect to θ. Then we can find $\Gamma_1(s)=\sum z_\nu s^\nu$, $z_\nu\in\mathbf{R}^{2n+2}$ which is meromorphic in a neighborhood of $s=0$, an open set $W\subset\mathbf{R}^{2n+2}$ and $-1<\alpha<0$ such that*

$$\tilde{P}_{z_0}(\Gamma_1(s)+s^\alpha z+s^\alpha\tau\theta)=cs^{-m^*}\left(R_1(\tau,z)+O(s^\delta)\right),\quad z\in W$$

with some $\delta>0$ where $c\neq 0$, $m^*\in\mathbf{R}$ and $R_1(\tau,z)$ is a monic polynomial in τ which has a non-real root for $\forall z\in W$.

Admitting Proposition 3.2 we prove Theorem 2.4. Taking $\lambda=s^{-1}$ and $\sigma/2=-\alpha$, Proposition 3.2 yields

$$\tilde{P}_{z_0}(\Gamma(\lambda)+(\lambda^{\sigma/2}x,\lambda^{\sigma/2}\xi))=c\lambda^{m^*}(R(x,\xi)+O(\lambda^{-\delta})).\tag{3.2}$$

It is clear that $R(x,\xi)$ is a monic polynomial in ξ_0 and $R(x,\xi)=0$ has a non-real root for every $(x,\xi')\in W'$ where W' is an open set in $\mathbf{R}^{2n+1}$. Therefore we may assume, by shrinking W' if necessary, that

$$R(x,\xi)=\prod_{j=1}^{l}(\xi_0-f_j(x,\xi'))^{r_j}$$

where $f_j(x,\xi')$ are real analytic and mutually different from each other in W' and $\mathrm{Im}f_1\neq 0$ in W'. Let $\phi(x)$ be a solution to

$$\phi_{x_0}(x)=f_1(x,\phi_{x'}(x)),\quad \phi(\hat{x}_0,x')=<\hat{\xi}',x'>$$

which is defined near $\hat{x}$ with $(\hat{x},\hat{\xi}')\in W'$. Set

$$E(x)=\exp(i\lambda^\sigma\phi(x))$$

and study

$$E(x)^{-1}\tilde{P}_{z_0}(\Gamma(\lambda)+(\lambda^{\sigma/2}x,\lambda^{-\sigma/2}D))E(x)\exp(i\lambda^{\sigma_1}w(x))$$

where $w(x)$ is a C^∞ function near $\hat{x}$ and $\sigma_1>0$ which will be determined in the following lemma.

Lemma 3.3. *There exists* $0<\sigma_1<\sigma$ *such that for every* $w(x)$ *we have*

$$\widetilde{P}_{z_0}(\Gamma(\lambda)+(\lambda^{\sigma/2}x, \lambda^{-\sigma/2}D))E(x)\exp(i\lambda^{\sigma_1}w(x))$$
$$=\lambda^{m^*-r_1(\sigma-\sigma_1)}\left(L(x, w_x(x))+o(1)\right)E(x)\exp(i\lambda^{\sigma_1}w(x))$$

where

$$L(x,\zeta)=\sum_{|\alpha|=r_1} R^{(\alpha)}(x,\zeta)/\alpha!+S(x,\zeta)$$

and $S(x,\zeta)$ *is a polynomial in* ζ *of degree less than* r_1.

Proof. Recall that

$$e^{-i\lambda^\sigma\phi(x)}\widetilde{P}_{z_0}(\Gamma(\lambda)+(\lambda^{\sigma/2}x,\lambda^{-\sigma/2}D))(a(x)e^{i\lambda^\sigma\phi(x)})$$
$$=\sum_\alpha(\alpha!)^{-1}\partial_\xi^\alpha\tilde{P}_{z_0}(\Gamma(\lambda)+(\lambda^{\sigma/2}x,\lambda^{-\sigma/2}\xi))|_{(\xi=\lambda^\sigma\phi_x)}D_z^\alpha(a(z)e^{i\lambda^\sigma\psi})|_{(z=x)}$$

where $\psi(x,z)=\phi(z)-\phi(x)-<\phi_x(x), z-x>$. On the other hand we have

$$\partial_\xi^\alpha\widetilde{P}_{z_0}(\Gamma(\lambda)+(\lambda^{\sigma/2}x,\lambda^{-\sigma/2}\xi))|_{(\xi=\lambda^\sigma\Xi)}=\lambda^{m^*-\sigma|\alpha|}(R^{(\alpha)}(x,\Xi)+O(\lambda^{-\delta}))$$

by (3.2) and hence the result.

Now to prove Theorem 2.4 it is enough to follow the same arguments as in Ivrii–Petkov [7] and Flaschka–Strang [3] (see section 6 in Ivrii–Petkov [7]).

It remains to prove Proposition 3.2. We first recall the following result of Svensson [9]. Let $q(z)=\sum_{j=0}^r q_{r-j}(z)$ be a polynomial of degree r in $z\in\mathbf{R}^N$, $N=2(n+1)$ where $q_{r-j}(z)$ stands for the homogeneous part of degree $r-j$. Let $f(t,s)$ be a polynomial in t with coefficients which can be expanded in a Puiseux series in $s\in\mathbf{R}$ in a neighborhood of $s=0$. We denote by $R(f(t,s))$ the Newton polygon of f (see Svensson [9]). Then the result of Svensson [9] asserts that

Theorem 3.4 (Svensson [9]). *The following two conditions are equivalent.*

(i) $\sum_{j=0}^r q_{r-j}(z)$ *is hyperbolic with respect to* $\theta=(0,e_0)$,

(ii) $q_r(\theta)\neq 0$ *and* $R(t^k q_{r-k}(\gamma(s)+t\theta))\subset R(q_r(\gamma(s)+t\theta))$, $1\leq k\leq r$, *for every* $\gamma(s)=\sum z_\nu s^\nu$, $z_\nu\in\mathbf{R}^N$ *which is meromorphic in a neighborhood of* $s=0$.

Lemma 3.5. *Assume that* q_r *is hyperbolic with respect to* θ. *Let* $\gamma(s)=\sum z_\nu s^\nu$, $z_\nu\in\mathbf{R}^N$ *be meromorphic in a neighborhood of* $s=0$. *Then there is a neighborhood* U *of the origin in* $\mathbf{R}^N$ *such that*

$$R(q_r(\gamma(s)+tz+t\theta))=R(q_r(\gamma(s)+t\theta)),\quad \forall z\in U.$$

Proof. We first show that

$$R(q_r(\gamma(s)+tz+t\theta))\subset R(q_r(\gamma(s)+t\theta)),\quad \forall z\in\mathbf{R}^N. \tag{3.3}$$

Since $q_r^{(\beta)}(z)=\partial_z^\beta q_r(z)$ is weaker than q_r it follows from Theorem 12.4.6 in Hörmander [6, II] that the hyperbolicity of q_r is not altered by adding any linear combination of $q_r^{(\beta)}(z)$, $|\beta|\geq 1$. Then Theorem 3.4 shows that

$$R(z^\beta t^{|\beta|}q_r^{(\beta)}(\gamma(s)+t\theta))\subset R(q_r(\gamma(s)+t\theta)),\quad \forall z\in\mathbf{R}^N$$

which proves (3.3) because

$$q_r(\gamma(s)+tz+t\theta)=\sum_\beta q_r^{(\beta)}(\gamma(s)+t\theta)z^\beta t^{|\beta|}/\beta!.$$

Using (3.3) we end the proof. Write

$$q_r(\gamma(s)+tz+t\theta)=\sum_k\Big(\sum_{i+|\beta|=k}\partial_{\xi_0}^i q_r^{(\beta)}(\gamma(s))z^\beta/\beta!i!\Big)t^k$$

and let (l,k) be any vertex of $R(q_r(\gamma(s)+t\theta))$. Note that

$$\partial_{\xi_0}^k q_r(\gamma(s))=cs^l(1+o(1)),\quad c\neq 0. \tag{3.4}$$

From (3.3) it follows that

$$\sum_{i+|\beta|=k}\partial_{\xi_0}^i q_r^{(\beta)}(\gamma(s))z^\beta/\beta!i!=O(s^l) \tag{3.5}$$

for every $z\in\mathbf{R}^N$ and hence $\partial_{\xi_0}^i q_r^{(\beta)}(\gamma(s))=O(s^l)$, $i+|\beta|=k$. Hence taking U sufficiently small we conclude that the left-hand side of (3.5) is equal to $cs^l(1+o(1))$ with $c\neq 0$ by (3.4). This together with (3.3) proves the assertion.

Lemma 3.6. *If z avoids the union of the zeros of finitely many polynomials in z then $R(q_j(\gamma(s)+tz+t\theta))$ is independent of z and*

$$R(q_j(\gamma(s)+t\theta))\subset R(q_j(\gamma(s)+tz+t\theta)).$$

Proof. Recall that

$$q_j(\gamma(s)+tz+t\theta)=\sum_k\Big(\sum_{i+|\beta|=k}\partial_{\xi_0}^i q_j^{(\beta)}(\gamma(s))z^\beta/\beta!i!\Big)t^k.$$

It is clear that

$$\sum_{i+|\beta|=k} \partial_{\xi_0}^i q_j^{(\beta)}(\gamma(s))z^\beta/\beta!i! = s^l(P_k(z)+o(1)), \quad s\to 0$$

with some polynomial $P_k(z)$ and an integer l if the left-hand side is not identically zero. This proves the assertion because $\partial_{\xi_0}^k q_j(\gamma(s))=O(s^l)$.

To simplify notations we write

$$q_{r-2j}(z)=P_{m-j,z_0}(z), \quad q(z)=\sum_{r-2j\geq 0} q_{r-2j}(z)$$

and assume that q is not hyperbolic with respect to θ. Then by Theorem 3.4 we can find a non-negative integer k and $\gamma(s)=\sum z_\nu s^\nu$, $z_\nu\in\mathbf{R}^N$ which is meromorphic in a neighborhood of $s=0$ such that

$$R(t^{2k}q_{r-2k}(\gamma(s)+t\theta))\not\subset R(q_r(\gamma(s)+t\theta)).$$

Hence by Lemmas 3.5 and 3.6 one can choose a neighborhood U of the origin in $\mathbf{R}^N$ and Z', the union of the zeros of several polynomials, so that

$$R(t^{2k}q_{r-2k}(\gamma(s)+tz+t\theta))\not\subset R(q_r(\gamma(s)+tz+t\theta)), \quad z\in U,\ z\notin Z'.$$

We now follow the proof of Theorem 1.1 in Svensson [9] to conclude that there are an integer p, a real constant $c\neq 0$ and a set Z which is the union of the zeros of several polynomials such that:

(a) $R(c^r s^{pr} q(c^{-1}s^{-p}(\gamma(s)+tz+t\theta)))$ is independent of $z\in U$, $z\notin Z$ and has a line segment with slope $-\mu$, $p-1<\mu<p$ as a part of the boundary.

(b) The right endpoint of the line segment is a vertex of the Newton polygon $R(q_r(\gamma(s)+tz+t\theta))$.

Let (l_0, k_0) be the right endpoint of the segment and set $k_0\mu+l_0=g$.

Lemma 3.7. *We have*

(3.6)

$$c^r s^{pr} q(c^{-1}s^{-p}(\gamma(s)+tz+t\tau\theta)) = \sum_{k\mu+l=g}\Big(\sum_{i=0}^{k} c_{ki}(z)t^k s^l \tau^{k-i}\Big) + \sum_{k\mu+l>g} \tilde{c}_{ki}(\tau,z)t^k s^l$$

for $z\in U$, $z\notin Z$ where $c_{ki}(z)$ are homogeneous of degree i in z and $c_{k_0 0}\neq 0$. Moreover $c_{ki}(z)$ is not identically zero for some (k,i) with $k<k_0$.

Proof. With $\gamma_1(s)=c^{-1}s^{-p}\gamma(s)$ we have

$$c^r s^{pr} q(c^{-1}s^{-p}(\gamma(s)+tz+t\tau\theta)) = \sum_k \Big((cs^p)^{r-k} \sum_{i+|\beta|=k} \partial_{\xi_0}^i q^{(\beta)}(\gamma_1(s))z^\beta\tau^i/\beta!i!\Big)t^k.$$

Taking $\tau=1$ we see that the coefficient of t^k in the right-hand side is $O(s^l)$ $(k\mu+l=g)$ for every z with $z\in U$, $z\notin Z$ by (a). This shows that

$$(cs^p)^{r-k}\partial_{\xi_0}^i q^{(\beta)}(\gamma_1(s))=s^l(b_{i\beta}+o(1)),\quad i+|\beta|=k,\ k\mu+l=g.$$

Thus the coefficient of t^k is equal to

$$s^l\sum_{i=0}^{k}\sum_{|\beta|=k-i}(b_{i\beta}z^\beta+o(1))\tau^i/\beta!i!$$

and hence the result. In particular, since (l_0,k_0) is a vertex of $R(q(\gamma(s)+t\theta))$ we have

$$(cs^p)^{r-k_0}\partial_{\xi_0}^{k_0}(\gamma_1(s))=cs^{l_0}(1+o(1)),\quad c\neq 0$$

which proves $c_{k_0 0}\neq 0$. The last assertion is clear because the line segment contains another vertex different from (l_0,k_0).

We now prove Proposition 3.2. Taking $t=s^\mu$ $(s>0)$ and changing $c^{-1}z$, $c^{-1}\tau$ to z and τ in (3.6) we have

$$\begin{aligned}q(\Gamma_1(s)+s^\alpha z+s^\alpha\tau\theta)&=c_1s^{-m^*}\Big(\sum_{k\mu+l=g}\sum_{i=0}^{k}c'_{ki}(z)\tau^{k-i}+o(1)\Big)\\&=c_1s^{-m^*}(R(\tau,z)+o(1))\end{aligned}\tag{3.7}$$

where $\Gamma_1(s)=c^{-1}s^{-p}\gamma(s)$, $\alpha=\mu-p$ and hence $-1<\alpha<0$, $m^*=pr-g$ and $c'_{ki}(z)=c^{k-i}c_{ki}(cz)$. On the other hand after changing s to $-s$ in (3.6) we take $t=s^\mu$ $(\mu>0)$ and change $c^{-1}(-1)^{-p}z$, $c^{-1}(-1)^{-p}\tau$ to z and τ. Then (3.6) turns out to be

(3.8)
$$\begin{aligned}q(\Gamma_2(s)+s^\alpha z+s^\alpha\tau\theta)&=c_2s^{-m^*}\Big(\sum_{k\mu+l=g}\sum_{i=0}^{k}c''_{ki}(z)(-1)^l\tau^{k-i}+o(1)\Big)\\&=c_2s^{-m^*}(R'(\tau,z)+o(1))\end{aligned}$$

where $\Gamma_2(s)=c^{-1}(-s)^{-p}\gamma(-s)$, $c''_{ki}(z)=(-1)^{p(k-i)}c^{k-i}c_{ki}(c(-1)^pz)$. Therefore to prove Proposition 3.2 it is enough to show that either $R(\tau,z)$ or $R'(\tau,z)$ has a non-real root for some $z\in U$, $z\notin Z$.

Set $\mu=a/b$ where a and b are relatively prime so that k with $k\mu+l=g$ takes the form $k=k_0-jb$, $j=0,1,\dots,j_0$. Thus $R(\tau,z)$ becomes

$$\sum_{j=0}^{j_0}\sum_{i=0}^{k_0-jb}a'_{ji}(z)\tau^{k_0-jb-i},\quad a'_{ji}(z)=c'_{k_0-jb,i}(z).$$

Recall that $a'_{00}\neq 0$ and hence we may assume that $a'_{00}>0$. Let S be the set of indices (j,i), $j+i\geq 0$ such that a'_{ji} is not identically zero and remark that S contains at least two elements. Set

$$\gamma=\min_{(j,i)\in S, jb+i<k_0}\frac{i}{jb+i}$$

which is less than 1 of course. Plugging $\tau=|z|^{\gamma}\tilde{\tau}$ into $R(\tau,z)$ it follows that

$$R(|z|^{\gamma}\tilde{\tau},z)=|z|^{\gamma k_0}\sum_{i=\gamma(jb+i)}a'_{ji}(z/|z|)\tilde{\tau}^{k_0-jb-i}+o(|z|^{\gamma k_0}).$$

If $\gamma>0$ then no terms $\tilde{\tau}^{k_0-1}$, $\tilde{\tau}^{k_0-2}$ occur in the first term of the right-hand side because $b\geq 2$ and $\gamma<1$. This implies that

$$\sum_{i=\gamma(jb+i)}a'_{ji}(z/|z|)\tilde{\tau}^{k_0-jb-i}=0$$

has a non-real root for every $z\in U$, $z\notin Z$. Then taking $z\notin Z$ sufficiently close to the origin we conclude that $R(|z|^{\gamma}\tilde{\tau},z)=0$ has a non-real root $\tilde{\tau}$ and so does $R(\tau,z)=0$. We turn to the case $\gamma=0$. This means that there is $j\geq 1$ with $a'_{j0}\neq 0$. Since

$$R(\tau,z)=\sum_{j=0}^{j_0}a'_{j0}\tau^{k_0-jb}+O(|z|)$$

the same argument can be applied if either $b\geq 3$ or $a'_{10}\geq 0$ to conclude that $R(\tau,z)=0$ has a non-real root for some $z\in U$, $z\notin Z$. It remains to examine the case $b=2$ with $a'_{10}<0$ and hence $a=1$ necessarily. In this case we employ $R'(\tau,z)$. Noting that the coefficient of τ^{k_0} and τ^{k_0-2} in $R'(\tau,z)$ are equal to $(-1)^{pk_0}a'_{00}$ and $-(-1)^{pk_0}a'_{10}$ respectively the proof is reduced to the preceding case. Thus we have proved Proposition 3.2.

4. Proof of Theorem 2.7

Our aim in this section is to prove Theorem 2.7. Let z_k be characteristics of $P_{(z_0,\dots,z_{k-1})}(z)$ of order r_k $(r_k\geq 2)$, $1\leq k\leq s$ and let $P_{m-j(z_0,\dots,z_k)}(z)$ be given by (2.4). We first give another formula which defines $P_{m-j(z_0,\dots,z_k)}(z)$ directly. Let $0<\mu_0<\mu_1<\dots<\mu_s$ be a sequence of positive parameters with $\mu_j=O(\mu_{j+1}^{m+1})$ as $\mu_{j+1}\to 0$.

Lemma 4.1. *Let μ_j be as above. Then*

$$P_{m-j}(z_0+\mu_0 z_1+\ldots+\mu_0\ldots\mu_{k-1}z_k+\mu_0\ldots\mu_k z) = \mu_0^{r_0-2j}\mu_1^{r_1-2j}\ldots\mu_k^{r_k-2j}(P_{m-j(z_0,\ldots,z_k)}(z)+O(\mu_k)).$$

Proof. Since z_0 is a characteristic of P_m of order r_0 it follows from Corollary 2.6 that $P_{m-j}(z_0+\mu_0 z)=\mu_0^{r_0-2j}(P_{m-j,z_0}(z)+O(\mu_0))$. Hence

$$P_{m-j}(z_0+\mu_0(z_1+\mu_1 z))=\mu_0^{r_0-2j}(P_{m-j,z_0}(z_1+\mu_1 z)+O(\mu_0)).$$

Since z_1 is a characteristic of P_{m,z_0} of order r_1 we see from Corollary 2.6 again that

$$P_{m-j,z_0}(z_1+\mu_1 z)=\mu_1^{r_1-2j}(P_{m-j(z_0,z_1)}(z)+O(\mu_1)).$$

Noting that $\mu_0=O(\mu_1^{m+1})$ we get the desired result with $k=1$. A repetition of the argument completes the proof.

Assume that $\Lambda_{(z_0,\ldots,z_s)}(P_m)^\sigma\subset\Lambda_{(z_0,\ldots,z_s)}(P_m)$ and recall that $P_{(z_0,\ldots,z_s)}(z)$ is an invariantly defined polynomial on $T_{z_0}(T^*\Omega)$. Then one can find local coordinates x preserving the plane $x_0=0$ such that

$$P_{m(z_0,\ldots,z_s)}(z)=q(\xi_a,x_b)$$

with a homogeneous polynomial q of degree r_s where $x=(x_a,x_b)$, $x_a=(x_0,\ldots,x_k)$, $x_b=(x_{k+1},\ldots,x_n)$ is a partition of the variables x and $\xi=(\xi_a,\xi_b)$ is that of ξ (see Proposition 2.6 in Nishitani [8]).

Lemma 4.2. *In the above local coordinates x we have*

$$P_{(z_0,\ldots,z_s)}(z)=\sum_{r_s-2j\geq 0}P_{m-j(z_0,\ldots,z_s)}(z).$$

Proof. By Lemma 2.5 it sufficies to show that

$$Q_{m-j(z_0,\ldots,z_s)}(z)=P_{m-j(z_0,\ldots,z_s)}(z) \tag{4.1}$$

in these coordinates. Since $P_{(z_0,\ldots,z_s)}(z)$ is hyperbolic it follows from Corollary 12.4.8 in Hörmander [6, II] that $Q_{m-j(z_0,\ldots,z_s)}(z)$ are polynomials in (ξ_a,x_b) and hence $Q^{(\alpha)}_{m-j(z_0,\ldots,z_s)(\alpha)}(z)=0$ if $|\alpha|\geq 1$. Recall that

$$Q_{m-j(z_0,\ldots,z_{k-1})}(z)=P_{m-j(z_0,\ldots,z_{k-1})}(z)+\sum_{i<j,|\alpha|=j-i}c_\alpha P^{(\alpha)}_{m-i(z_0,\ldots,z_{k-1})(\alpha)}(z)$$

with some constants c_α. Using this formula and the fact

$$Q_{m(z_0,\dots,z_{k-1})}(z)=P_{m(z_0,\dots,z_{k-1})}(z)$$

we conclude (4.1) by induction on j.

It is clear that what we shall prove is that

$$P_{m-j(z_0,\dots,z_s)}(z)=0,\quad j\geq 1,\ r_s-2j\geq 0. \tag{4.2}$$

Assuming that (4.2) is false we construct an asymptotic solution to $Pu\sim 0$ contradicting the a priori estimate in Proposition 3.1. Let us take

$$n_j=(m+1)^{s-j},\quad 1\leq j\leq s,\quad n_0=\sum_{k=1}^{s} n_k$$

so that with $\mu_0=\lambda^{-1/2+n_0\delta}$, $\mu_j=\lambda^{-n_j\delta}$, $j\geq 1$ we have

$$\mu_0=O(\mu_1^{m+1}),\quad \mu_j=\mu_{j+1}^{m+1}$$

for sufficiently small $\delta>0$. Note that

$$\mu_0^{r_0-2j}\dots\mu_l^{r_l-2j}=\lambda^{m_l},$$

$$m_l=(-\tfrac{1}{2}+n_0\delta)r_0+j-\sum_{k=1}^{l} n_k r_k\delta-2j\delta\sum_{k=l+1}^{s} n_k.$$

Let us set

$$\begin{aligned}\gamma(\lambda)_\nu&=z_{0\nu}+\lambda^{-1/2+n_0\delta}z_{1\nu}+\lambda^{-1/2+n_0\delta-n_1\delta}z_{2\nu}+\dots+\lambda^{-1/2+n_0\delta-\sum_{k=1}^{s-1}n_k\delta}z_{s\nu}\\&=(y(\lambda),\nu\eta(\lambda))\end{aligned}$$

where $z_{j\nu}=(x_j,\nu\xi_j)$ for $z_j=(x_j,\xi_j)$ and $1\ll\nu\ll\lambda^{\delta/2m}$. When $\nu=1$ we write $\gamma(\lambda)$ for $\gamma(\lambda)_\nu$ dropping ν. We now study

$$P(y(\lambda)+\lambda^{-\sigma}x,\lambda\nu\eta(\lambda)+\lambda^{\sigma}\xi)=\sum\lambda^{m-j}P_{m-j}(\gamma(\lambda)_\nu+(\lambda^{-\sigma}x,\lambda^{\sigma-1}\xi)) \tag{4.3}$$

with

$$\lambda^{-\sigma}x=(\lambda^{-1/2+\varepsilon}x_a,\lambda^{-1/2-\varepsilon}x_b),\quad \lambda^{\sigma-1}\xi=(\lambda^{-1/2-\varepsilon}\xi_a,\lambda^{-1/2+\varepsilon}\xi_b)$$

where $0<\varepsilon<\delta$. Setting $X=(\xi_a,x_b)$, $Y=(\xi_b,x_a)$ and taking the homogeneity into account the right-hand side of (4.3) is written as

$$\sum\lambda^{m-j}\nu^{m-j}P_{m-j}(\gamma(\lambda)+\lambda^{-1/2+\varepsilon}Y_{\nu^{-1}}+\lambda^{-1/2-\varepsilon}X_{\nu^{-1}}).$$

Lemma 4.3. *Let $r_s-2j<0$. Then taking $\delta>0$ small we get*

$$\lambda^{m-j}\nu^{m-j}P_{m-j}(\gamma(\lambda)+\lambda^{-1/2+\varepsilon}Y_{\nu^{-1}}+\lambda^{-1/2-\varepsilon}X_{\nu^{-1}})=O(\lambda^{m+m^*-\delta/2}) \tag{4.4}$$

where $m^*=(-\frac{1}{2}+n_0\delta)r_0-\sum_{k=1}^{s}n_kr_k\delta$.

Proof. When $r_0-2j<0$ the assertion is clear because $m-j\le m-r_0/2-\frac{1}{2}$. Let $r_0-2j\ge 0$ and $r_s-2j<0$. Choose l to be the smallest integer satisfying $r_l-2j\ge 0$. Since

$$\gamma(\lambda)=z_0+\ldots+\lambda^{-1/2+n_0\delta-\sum_{k=1}^{l-1}n_k\delta}(z_l+\lambda^{-n_l\delta}z_{l+1}+\ldots)$$

and $P_{m-j(z_0,\ldots,z_l)}(z)$ exists by Corollary 2.6 the left-hand side of (4.4) becomes

$$\lambda^{m-j}\nu^{m-j}\lambda^{m_l}(P_{m-j(z_0,\ldots,z_l)}(z_{l+1}+\ldots)+O(\lambda^{-n_l\delta})).$$

Since $m^*-(m_l-j)=\delta\sum_{k=l+1}^{s}(2j-r_k)n_k\ge\delta$ and $\nu^m\ll\lambda^{\delta/2}$ we obtain the assertion.

We turn to the case $r_s-2j\ge 0$. From Lemma 4.1 with $k=s$ it follows that

$$\begin{aligned}&\lambda^{m-j}\nu^{m-j}P_{m-j}(\gamma(\lambda)+\lambda^{-1/2+\varepsilon}(Y_{\nu^{-1}}+\lambda^{-2\varepsilon}X_{\nu^{-1}}))\\&=\nu^{m-j}\lambda^{m-j+m_{s-1}-(\delta-\varepsilon)(r_s-2j)}(P_{m-j(z_0,\ldots,z_s)}(Y_{\nu^{-1}}+\lambda^{-2\varepsilon}X_{\nu^{-1}})+O(\lambda^{-\delta+\varepsilon})).\end{aligned}$$

Noting that $m_{s-1}-j-(\delta-\varepsilon)(r_s-2j)=m^*+\varepsilon(r_s-2j)$ and $P_{m-j(z_0,\ldots,z_s)}(z)$ is independent of Y (see the proof of Lemma 4.2) the right-hand side yields

$$\nu^{m-j}\lambda^{m+m^*-\varepsilon r_s+2\varepsilon j}(P_{m-j(z_0,\ldots,z_s)}(X_{\nu^{-1}})+O(\lambda^{-\delta+(2r_s+1)\varepsilon})).$$

Taking $\varepsilon>0$ so that $2\varepsilon(2m+1)<\delta$ we summarize what we have proved.

Lemma 4.4. *Let $\nu=\theta\lambda^{2\varepsilon}$ with $\theta\in\mathbf{R}\setminus 0$. Then we have*

$$\begin{aligned}&P(y(\lambda)+\lambda^{-\sigma}x,\lambda\nu\eta(\lambda)+\lambda^{\sigma}\xi)\\&\qquad=\lambda^{m^*+m-\varepsilon r_s}\nu^m\Big(\sum_{r_s-2j\ge 0}\theta^{-j}P_{m-j(z_0,\ldots,z_s)}(X_{\nu^{-1}})+O(\lambda^{-\delta/2})\Big)\end{aligned}$$

where $\delta>0$.

Let us set

$$R(X)=\sum_{r_s-2j\ge 0}\theta^{-j}P_{m-j(z_0,\ldots,z_s)}(X)$$

where $X=(\xi_a,x_b)=(\xi_0,X')$.

Lemma 4.5. *Assume that some $P_{m-j(z_0,\dots,z_s)}(X)$ with $j\geq 1$, $r_s-2j\geq 0$ is not identically zero. Then with a suitable $\theta\in\mathbf{R}\setminus 0$*

$$R(X)=0$$

has a non-real root ξ_0 for some X'.

Proof. It is clear with a small positive ε that

$$R(\xi_0,\varepsilon X')=c\xi_0^{r_s}+O(\varepsilon)+\sum_{r_s-2j\geq 0,j\geq 1}\theta^{-j}\varepsilon^{p_j}(R_j(X)+O(\varepsilon))$$

with a constant $c\neq 0$ where p_j are non-negative integers. Recall that there is $j\geq 1$ with $R_j(X)\neq 0$. Letting

$$\gamma=\min_{j\geq 1,R_j(X)\neq 0}\frac{p_j}{j}$$

and setting $\theta=\varepsilon^{\gamma}\tilde{\theta}$ we have

$$R(\xi_0,\varepsilon X')=c\xi_0^{r_s}+\sum_{p_j-j\gamma=0}\tilde{\theta}^{-j}R_j(X)+o(1),\quad \varepsilon\to 0.$$

Since the degree of R_j are less than or equal to r_s-2j, $j\geq 1$ it is clear that

$$c\xi_0^{r_s}+\sum_{p_j-j\gamma=0}\tilde{\theta}^{-j}R_j(X)=0$$

has a non-real root for some X' changing $\tilde{\theta}$ to $-\tilde{\theta}$ if necessary. Taking $\varepsilon>0$ sufficiently small we get the desired assertion.

The rest of the proof of Theorem 2.7 is a repetition of that of Theorem 2.4.

Remark. If we are interested in the microlocal Cauchy problem near z_0 then the wave front set of the asymptotic solution that we have constructed should be contained in a conic neighborhood of z_0. Hence the sign of θ in Lemma 4.5 is limited to be positive. In this case we could have a weaker assertion: $P_{(z_0,\dots,z_s)}(z)$ has only real zeros ξ_0 for every (x,ξ').

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Tatsuo Nishitani
Department of Mathematics
College of General Education
Osaka University
Toyonaka
Osaka 560
Japan